3. The Clockwork Universe

1583 CE–1819 CE

Beyond the Greeks

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Rise of Mechanics

The New Astronomy

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Stars' orbits you will know; and bold,
You learn what nature has to teach;
Your soul is freed, and you behold
The spirits' words, the spirits' speech.
Though dry reflection might expound
These holy symbols, it is dreary:
You float, oh spirits, all around;
Respond to me, if you can hear me.

What jubilation bursts out of this sight
Into my senses – now I feel it flowing,
Youthful, a sacred fountain of delight,
Through every nerve, my veins are glowing.
Was it a god that made these symbols be
That soothe my feverish unrest,
Filling with joy my anxious breast,
And with mysterious potency
Make nature's hidden powers around me,
manifest?

Am I a god? Light grows this page –
In these pure lines my eye can see
Creative nature spread in front of me.
But now I grasp the meaning of the sage:
"The realm of spirits is not far away;
Your mind is closed, your heart is dead.
Rise, student, bathe without dismay
In heaven's dawn your mortal head."

All weaves itself into the whole,
Each living in the other's soul.
How heaven's powers climb up and descend.
Passing the golden pails from hand to hand!
Bliss-scented, they are winging
Through sky and earth – their singing
Is ringing through the world.

What play! Yet but a play, however vast!
Where, boundless nature, can I hold you fast?
And where you breasts? Wells that sustain
All life – the heaven and the earth are nursed.
The wilted breast craves you in thirst –
You well, you still – and I languish in vain?

From Goethe's '*Faust*'.

Translated by Walter Kaufmann, 1961.

1583–1600 CE **Giordano Bruno** (1548–1600, Italy). Philosopher of the Renaissance. Adopted the view that the universe is infinite with innumerable stars and planetary systems. In his publication "Dell' infinito universo e mondi" ("Of infinity, the universe and the world") he criticized the doctrines of Aristotle and Ptolemy that there was an absolutely fixed center in the universe.

Bruno was christened Filippo. In his 15^{th} year he entered the order of the Dominicans at Naples. But from an early age he was on the move through the cities of Europe: Rome (1576), Geneve (1579), Paris (1581), Oxford (1582), Wittenberg (1587), Prague (1588), Frankfurt (1591), Zürich (1592), and Venice (1593).

He was burned at the stake in Rome by the Inquisition on the charge of believing in the nonexistence of the absolute truth. His last cry from the burning stake was "Eppur si muove!" ("and nonetheless it moves!").

Beyond the Greeks – The Emergence of Modern Science

* * *

"Until the Scientific Revolution of the 17^{th} century, meaning flowed from ourselves into the world; afterwards, meaning flowed from the world to us".

(Chet Raymo, 1999)

In some branches of science, notably astronomy, the ancients made substantial contributions. In physics, the most fundamental of the sciences, the record is scant. Simple engineering tools like the lever, the wheel, and the inclined plane were known before recorded history; with them, by 3000 BCE, *the Egyptians had built such magnificent structures as the pyramids. The functioning of these tools, of course, depended on unsuspected physical principles.*

The first faint stirring of the science itself — the rigorous examination of physical principles — was apparently a product of Greek civilization. From the contemporary physicist's viewpoint, its most interesting legacy was one of the first recorded scientific controversies, having to do with the fundamental nature of matter. ***Democritos****, who lived about* 400 *BCE, was the leader of an "atomistic" school which held that all matter was composed, in varying combinations, of four different kinds of particles, tiny and indivisible. He believed that their existence was literally a fact.* ***Plato*** *the philosopher, his foremost opponent, conceived of fundamental matter in terms of mathematical patterns, forms, and "ideas". This ancient controversy between materialism and idealism, has, oddly enough, been revived recently in a quite specific way by modern atomic physics, and especially by the quantum theory.*

The activity of Greek physicists was not limited to theoretical and philosophical problems. Among the earliest experimental physicists was ***Pythagoras****, the* 6^{th}*-century philosopher and mathematician. He and his school attempted to formulate a theory of musical harmony by experimenting with strings of different lengths, thicknesses and tensions. It was indeed the first instance of the application of mathematics to a basic physical phenomenon.* ***Euclid*** *the geometer, who flourished at Alexandria about* 300 *BCE, made studies in the laws of perspective and reflection, and is said to have written on music and mechanics.* ***Hero of Alexandria****, who lived probably about* 150 CE*, made pulleys, gears, siphons, and an engine which used steam to rotate a hollow sphere — the first known utilization of the law of action and reaction.* ***Ptolemy****, an Alexandrian of about the same period, whose cosmology was accepted for many centuries, wrote on reflection and refraction. Unquestionably the greatest ancient figure in both physics and mathematics was* ***Archimedes of Syracuse****, killed by a Roman soldier in 212 BCE. He was famous for his engineering and military inventions. More important, he founded the sciences of statics and hydrostatics.*

About the second half of the first century BCE, the Roman ***Vitruvius*** *wrote De Architectura, an encyclopedia of useful knowledge in the fields of architecture, engineering and construction. He investigated such matters as the measurement of time and acoustics, comparing the waves of sound to those caused by a stone thrown into a pond. Like his fellow countrymen, he laid emphasis on practical applications rather than on theoretical scientific knowledge.*

The above names and a few others make up the meager roster of ancient physicists. Their influence on the main stream of scientific history was slight. The monasteries, the cultural centers of the Middle Ages, were concerned primarily with questions of philosophy and religion — for example, whether God could create a stone so heavy that the Himself could not lift it, a problem which does not lend itself to experimental verification. Over all science lay

*the shadow of **Aristotle**, the Greek scientist of the 4^{th} century BCE. His interests were universal, embracing logic, philosophy, history, politics and the biological and natural sciences. He was the apostle par excellence of rationalism, the belief in logical rather than experimental explanations. From a purely philosophical viewpoint, **Francis Bacon**, Galileo's contemporary, did much to overthrow this doctrine. But in actual practice, it was **Galileo** who who sounded its death knell.*

1583–1637 CE **Galileo Galilei** (1564–1642, Italy). Pioneer of modern applied mathematics, physics and astronomy. The founder of modern physics on account of his willingness to replace old assumptions in favor of new scientifically deduced theories.

Supported the Copernican Revolution and paved the road for Newton's laws of motion. Introduced the method of mathematical analysis for the solution of physical problems.

His major achievements are these:

(1) Originated (1583) modern accurate time-keeping through the discovery of a natural periodic process that can be repeated indefinitely and counted — the swinging pendulum. He found that each simple pendulum has its own period, depending on its length. [The actual step of applying the pendulum to clockwork, so as to record mechanically the number of swings, was taken by **Christiaan Huygens** in 1656.]

(2) First pointed his self-made telescope at the sky in 1609. With this instrument he extended our knowledge by observing many stars which are too faint to be seen directly. Discovered the satellites of Jupiter[1], and phases of Venus[2]. This small-scale model of the solar system convinced him of the truth of the Copernican theory.

[1] **Simon Mayr** (Mair, Meyer, Marius; 1573-1624, Germany). Astronomer. Assistant to Tycho Brahe (1601). Claimed to have discovered (ca 1610) largest moons of Jupiter and named them: *Io*, *Europa*, *Ganymede* and *Callisto* (a discovery, generally credited to Galileo). Made first telescopic observations of Andromeda spiral nebula (1611).

[2] It was these observations, rather then the ideas of Copernicus, that dealt a deathblow to the Ptolemaic geocentric system: *Both* the Ptolemaic and the Copernican views described the motions of the planets. The heliocentric view of Copernicus was a simpler hypothesis (*Ockham's razor!*). This in itself, however, is not a

(3) While investigating the motion of objects in *free fall*[3] he was first to realize that it is not the velocity of a body, but its acceleration which signifies that there are forces acting on it (1604–1609).

(4) Showed by experiments (or thought-experiments as some claim) that bodies of different constitution and weight (mass) are equally accelerated by gravity (1609), i.e. fall with the same terminal velocity. Formulated correctly the basic *kinematic* laws of falling bodies.

(5) Recognized the concepts of parallelograms of forces and velocities, and with it the separation of projectile motion into horizontal and vertical *components*.

(6) Formulated the restricted *mechanical* '*principle of relativity*', stating that no mechanical experiment will reveal whether a system is at rest, or is moving uniformly in a straight line. In other words — the laws of mechanics are invariant under a 'Galilean transformation'. [In modern notation, $\mathbf{r}' = \mathbf{r} - \mathbf{v}_0 t,\ t' = t$: a coordinate transformation that connects observers in different frames.]

(7) Initiated the modern attitude toward the actual *infinite* in mathematics (1638). Asserted that infinite numbers obey a different "arithmetic" from finite numbers: If using the ordinary notions of "equal" and "less than" on infinite sets leads to contradictions, this is not a sign that infinite sets cannot exist, but rather, that these notions do not apply without modifications to infinite sets. Galileo himself did not see how to carry

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demonstration of its validity. Nature may, after all, be complex.

However, both systems differed in another aspect: According to Ptolemy, the sun circled about the earth, and inside of the sphere of the sun lay the spheres of Venus and Mercury. With such a geometry, it would be impossible for us ever to see the entire bright side of Venus! According to Copernicus however, both Venus and earth circled the sun. Since Venus was sometimes beyond the earth and the sun, it would be possible for us to see its bright side. Thus, when Galileo turned his telescope to Venus, and *saw* that its disk underwent phases from a '*full Venus*' (similar to our full moon), to a *new Venus* (the dark side of Venus, corresponding to our new moon), it was clear that the Copernican hypothesis stood vindicated. *Sunspots* were reported independently in 1611 by Galileo and **Christoph Scheiner** (1573–1650, Germany). However, Galileo and the other "discoverers" of sunspots were well aware of the existence of sunspots, and naked-eye reports of them, *before* they looked at the sun through telescopes.

[3] **Giovanni Battista Benedetti** (1530–1590, Italy) was an important forerunner of Galileo. Studied under Tartaglia. He worked on the *free fall* of bodies and proposed (ca 1560) a theory almost identical to that which Galileo published in *De motu* (1590).

out such a modification of these notions (this was left to **Georg Cantor** some 250 years later).

(8) Invented the *thermoscope* (1556), a prototype of the later thermometer, that could only indicate relative *changes* in temperature: an inverted flask with a narrow neck was lowered into a shallow bowl containing liquid. The liquid would go part way up the flask's neck; changes in the surrounding temperature would either raise or lower the liquid. There was no way to tell what the temperature actually was. It was later improved by his friend **Sanctorius** (1611). Only in 1709 did **Daniel Fahrenheit** invent the calibrated mercury thermometer.

Although Galileo[4] did not define *inertia*[5], he came close to it by understanding that one must exert a force on a body in order to accelerate or decelerate it. It remained for Newton to generalize Galileo's results to forces, in general, and to define mass and inertia.

Galileo's experiments with falling bodies[6] were a crucial landmark in physics in the sense that they marked the *demise of Aristotelian physics*. In 1907, Einstein elevated Galileo's *experiment* into a *principle*, just as earlier in 1905 he had generalized Galileo's principle of relativity to incorporate all laws of physics.

After Galileo's work became known, philosophers and scientists slowly began to realize that the behavior of physical objects could be described in mathematical terms. This led to the idea that there existed laws that had been established by God to regulate his creation. The new idea that nature itself was subject to laws, depended upon the implicit assumption that such laws did not change with time, and that their divine origin guaranteed their eternal endurance.

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[4] Often just called by his first name.

[5] **Nicole Oresme** and **Johannes Buridanus** (ca 1350), in Paris, criticized Aristotle's doctrine of motion. They in turn were influenced by the idea of *impetus* introduced ca 530 by **John Philoponus** (John the Grammarian). Philoponus was a Greek philosopher of Alexandria who speculated that a projectile would gain momentum from the mechanism that fired it, thus arriving at a crude idea of inertia. Galileo amalgamated these notions into his theory of motion.

[6] In 1328, **Thomas Bradwardine** (1290–1349, England) discussed the issue of the hypothetical free fall of bodies in void and concluded that two bodies of the same material but different size, will fall with the same terminal velocity, contradicting the Aristotelian view that the heavier body falls faster. Bradwardine was a theologician and mathematician who became the Archbishop of Canterbury.

Like Copernicus and Kepler, and in contradistinction to the Greek doctrine, Galileo realized that the 'mysteries' of nature could be illuminated only with the assistance of accurate observations and experiments. It was just this approach that was responsible for the epochal success of these scientists. In fact, as soon as Galileo and others began to apply the scientific method in physics and astronomy, a chain of discoveries resulted.

These discoveries fired the imagination and enthusiasm of European thinkers. Scientific societies were organized, scientific journals began to appear. Science, hitherto the pursuit of occasional lone individuals, became a social enterprise and has continued to be so to the present. Furthermore, it became fashionable. Newton's work, for example, made a profound impression on every writer in Europe.

Galileo was born at Pisa. His father Vincenzio was an impoverished descendant of a noble Florentine house, which had exchanged the surname of Bonajuti for that of Galilei. Vincenzio was a competent mathematician and a musician. By his wife, Giulia Ammannati of Pescia, he had 4 daughters and 3 sons, the eldest of which was Galileo. From his earliest childhood, Galileo was remarkable for intellectual aptitude as well as for mechanical invention. His education was principally conducted in the monastery of Vallombrosa, near Florence. There he acquired a fair command of Latin, Greek and logic.

He was at this time attracted toward a religious life, but his father withdrew him permanently from the care of the monks and placed him in 1581, at the University of Pisa on a course of medical studies. In that year, while watching a lamp set swinging in the cathedral of Pisa he observed that, whatever the range of its oscillations, they were invariably executed in equal times. The experimental verification of this fact led him to the important discovery of the isochronism of the pendulum. (More than 50 years later he turned it to account in the construction of an astronomical clock.)

Up to this time he was entirely ignorant of mathematics, his father having carefully held him aloof from a study which he rightly apprehended would lead him to forsake medicine. Listening one day to an accidental lesson in geometry, his attention was riveted and he threw all his energies into the new pursuit. He rapidly mastered the elements of the science, and eventually extracted his father's reluctant permission to exchange **Hippocrates of Cos** and **Galen** for **Euclid** and **Archimedes**. For lack of means, he withdrew from the university in 1585 before obtaining a degree, and returned to Florence. Shortly afterward he invented the *hydrostatic balance*, which he used to find the specific gravity of objects by immersing them in water.

In 1588 he wrote a treatise on the center of gravity in solids, which together with his former invention, made his name known throughout Italy and secured him, at age 24, the post of professor of mathematics at the University of

Pisa. During the ensuing two years (1589–1591) he carried out a series of experiments by which he established the fundamental principles of dynamics. His new views put him on a collision course with Aristotelian physics and he was forced to leave the university. Through the death of his father in July 1591, family cares and responsibilities devolved upon him, and thus his nomination to the chair of mathematics at the University of Padua, was welcome both for the relief it offered from pecuniary need, and as opening the road to scientific distinction.

His residence at Padua (1592–1610) was a course of uninterrupted prosperity. His appointment was renewed three times, on each occasion with the expressions of the highest esteem. His lectures were attended by persons of the highest distinction from all parts of Europe, and such was the charm of his demonstrations, that a hall capable of containing 2000 people had eventually to be assigned for the accommodation of the overflowing audiences which they attracted.

In 1593 he constructed the first *thermoscope*, consisting of a bulb filled with air and water and terminating in a vessel of water.

In spite of his adherence to the Copernican theory, he continued to conform, in his public teaching, to Ptolemaic principles, waiting for the proper opportunity to make an open onslaught upon the Aristotelian axioms[7]. The discovery of the telescope by the obscure Middleburg optician, **Jan (Hans) Lippershey** (1608), provided him with that opportunity. Galileo's direction of his new instrument [after one night of profound meditation on the principles of refraction in June 1609 in Venice, he was able to produce a telescope with a magnifying power of 32] to the heavens opened an era in the history of astronomy.

Discoveries followed upon it with astounding rapidity and in bewildering variety: During 1609–1613 his discoveries indicated: that the *moon* (contrary to the teachings of Aristotle) was *not* a smooth sphere shining by its own light, that the *Milky way* was a mass of numerous stars, that *Jupiter* has 4 bright satellites (which he named after the Medici family, who ruled the province of Tuscany where he was born), the peculiar form of *Saturn*, the phases of *Venus* and finally the *sunspots*[8].

[7] Nevertheless, he was not *completely* free of the Pythagorean and Neo-Platonic doctrines, which had been disseminated during the medieval period and early renaissance: his work is premised on the deep-seated conviction of a *simple*, *ordered world*, free from arbitrariness and disclosing geometrical regularity.

[8] It was discovered earlier (1611) by the German astronomer **Johannes Fabricius** (1587–1615). He concluded from his observations that the spots were integral parts of the sun and that their movement was caused by the *sun's rotation about*

In September 1610 Galileo finally abandoned Padua for Florence. His researches with the telescope had been rewarded by the Venetian senate with the appointment for life to his professorship, at an unprecedented high salary. His discovery of the 'Medicean Stars' earned him the title of 'philosopher and mathematician extraordinary' to the grand duke of Tuscany.

When Galileo firmly upheld the Copernican theory that the earth moves around the sun, Church officials warned him to abandon this 'heretical' system. At the same time (1616), the Church placed the work of Copernicus on the *Index* of prohibited books, where it remained for 200 years.

In 1632, Galileo published his masterpiece, *A dialogue on the Two Systems of the World*[9]. After a long trial, Church officials forced him to say that he gave up his belief in the Copernican theory, and sentenced him to an indefinite prison term, which he spent in his villa near Florence.

Domestic afflictions combined with numerous and painful infirmities to embitter his old age. His sister in law and her whole family, who came to live with him on his return from Rome, perished shortly thereafter in the plague. In 1634, his eldest and best-beloved daughter, a nun in a convent, died.

Galileo was never married, but by a Venetian woman named Marina Gamba he had three children — a son who married and left descendants, and two daughters who took the veil at an early age.

His prodigious mental activity continued undiminished to the last. In 1636 he completed his *Discorsi a due nuove scienze* (Discourses on the Two New Sciences) in which he recapitulated the results of his early experiments and presented mature meditations on the principles of mechanics. It summed up

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its axis. In 1612 Galileo in Italy, **Thomas Harriot** in England, and the German Jesuit **Christoph Scheiner** published their own observations. Galileo came forth with the same explanation of the movement of the spots as did Fabricius, whereas Scheiner said they were small planets revolving around the sun. In his 1613 publication *Istoria e dimostrazioni intorno alle macchie solarie e loro accidenti*, Galileo disproved Scheiner's reasoning and, for the first time, publicly supported the heliocentric theory of Copernicus. Scheiner, who eventually conceded that Galileo was right, went on to make much more accurate observations than Galileo had, and he found that the sun completes a full rotation in 27 days.

[9] For further reading, see:

- Galilei, G., *Dialogues Concerning Two New Sciences* (*1638*), Dover Publications: New York, 1954, 300 pp.
- Drake, S., *Galileo at Work, His Scientific Biography*, University of Chicago Press: Chicago, 1978, 536 pp.

his life's work on motion, acceleration and gravity, and furnished a basis for the three laws of motion laid down by Newton in 1687. His last telescopic discovery — that of the moon's diurnal and monthly librations — was made in 1637, only a few months before he became blind. It was in this condition that **Milton** found him when he visited him in Arcetri in 1638. He continued his scientific correspondence and thought out the application of the pendulum to the regulation of clockwork, which Huygens successfully realized 15 years later.

He was also engaged in dictating to his disciples, **Viviani** and **Torricelli**, his latest ideas on the theory of impact, when he was seized with the slow fever which, within two months, brought him to the grave. He was buried in the Church of Santa Croce in Florence. Fifty years after his death, the city erected a monument at the church in his honor.

Science Progress Report No. 6

"Eppur si muove" (1600–1633)

*The Catholic hierarchy recognized, as had the Protestants earlier, that the new cosmology was subversive — incompatible with the traditional, authoritarian society. One of the first victims of the Counter-Reformation was **Giordano Bruno**, a former monk. Bruno traveled to England and befriended its leading political and scientific figures; and when he returned, he popularized Copernican theory on the continent. Bruno took Digges' version of the infinite, Copernical universe and purged it of remaining Ptolemaic elements, such as the perfect spheres that carried the planet's orbits. He made this infinite universe, with its infinity of inhabited worlds, the basis of his philosophy, incorporating Nicholas of Cusa's thinking and going beyond it. Bruno explicitly challenged the idea of creation* ex nihilo*, arguing that the universe must be unlimited in both space and* time*, without beginning or end.*

Bruno was a philosopher, not a scientist, and he used the tradition of logical argument to support the Copernican worldview. Above all, he considered himself a loyal Catholic bent on reforming, not rejecting, the church. Yet on his return to Catholic territory in 1592*, he was promptly arrested. Cardinal*

Bellarmino, a prominent leader of the Counter-Reformation and the pope's own theologian, saw in Bruno's writing an effort to subvert the church from within. The idea of an infinite number of worlds not only undermined the primacy of the church hierarchy, it contradicted as well sources of authority — the idea was found neither in the Bible nor in Aristotle or Plato. Moreover, it very obviously destroyed the Catholic vision of a material, subterranean hell and an ethereal heaven beyond the cosmic spheres: it portrayed a cosmos in which these threats and enticements would have no place, and would be comprehensible to only a few — certainly not to the ill-educated peasants, as the simple picture of a heaven above and a hell below certainly was.

Over seven years of imprisonment Bellarmino labored to get Bruno to recant the doctrine of the infinite plurality of worlds. Bruno refused, and in 1600 *he was burned at the stake.*

Since the charges against Bruno were never made public, other Catholic scientists, including Galileo, did not take his execution as a sign of Catholic hostility to Copernicus. But this hostility was confirmed even as the new theory triumphed.

The astronomical discoveries of Galileo added support to the Copernican system and brought him more fame. But the new views of the solar system promulgated by Galileo's formidable dialectic zeal alerted the Church, which saw in his scientific teaching a danger to religion. The new astronomy was publicly denounced by the Church, and on February 1615 *the matter was brought before the Inquisition. Consequently, Galileo received a semi-official warning to avoid theology and limit himself to physical reasoning.*

However, Galileo had already committed himself to dangerous grounds. In December 1615 *he lectured before the entire pontifical court, full of confidence that the weight of his arguments and the vivacity of his eloquence could not fail to convert them to his views. He was cordially received, and eagerly listened to, but his imprudent ardor served but to injure his cause.*

On the 24^{th} *of February* 1616*, the consulting theologians of the Holy Office characterized the proposition that the sun is immovable in the center of the world as "absurd philosophy and formally heretical, because expressly contrary to Holy Scripture". The proposition that earth has diurnal rotation was described by the Church as "open to the same censure in philosophy, and at least erroneous to faith". Two days later Galileo was summoned to the palace of Cardinal Bellarmino (1542–1621), and there officially admonished not thenceforward to "hold, teach or defend" the condemned doctrine. This injunction he promised to obey. However, he trusted his dialectical ingenuity to find the means of presenting his scientific convictions under the transparent veil of an hypothesis.*

*During 1616–1623, Galileo led a life of studious retirement in the Villa Segni at Bellosguardo, near Florence, and maintained an almost unbroken silence. When a new pope was elected in 1623, Galileo visited Rome with the hope to obtain the revocation of the decree of 1616, through personal influence. Although he failed to achieve this, he expected that the decree would at least be interpreted in a liberal spirit. On his return to Florence, he therefore set himself to complete his work "*Dialogo dei due massimi sistemi del mondo*". It emerged from the press in 1632. A tumult of applause from every part of Europe followed its publication.*

It was at once evident that the whole tenor of this work was in flagrant contradiction with the edict passed 16 years before its publication, as well as with the author's personal pledge of conformity to it. Toward the end of August 1632 the book's sale was prohibited, and on the 1^{st} of October the author was summoned to Rome by the Inquisition. He pleaded his age, now close to 70 years, his infirm health, and the obstacles to travel caused by quarantine regulations, but the pope was indignant at what he held to be his ingratitude and insubordination, and no excuse was admitted. He arrived on 13 February 1633 and was detained until 21^{st} of June, when he was finally examined under threat of torture.

On the 22^{nd} of June 1633, in the church of Santa Maria Sopra Minerva, Galileo read his recantation and received his sentence: He was condemned, as "vehemently suspected of heresy", to incarceration at the pleasure of the tribunal, and by way of penance was enjoined to recite once a week for three years the seven penitential psalms. This sentence was signed by 7 cardinals, but did not receive the customary papal ratification. He was held by the Inquisition until December 1633, when he was allowed to return to his villa. There he spent the remaining 8 years of his life — practically under house arrest, in strict seclusion, constantly watched by the Inquisition.

Galileo did not make the trek to the stake, because he was sensible enough not to die for his beliefs but to live for them — he recanted in public and went on with his studies in private.

He died in 1642, the year Newton was born, surrounded by friends and pupils. His epitaph was written for him by posterity: eppur si muove *— the famous words which he never uttered at his trial. When his friends wanted to erect a monument over his grave, Urban told the Tuscan Ambassador that this would be a bad example for the world, since the dead man 'had altogether given rise to the greatest scandal throughout Christendom'.*

In the year 1992, the Roman Catholic Church did in its acknowledgment that Galileo was right after all, that the earth does revolve around the sun[10].

1583–1592 CE **Simon Stevin** (Stevinus, 1548–1620, Netherlands). Versatile mathematician, scientist and engineer. The greatest mechanician of the long period extending from Archimedes to Galileo. A leading figure in the Dutch school of mathematics and science, and an outstanding representative of the great scholars of the closing years of the Late Renaissance; he combined a capability for theoretical investigation with practical skill and inventiveness.

Stevinus originated the study of modern statics and distinguished stable from unstable equilibrium. He demonstrated (1586) how to *resolve* a force according to the parallelogram law[11] (vector *decomposition*; their composition was known to Galileo). He discovered the hydrostatic 'paradox' that the downward pressure of a liquid is independent of the shape of the vessel, and depends only on its height and base. He also gave the measure of the pressure on any given portion of the side of the vessel. He had the idea of explaining the tides by the attraction of the moon.

In 1586, Stevinus performed experiments concerning the effect of gravity on falling bodies. He made a noteworthy contribution to trigonometry, using the *unit circle*. His greatest success, however, was a small pamphlet, first published in Dutch in 1586 (under the name *De Thiende*, i.e. the tithe), and not exceeding seven pages in the French translation: *La Disme enseignant facilement expédier par Nombres Entiers sans rompuz tous Comptes se recontrans aux Affaires des Hommes*. It presented first systematic account of *decimal fractions* and strongly advocated their usage.

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[10] *Modern* Roman Catholicism has no quarrel with the Big Bang, with a universe 15 billion or so years old, with the first living things arising from prebiological molecules, or with humans evolving from apelike ancestors, although it has special opinions on "ensoulment"

[11] This was rediscovered by **Bernard Lamy** (1640–1715, France) in 1676 and again by **Pierre Varignon** (1654–1722, France) at about the same time. The availability of the works of Archimedes was a boon to the study of *statics*, but it is regrettable that the popularity of these so heavily overshadowed medieval steps toward *dynamics* had to await the genius of Newton a century later (1687).

Decimal fractions had been employed for the extraction of square roots since some five centuries before his time, but nobody before Stevinus established their daily use. So well aware was he of the importance of his innovation that he declared the universal introduction of decimal coinage, measures and weights to be only a question of time. Not until the *French Revolution*, more than a two centuries later, did the large scale use of decimals come into vogue[12]. During the last century it spread all over the world, except, strangely enough in the Anglo-Saxon countries, where it met — and still meets — with resistance, which is the stronger in that it is irrational.

His *notation* (1585) was rather unwieldy: he printed little circles round the exponents of the different powers of one-tenth. For instance $237\frac{578}{1000}$ was printed 237 ⓪ 5 ① 7 ② 8 ③ which was a regression from the early use of 237/578 of **Christoff Rudolff**[13] (1500–1545, Austria) in 1530.

Stevinus was born in Bruges and died at the Hague. He began life as a merchant's clerk in Antwerp and traveled in Poland, Denmark and other parts of Northern Europe. He was an adviser to Prince Maurice of Orange (son of William the Silent and great uncle of William of Orange), who made him a Quartermaster General.

Late in life (at an age of 64) he married a young woman who bore him four children. She remarried in 1623 and died half a century later (1673). In July 1846 a modest monument was erected to Stevin's memory in his native city, Bruges.

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[12] The *decimal Dollar* became the basis unit of money in the United States through the Coinage Act of 1792.

[13] The introduction of the *decimal point* to mark the gap between the integral and fractional part is attributed to **Pelazzi of Nice**, about 1492.
The decimal point separatrix was reinstituted by **G.A. Magini** (1555–1617, Italy) in 1592 and by **Christoph Calvius** (1537–1612, Germany) in 1593, both friends of Kepler.
Finally, it reappeared in the trigonometric tables of **Bartholomaeus Pitiscus** (1612) and was accepted by **John Napier** in his logarithmic papers (1614 and 1619).

History of Number Representations

Our decimal system of numerals involves three distinct ideas:

- *There are only* 10 *symbols to write any number. The choice of the base* 10 *is due to the fact that our ancestors made their family accounts on their fingers or on their toes [the* Babylonians, *however, used the base* 60 *and the* Mayas *— the base 20].*
- *The representation of numbers employs the* principle of local value, *i.e., the* position *of any digit in the number determines its value.*
- *There is a special symbol (zero) representing a vacant position (no number). It seems that the Maya knew the use of it, but they did not think of the decimal system.*

All sizable calculations in the ancient world were performed with the aid of some kind of abacus. A written *number representation was needed for record purposes only.*

The earliest method of recording numbers *— either by writing or by notches on a tally stick — was simply to make the requisite number of strokes. This procedure sufficed for small numbers. It was supplemented as early as the first Egyptian Dynasty (ca 3400 BCE) by the use of an additional symbol for ten. Further symbols were introduced for* 100 *and* 1000 *and the method of grouping by tens was a feature of most of the early civilizations of the Mediterranean. In some cases (Etruscan, early Greek and Roman), additional symbols for* 5 *and* 50 *were incorporated for brevity.*

The Greeks used two number representations. Some time in the 3^{rd} *century BCE, they abandoned their* Roman*-type notation in favor of another — known as the* Alexandrian; *the numbers* 1 *to* 9 *were represented by the first nine letters of the Greek alphabet, the numbers* $10, 20, \ldots, 90$ *by the next nine letters, and the numbers* $100, 200, \ldots, 900$ *by the next nine letters. (The Greek alphabet contained* 24 *letters, so* 3 *additional symbols were borrowed from other alphabets). The notation was extended by various artifices to enable numbers greater than* 999 *to be represented. This system was in fact used for business purposes in the Byzantine Empire until its collapse in* 1453. *All such number representations are non-positional; the position of any symbol in the group is without numerical significance. Thus* 183, *for example, is represented in the Roman system as CLXXXIII, but the order of the symbols*

is irrelevant[14]*. The notation merely expresses the fact that* 183 *is the sum of hundred, on fifty, three tens and three units.*

Apparently, the first people to use a positional system for writing numbers were the Babylonians. They employed the rather odd radix of 60*, which we still retain in our method of expressing* angles *and* time*. The Babylonians had separate symbols for* 1 *and* 10*, and also one for* 100 *which was seldom used. The symbol* $\vee$ *for* 1 *served also for* 60*, for* $60 \times 60 = 3600$*, and in general for any power of* 60*; while the symbol* $<$ *for* 10 *also served for 10 multiplied by any power of* 60*. It seems that the number of powers of* 60 *in any particular case had to be deduced from the context.*

The commercially minded Babylonians were the great computers of antiquity, and modern research enables us to appreciate the extent of their achievements. For example, they extended the positional notation to deal with fractional numbers*, and some later Babylonian records even contain a symbol for zero. So far there is no evidence that this symbol was used in computation.*

The Mayan civilization of Central America, with its highly developed observational astronomy and its preoccupation with the calendar, also used a positional notation. It was more highly developed than the Babylonian, although it was encumbered with a clumsy mixture of radices: 5, 20, and 360*. The Mayas even had a symbol, resembling a half closed eye, for denoting zero.*

Although the first steps towards the use of a radix notation were taken by the Babylonians in the 3^{rd} *millennium BCE, the logical culmination of this approach was not reached for another* 2000 *years. If we leave aside the Mayas on the other side of the world, the credit for this achievement (which cannot be precisely dated) must be given to the Hindus.*

The earliest preserved examples of our present number symbols are found on some stone columns erected in India about 250 *BCE by King Aśoka. Other early examples in India, are found among records carved about* 100 *BCE on the walls of a cave in a hill near Poona, and in some inscriptions of about* 200 CE*, carved in the caves at Nasik. These early specimens contain no zero and do not employ positional notation.*

Probably about 600 CE*, the Hindus found a way of eliminating place names. They invented a symbol* sunya *(meaning* empty*), which we call* zero*. With this symbol, they could write "105" instead of "1 sata, 5". This revolution must have been effected prior to* 800 CE*, for the Persian mathematician*

[14] The late Roman use of the *subtractive form* (e.g. IV instead of IIII) provides an exception to this statement. The absolute position of the pair of symbols I and V is not important, but the relative position is.

Al-Khowarizmi describes such a complete Hindu system in a book of 835 CE. *Thus, the Hindu mathematicians took the two concepts (both known much earlier) of the positional representation of numbers and the decimal scale, and added their own contribution — the concept of zero as one of the basic digits.*

This recognition of the need to provide a special symbol to represent an empty column in the abacus — was a crucial step. It provided the world with a flexible and convenient notation whereby any number, however large, could be represented uniquely by an ordered sequence of symbols drawn from a set of ten. It set the stage for the development of arithmetic during the next few centuries.

By the 7^{th} *century* CE, *the focus of our interest shifts to the Arabs, who by then had established a vast empire with its capital at Baghdad. The Arabs had substantial commercial dealings with India: they found the Hindu merchants using the decimal notation and soon adopted it themselves. We know, for instance, that some Indian astronomical tables in which decimal digits are employed, were brought to Baghdad and translated into Arabic in the year* 773. *By the end of the* 8^{th} *century, the Arabs had absorbed the main body of Indian mathematics; during the following century they became acquainted with the works of the Greek masters.*

The transfer of the mathematical lore from India to Baghdad was effected by ***Al-Khowarizmi****, who visited India in 830* CE *and then based his algebra treatise on the work of* ***Brahmagupta****. His book was the main source whereby the decimal notation was introduced, some 300 years later, into the West. At that time no clear distinction was made between the disciplines now known as arithmetic and algebra. The new arithmetic — that is to say, the arithmetic based on the Hindu-Arabic notation instead of the Roman — was indeed known for several centuries as algorithm, or the art of Al-Khowarizmi.*

The Hindu-Arabic mathematics, and with it Greek mathematics as well, diffused slowly to Western Europe via Spain. The Moorish rule in Spain attained its Zenith in the 10^{th} *and* 11^{th} *centuries, but Islamic culture was carefully guarded from Christians and few breaches were made before the* 12^{th} *century.*

One of the first Christians to penetrate the Muslim curtain was the monk ***Adelard of Bath****, who disguised himself as a Muslim and studied at the University of Cordova. In about* 1120 CE *he translated some of the works of Al-Khowarizmi and Euclid from the Arabic into Latin.*

The earliest coin bearing the Hindu numerals is one with an Arabic legend struck in 1138 *to commemorate the reign of Roger of Sicily. But the conditions prevailing in Sicily, where Byzantines, Latins and Moslems met on an equal footing, were too exceptional to be representative of Western Europe.*

However, by the end of the 12^{th} *century a small élite was apparently familiar with the new system.*

A pioneer in spreading the new knowledge in Europe was the mathematician ***Leonardo Fibonacci****. His father was sent by his fellows merchants to control a custom house in Barbary, and Leonardo grew up in an Arab cultural environment and became acquainted with the work of Al-Khowarizmi. He returned to Italy as a young man, and in* 1202 *he published his* liber Abaci *in which he explained the Arabic system "in order that the Latin race might no longer be deficient in that knowledge".*

Leonardo was a vigorous propagandist for the use of Arabic numerals in commercial affairs. By the middle of the 13^{th} *century, a large proportion of Italian merchants were employing the new system alongside the old. The changeover was, of course, not achieved without some opposition. In* 1299, *for example, an edict was issued at Florence forbidding the bankers to use the infidel symbols!*

Outside Italy the new notation gained ground more slowly, and merchants throughout most of Europe continued to keep their accounts in Roman numerals until the middle of the 16^{th} *century. The Arabic system was, however, in general use for* scientific *purposes throughout Europe by about the year* 1400. *While no Arabic numerals are to be found in English parish registers or Manor Court rolls before the* 16^{th} *century, a popular account of the new algoristic arithmetic entitled* The Craft of Nombrynge *appeared as early as about* 1300 CE *— one of the first books to be written in the English language.*

Although the new decimal system was a time- and labor-saving invention of the first magnitude, more than 1000 *years elapsed between the discovery and its general acceptance: not until the beginning of the* 17^{th} *century was it finally established in civilized Europe. Even then there were still learned doctors and professors who claimed that the Roman letters were much clearer than the Hindu numerals. Was it not much simpler to write* CCCXLVIII *than 348?*

The Hindus had made to mankind a gift of inestimable value. No strings of any kind were attached to it, nor was the suggested improvement entangled with any sort of religious or philosophic ideas. Those proposing to use the new numerals were not expected to make any disavowal or concession; nor could their feelings be hurt in any way. They were asked simply to exchange a bad tool for a good one. The history of our numerals is but one example, among so many others, of the difficulty of overcoming the enormous inertia of rested traditions.

The advantages of the new system were so great that its universal adoption was only a matter of time. The invention of printing hastened the process.

The first manual on arithmetic to come off the press of Renaissance Italy was printed in Treviso, Venice, in 1478.

The 16^{th} and 17^{th} centuries saw a number of important advances in the technique of practical calculation; mathematical rigor came later. Arithmetical procedures were simplified, additional signs were introduced and the decimal notation was extended to represent fractions. The introduction of the decimal point was finalized at the turn of the 17^{th} century, just before the appearance of logarithms.

Such is the story of the representation of numbers. One of its most arresting features is the length of time that elapsed (at least 3000 years) between the coming into use of the abacus, a concrete embodiment of the positional decimal notation, and the introduction of the same system for the representation of numbers in writing. The whole sequence of events provides a striking illustration of the importance of notation in mathematics.

Even the Greeks, with their unrivaled intellectual prowess, could make little progress in arithmetic because of the unsuitable number representations with which they were burdened. Why did the Greek miss the crucial idea which appears so simple to us now?

A partial answer may be attempted in terms of the social and economic climate of classical Greece, which emphasized the gulf between theory and practice, between the intellectual and the artisan. The point, however, is a wider one, and can be applied to all the ancient civilizations of the Mediterranean and Near East. The very efficiency of the abacus as a computing tool weakened the practical need for an efficient written number representation which would facilitate arithmetical calculations. The calculations of everyday life could be carried on quite satisfactorily with the aid of the abacus. The written symbols were used merely as labels for recording the results. If the records were somewhat cumbersome, no great harm was done. The Greek philosopher with an interest in mathematics could happily devote himself to geometry, with its superior aesthetic and intellectual fascination.

So the Greeks missed their opportunity and it was left to the Hindus to take the crucial step. It is interesting to note that the Hindus and the Arabs made comparatively little use of the abacus; so much more acute was therefore their need for an effective written *number representation.*

The utilitarian motive appears, indeed, to dominate the situation throughout. The main stimulus to the spread of the new notation throughout Europe came from the merchants and traders, with the 'establishment', both lay and clerical, usually fighting a rearguard action against the forces of change.

1584–1603 CE **Walter Raleigh** (1552–1618, England). Poet, historian, scholar, soldier, navigator and explorer. One of the most flamboyant characters in the colorful reign (1558–1603) of Elizabeth I. Raleigh sent an expedition which explored the North American coast from Florida to North Carolina (1584) and named the coast north of Florida "Virginia". He succeeded in introducing potatoes and tobacco into England and Ireland.

He fitted out an expedition to seek the fabulous wealth of Guiana (gold mines), explored the coasts of Trinidad and sailed up the Orinoco river (1595).

Raleigh's daring expeditions to the New World, along with his quick wit, handsome face and ostentatious gallantry, made him a favorite with the Queen, but after the accession of James I (1603) Raleigh fortunes changed. He was accused of treason, committed to the Tower of London and executed in 1618.

1585–1641 CE **Pedro Teixeira** (ca 1570–1650, Portugal). Explorer and author. One of the greatest travelers of his age; circumnavigated the globe during 1585–1601 and commanded an expedition that made the first documented round trip voyage up the *Amazon* (1637–1638) from Pará to Quito and back.

Born in Lisbon of *Marrano parents*. A man of education and a close observer, he traveled on his first journey for 18 months (1585–1586) through the Philippines, China, the Americas and finally back to Lisbon (1601). His second journey took him to India, Persia and other parts of the Orient (1603–1607). He then settled in Antwerp and published a detailed account of these travels, *Relaciones de Pedro Teixeira* ... (Antwerp, 1610), containing data long considered authoritative. If was translated into French in 1681 and the first English version appeared in 1708-10. A complete English translation, *The Travels of Pedro Teixeira*, was published in 1902. The book is still held to be one of the most important sources of information about the Orient at the beginning of the 17th century.

Teixeira arrived in Brazil in the early 1620s and led successful forays against the English and the Dutch. In July 1637, at the request of Philip III of Portugal (Philip IV of Spain), he undertook a journey of exploration in the country. In what was to be his last expedition, Teixeira set our from Pará (Belém) with a party of 2,000 men and *made the first*[15] *documented continuous voyage* up the *Amazon*, finally reaching Quito after an adventurous trip

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[15] The first European to see the Amazon (1500) was the Spanish explorer **Vicente Yanẽz Pinzon** 1460–1523), who during 1487–1500 explored the coast of Brazil. Another Spaniard, **Francisco de Orellana** (c. 1490–c. 1546) let the first exploration of the river by a European. His expedition followed the Amazon from

lasting ten months. In the course of this journey he extended the boundaries of Brazil and established a line of demarcation between the Spanish and Portuguese possessions in South America.

Teixeira returned to Europe (1640) and settled in Antwerp, where he reverted to Judaism. A description of his expedition to the source of the Amazon is found in the *Nuevo descubrimiento del Gran Rio de la Amazonas* (1641).

1588 CE **Thomas Harriot** (1560–1621, England). Mathematician, astronomer and geographer. One of Britain's greatest mathematical scientists before Newton. He remained comparatively obscure, because he did not publish his work during his lifetime. His achievements are summarized as follows:

- First European to consider the idea of a binary number system.
- In algebra, he introduced the signs for greater than ($>$), less than ($<$), and the raised dot ($\cdot$) to signify multiplication.
- In optics, he discovered the sine law of refraction, ahead of the Dutch mathematician Willebrod van Roijen Snell.
- Harriot made telescopes in the same year Galileo did, and he used them to observe the moon, sunspots, comets and the satellites of Jupiter.
- Investigated the ballistic trajectory of a projectile under the influence of gravity, a decade before Galileo did. He concluded that the path was a parabola.
- Was interested in the atomic theory of substances. He believed in the hypothesis that substances consist of atoms was plausible, and capable of explaining some of the properties of matter.

His writings contain the following propositions (in his own Elizabethan style and spelling):

(1) "*The more solid bodies have Atoms touching on all Sydes*".

(2) "*Homogeneall bodies consist of Atoms of like figure, and quantitie*".

(3) "*The waight may increase by interposition of lesse Atoms in the vacuities betwine the greater*".

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the mouth of the Napo River in Peru to the Atlantic Ocean (1541–1542).

Orellana took part in the conquest of Peru under Pizarro, and Pinzon Commanded the Ninã on Columbus' first voyage. His brother **Martin Alonso Pinzon** (1441–1493) commanded the Pinta on that voyage, and a third brother **Francisco Martin Pinzon** (1440–1493) was master of the Pinta under Martin Alonso.

(4) "*By the differences of regular touches* (*in bodies more solid*), *we find that the lightest are such, where euery Atom is touched with six others about it, the greatest* (*if not intermingled*) *where twelve others do touch euery Atom*".[16]

Harriot was born in Oxford. He studied at St. Mary Hall, Oxford and received his bachelor of arts there in 1580. He then became tutor and scientific adviser to Sir Walter Rayleigh, who appointed him in 1585 to the office of geographer to the second expedition to the newly founded colony of Roan Island in what is now North Carolina (Harriot published an account of this expedition in 1588). On his return to England in 1587, he resumed his mathematical studies and secured the patronage of Henry Percy, Earl of Northumbria, which yielded him a yearly pension of £300, on which he lived.

But Henry was suspected of complicity in the gunpowder plot and in 1606 was jailed. Harriot remained with his patron, and illness and political turmoil prevented him from completing the promising projects he has undertaken. Given more favorable circumstances, he might have become known as the inventor of analytic geometry or as one who solved the rainbow problem.

Harriot was one of the first algebraists who occasionally placed a purely negative quantity by itself on one side of an equation. **Viète** (1600) discarded negative roots of equations. Indeed we find few algebraists before and during the Renaissance who understood the significance of negative quantities. **Fibonacci** (1202) seldom used them. **L. Pacioli** (1494) stated the rule that "minus times minus gives plus", but applied it only to the development of the product $(a-b)(c-d)$.

The first use of $+$ and $-$ as symbols of algebraic notation was due to the Dutch mathematician **van der Hoecke** (1514). **Stifel** (1544) spoke of

[16] The structure in which every atom is in contact with six others about it that Harriot had in mind, is probably the simple cubic arrangement; in this arrangement of atoms the unit of structure is a *cube* that contains one atom, which can be assigned the coordinates $(0, 0, 0)$. Each atom is then in contact with six other atoms, which are at the distance d from it. The volume of the unit cube is accordingly d^3. If the mass of the atom is M, the density for this arrangement is $\frac{M}{d^3}$.
The denser structure referred by Harriot, where twelve atoms are in contact with each atom, is the *cubic closest-packed arrangement*. The cubic unit of structure for this arrangement contains four atoms. Its edge a is equal to $d\sqrt{2}$ and its volume to $2\sqrt{2}d^3$. The mass contained in the unit cube is $4M$ and the density is accordingly $\sqrt{2}\frac{M}{d^3}$, that is 41 percent denser then the simple cubic packing. Harriot had apparently discovered that there is no way of packing *equal* hard spheres in space that gives a greater density.

numbers which are "*absurd*" or "*fictitious below zero*". However, these ideas remained sparse, and until the beginning of the 17^{th} century, mathematicians dealt exclusively with absolute positive quantities.

As regards the recognition of negative roots, **Cardano** (1545) and **Bombelli** (1572) were far in advance of all writers of the Renaissance, including **Viète**. Yet even they mentioned these so-called false or fictitious roots only in passing, and without grasping their real significance and importance. On this subject Cardano and Bombelli had advanced to about the same point as had the Hindu **Bhaskara** (1150), who saw negative roots, but did not approve of them.

The generalization of the concept of algebraic quantity, so as to include the negative, was an exceedingly slow and difficult process in the development of algebra.

1588 CE The Spanish *Armada* was defeated by the English fleet under Howard of Effingham, Francis Drake and John Hawkins.

How Thomas Digges defeated the Spanish Armada

It was in England that the two streams of Nicolas of Cusa's influence – scientific method and the new infinite cosmology – first merged. England had nurtured its own scientific tradition from the time of Bacon, and English scholars and politicians kept abreast of the latest developments in Italian philosophy. The practical impetus for astronomical and general scientific research was stronger in England than anywhere else.

After the feudal nobility had killed themselves off in the War of the Roses, a collateral royal line, previously involved in trade rather than landholding, came to power with Henry VII. By the time Elizabeth became Queen in 1558, English navigation was in a state of fevered expansion, attempting to wrest control of trade from Catholic Spain.

Elizabethan England, recently freed from the intolerance of Mary's rule, welcomed that antihierarchical and anti-authoritarian teaching of the Copernican system. ***Thomas Digges****, a leading English astronomer, became the*

first to popularize Copernicus' ideas to a broad audience, writing a book about it in English, not scholarly Latin, in 1576.

Already in 1572, Digges and other astronomers had studied the supernova of that year, showing that the heavens do in fact change, contrary to tradition – a sight visible to all. Indeed, Copernicus' ideas, backed by Digges' prestige as a leading scientist, became the property of the common man.

Digges synthesized Copernicus' and Nicholas of Cusa's work, proclaiming the universe to be infinite, populated with innumerable suns and worlds. But above all he explicitly criticized the ancients' method:

"I have perceived that the ancients progressed in reversed order from theories, to seek after true observations, when they ought rather to have proceeded from observation and then to have examined theories."

In a country where free labor was increasingly drawn into manufacture, and the need for both technological advances and an educated work force became acute, Digges championed the idea that scientific and technological advances are welded together, and that scientific knowledge must become common to all.

Since technological advance would be most rapid when the common workers had combined scientific knowledge with practical experience, Digges vowed to write all his work in English. Digges and others began a series of practical scientific manuals aimed at the widest audience. By 1589 publicly sponsored scientific lectures drew crowds of artisans, soldiers, and sailors eager for knowledge.

The conflict between the old and the new cosmologies was not settled by scholarly argument, but by the battles of the old and new societies – embodied in the struggles of nations. Protestants, in manufacturing Holland, revolted against its Catholic imperial ruler, Spain; and in 1584 the main Protestant power, England, allied with Holland. The Spanish empire was based on forced labor – serfs at home and serfs and slaves in the huge empire of the New World. The English and Dutch relied mainly of free labor.

The Copernican scientific worldview gave not only ideological justification to the Protestant side, but also decisive technological advantage. By synthesizing theoretical science with craft skill, English industry moved ahead of Spain in critical areas, such as the casting of naval artillery, producing lighter guns with greater range and accuracy.

The Copernican revolution had also meant throwing out Aristotelian physics – based on the idea that moving objects sought their "proper" place in the hierarchy. This had significant application in the science of ballistics. Aristotle had taught, and the medieval scholars accepted, that a projectile flew upward in a straight line, then fell vertically to earth. Leonardo and his

successor in engineering, Tartaglia, showed by experiment that the trajectory is a curve, and compiled a gunnery table linking the elevation of the gun to the range of the shot.

Digges and other English scientists systematized their results, producing widely read manuals of naval gunnery. English ships, manned by draftees drawn from the artisan and working classes, had by 1588 both seamen and officers on board trained in the basics of the new ballistics. Spain, by contrast, had no use or interest in the new sciences. Nor could their uneducated sailors use them[17].

The related differences in social structure, technology, and training proved decisive when the Spanish Armada sailed to invade England. The English ships mounted mostly small guns, called culvetines, whose effective range was one thousand meters. The Spaniards had crude cannons, effective only at point-blank range – that is, before the shot began to fall significantly, perhaps three hundred meters. With this and other advantages the English battered the Spaniards at long range, while the Spaniards' ammunition fell far short of the targets. For one hundred thousand cannonballs fired, the Spaniards killed one English officer and two dozen seamen, sinking no vessels. The English, with about half as many shots and lighter guns, sank or disabled seventeen Spanish ships and inflicted thousands of causalities. When the Spanish ran out of ammunition, the English chased the shattered Armada out of the channel.

Thus, in a very practical way, the superiority of the empirical worldview was demonstrated – with cannon, not with debate. In fact, the defeat of the Armada determined which worldview would triumph, since it determined which society would survive.

The revolutionary changes of the last quarter of the eighteenth century were not universally hailed, and neither were the new scientific theories. The capitalists who ruled Great Britain owed their power to the social revolutions of the seventeenth century and the industrial revolution of the eighteenth, but they had no desire to lose that power in further social upheavals. Great Britain became the major foe of all social change, fearing the development of rival industrial powers abroad and a continual evolution of social structure at home. From Britain, religious and philosophical replies were launched against the ideas of human and natural progress.

***Thomas Malthus**, rebutting the **Marquis de Condorcet**, the French theorist of progress, argued that population growth will always outstrip agricultural production, condemning most people to hunger and blocking material progress. Geologist **John Williams** blasted **Hutton**'s theories on theological grounds. Hutton's "wild and unnatural notion of the eternity of the earth*

[17] Had they not expelled the *Jews* in 1492, they could have fared better in 1588!

leads first to skepticism and at last to downright infidelity and atheism. If we once entertain a firm persuasion that the world is eternal, and can go on itself in the reproduction and progressive vicissitudes of things, we may then suppose that there is no use of the interposition of a Governing Power," he wrote, concluding that "all rebellions soon end in anarchy, confusion and misery and so does our intellectual rebellion."

But these efforts proved generally unsuccessful: in the course of the first half of the nineteenth century, Europe continued to be rocked by repeated popular revolutions, and the industrial revolution transformed British society as well.

While many of the social gains at the height of the revolution were subsequently rolled back, the fundamental outlook and goals of society had been irreversibly transformed. The English government's sponsorship of scientific research put English science far ahead of that of any other country; this, together with England's swift economic development, propelled it a century later into the industrial revolution.

The scientific revolution was thus not an inevitable process, a natural outgrowth of human intellectual development. It was the result of a fierce social conflict, in which cosmological questions were matters of life or death for individuals and whole societies.

Certainly the people of the time did not think that the defeat of Spain, the victory of England and Holland, and later the victory of the English revolution were at all inevitable. Yet without those victories, the scientific revolution would certainly have not occurred. Only the open society born in the sixteenth and seventeenth centuries could have nurtured the infinite unlimited cosmos of modern science. And only such a worldview could have given the new society the moral and material strength to prevail.

1588–1613 CE **Pietro Antonio Cataldi** (1548–1626, Italy). Mathematician and astronomer. Took the first steps in the theory of *continued fractions* and made contributions to the early theory of numbers, especially Mersenne primes and perfect numbers. Wrote a number of mathematical works.

Cataldi was born in Bologna and taught mathematics and astronomy in Florence, Perugia and Bologna, where he died.

1588–1623 CE **Gaspard Bauhin** (1560–1624, Switzerland). Physician, botanist and anatomist. Introduced (1623) a binomial system of nomenclature for botany. One of the first to describe *ileocecal valve* (1588), known as the *Bauhin valve*.

His book *Pinax Theatri Botanici* (1623) contains classification and description of over 6000 plants and was much used by *Linnaeus*.

Bauhin was born in Basel. Studied at Padua, Montpellier, Paris and Tübingen. Professor at Basel University.

1589–1606 CE **Giovanni Battista, della Porta** (1535–1615, Italy). Natural philosopher and inventor. Made the first distinct step from Hero's *aeolipile* toward the steam engine, by using steam instead of air as the displacing fluid (1601). In his *Magia Naturalis* (1589) he describes a number of optical experiments, including a description of the *camera obscura* to which he proposed to add a convex lens. He claimed to be the inventor of the telescope although he does not appear to have constructed one before Galileo. He was however first to recognize the heating affects of light rays (1589).

The *Inquisition* banned the publication of his works for a number of years. Although Porta made important physical observations, much of his work was from point of view of magic and alchemy.

Della Porta founded in Naples the *Accademia Secretorum Naturae* and in 1610 became a member of the *Accademia dei Lincei* at Rome.

1592–1613 CE **David (ben Shlomo, Seligman) Gans** (1541–1631, Germany and Prague). Chronologist. In his book *Zemach David* compiled a chronology of ancient and medieval events up to 1592. Author of textbooks on astronomy, mathematics, geography and cosmography.

Studied under the **Maharal of Prague** and interacted with **Kepler**, **Regiomontanus** and **Tycho Brahe**.

Gans was born in Lippstadt (Westphalia) and died in Prague.

1591–1639 CE **Tommaso Campanella** (1568–1639, Italy). Italian Renaissance philosopher. A precursor of modern empirical science. His work was a source of inspiration for **Descartes**, **Spinoza** and **Leibniz**. Many of his ideas are similar to those of modern-day existentialists. Though neither an original nor a systematic thinker, he stands in the uncertain half-light which preceded the dawn of modern philosophy and science.

Campanella was born in Stilo, Calabria. Before he was 13 years of age he had mastered nearly all the Latin authors presented to him. He entered the Dominican Order (1582), but in 1599 was sentenced to life imprisonment

during the Spanish rule for political plotting and heresy. During his stay in prison he wrote a valiant vindication of Galileo. After 30 years of incarceration, Campanella succeeded in escaping to France, where he remained for the remainder of his life under the aegis of Cardinal Richelieu.

His philosophy is a blend of medieval thought combined with the methods of modern science: he rejected Aristotelian scholasticism and insisted that knowledge should be based on close observation of the natural world. His views were strongly influenced by **Bernardo Telesio** (1509–1588, Italy) and also by those of **Nicolas of Cusa** (1401–1464). Telesio founded and directed the *Academia Telesiana*, a school in Naples that propagated the scientific approach to knowledge and advanced the scientific movement in the Renaissance.

1591–1603 CE **Prospero Alpini** (1553–1616, Italy). Physician and botanist. Studied plants for their therapeutic medicinal use. Introduced (1591) the first European descriptions of the coffee bush and the banana tree. First to establish the sexual difference of plants.

Alpini was born at Marostica, in the Republic of Venice. In his youth he served in the Milanese army and in 1574 went to study medicine at Padua, taking his doctor's degree in 1578. To extend his knowledge of exotic plants he traveled to Egypt (1580) as physician of the Venetian consul in Cairo. On his return (1583) he resided in Genoa and then (1593) was appointed professor of botany at Padua. Published *De Plantis Agyptic Liber* (1592).

1595–1620 CE **Francis Bacon** (1561–1626, England). A forerunner of the scientific revolution. Lawyer, essayist, statesman and philosopher. The first union in English literature of the man of letters and the man of science (there have been only a few striking examples since).

Bacon was not himself an active scientist, yet he can be likened to a signpost which shows the way. He had an enduring influence on an entire generation of great scientists. His chief works include: *Essays* (1597) — concise expressions of practical wisdom and shrewd observations; *Advancement of Learning* (1605) — a survey in English of the state of knowledge (incomplete project); *Novum Organum* (1620), in Latin, key to his system for the new systematic analysis of knowledge, intended to replace the deductive logic of Aristotle with inductive methods in interpreting nature. In his utopian tale *New Atlantis*, published posthumously (1627), he predicted robots, telephones, tape recorders and electric motors.

Until the time of Bacon, man had more or less 'drifted' in the natural world. His culture had grown up without conscious self-examination or attention to the fact that man might improve his own society through science. People decided all questions not by investigating the observable facts, but by appealing to a priori reasoning, received folk wisdom, religious dogmas, and the teachings of infallible authorities, living and dead – for instance in medieval Europe, the Church fathers and Aristotle. Education in Bacon's day was largely confined to metaphysical arguments along with the readings of Greek and Roman classics. At Cambridge, learning was largely pretense that all was of the past[18]. Men endlessly wove and rewove a gossamer webs of ideas derived from Greek and Roman sources. The world of Bacon and Shakespeare was only semiliterate, steeped in religious contentions, with its gaze turned backwards in wonder upon the Greco-Roman past.

Bacon waged a vigorous battle against the deductive method of scholasticism. He went much further beyond that to outline, with unique prophetic vision, the future of science and its role in the affairs of man. Bacon:

- Recognized that the triumph of the experimental method demands the thorough institutionalization of science *at many levels of activity and ability*. He eliminated reliance upon the rare elusive genius as a safe road into the future. It involved of too much risk and chance to rely upon such men alone.
- Grasped the cumulative nature of culture and the fact that inventions multiply in a favorable social environment. Science and its traditions had to be transmitted through the universities, and its efforts had to be *publicly supported*. He studied ways by which Cambridge and Oxford might be encouraged toward fostering laboratories and other educational tools.
- Recognized the value of the history of science.
- Observed that the lower organisms might reveal secrets of life which in the higher organisms lay more hidden. (This biological observation, and others in the social sciences, were made too early. By the time these subjects had emerged as recognized disciplines, his far-reaching, anticipatory insights were submerged in a welter of new books and newer phrasing.)
- Foresaw the necessity of using mathematics in the examination of nature.
- Entertained the idea of the universe as a problem to be solved.

For all his interest in scientific inquiry and the proper pursuit of science, Bacon missed practically all the most important developments of his own

[18] Even toward the close of the 19^{th} century, the greatest universities in England were still primarily devoted to the classical education of gentlemen!

time. He was unaware of the work of Kepler; and, though he was a patient of Harvey, did not know of the doctor's researches on the circulation of the blood. The rejection of the centrality of the syllogism in what still passed for natural philosophy in his day, led him to underestimate the function of *deduction* in scientific inquiry. In particular, he had little appreciation of the mathematical methods that were actually developing in his time. The role of *induction* (in itself a notion that was not new — Aristotle had already used it) in the framing of hypotheses is but one facet of the scientific method. Without the mathematical and logical deductions which lead from the hypotheses to concrete, testable predictions, there would be no knowing what to test against experiment.

Bacon was born in London, the son of an important government official. He attended Trinity College, Cambridge, from 1573 to 1575. In 1576, he joined the staff of England's ambassador to France. Bacon was elected to Parliament in 1584 and knighted in 1603. He held several high government positions[19] until 1621, when he was framed[20] and convicted of taking bribes and briefly imprisoned.

Bacon was never free of financial insecurity. In a mercenary age he lacked the means to buy advancement. Although the prestige of his final offices [Attorney General (1613); Lord Keeper (1617); Lord Chancellor (1618)] gave greater weight to his literary pronouncements and financed his publications, he was nonetheless a stranger in his own age — a civilized man out of his time and place. Even when one has measured the three sides of his triangular life (as man of letters, man of science and public servant), one is still at a loss to understand all the motives governing him in his contradictory actions.

Rumors persist that he did not die in the year 1626 but rather escaped to Holland[21]; that he was the real author of Shakespeare's plays; and that he was the unacknowledged son of Queen Elizabeth. These rumors are a measure of

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[19] In 1605, Bacon devised a code for sending secret diplomatic messages. Each letter of the alphabet was represented by a five-letter group of a's and b's. For example A=aaaaa, B=aaaab, C=aaaba, D=aaabb, ..., X=babab, Y=babba, Z=babbb.

[20] He became a victim of the conflict between King James I and his Parliament. Bacon's enemies, in frustration at their inability to vent their rage on the King, set to destroy the one man who had sought to temper the royal excess and preserve the state. Traditional homage was deliberately redescribed as bribery. King James advised him to avow his guilt and trust his protection to the Crown.

[21] He went to a farmhouse in a snowstorm to get a chicken to test his idea that snow could be used as a preservative instead of salt. The exposure to which the experiment subjected him caused his death soon after.

his power to captivate the curiosity of men — a power that has grown century by century since his birth in 1561.

In spite of certain mystifying aspects of his life, there is no evidence sufficient to justify these speculations, though a vast literature betokens their fascination and appeal.[22]

[22] For further reading, see:

- Eiseley, L., *The Man Who Saw Through Time*, Charles Scribner Sons: Ne York, 1973, 125 pp.
- Bacon, Francis, *The Essays*, Penguin Books, 1985, 285 pp.

Worldview VII: Francis Bacon

* * *

"Man can only conquer nature by obeying her".

* * *

"We are not to imagine or suppose, but to discover*, what nature does or may be made to do".*

* * *

"For the history that I require and design, special care is to be taken that it be of wide range and made to the measure of the universe. For the world is not to be narrowed till it will go into the understanding (which has been done hitherto), but the understanding is to be expanded and opened till it can take in the image of the world".

* * *

"I say without any imposture, that I… frail in health, involved in civil studies, coming to the obscurest of all subjects without guide or light, have done enough, if I have constructed the machine itself and the fabric, though I may not have employed or moved it".

* * *

"Science is not a belief to be held but a work to be done".

* * *

"Make the time to come the disciple of the time past and not its servant".

* * *

“Many parts of nature can neither be observed with sufficient subtlety, nor demonstrated with sufficient perspicuity without the aid and intervening of mathematics”.

* * *

“This third period of time will far surpass that of the Grecian and Roman learning only if men will employ wit and magnificence to things of worth, not to things vulgar”.

* * *

“Many things are reserved which kings with their treasures cannot buy, nor with their force command, their spies and intelligencers can give no news of them, their seamen and discoverers cannot sail where they grow”.

* * *

“Every act of discovery, advances the art of discovery”.

* * *

“Mere power and mere knowledge exalt human nature but do not bless it; we must gather from the whole store of things such as make most for the uses of life”.

* * *

“Books must follow sciences, and not sciences books”.

* * *

“If a man will begin with certainties, he shall end in doubts; but if he will be content to begin with doubts, he shall end in certainties”.

* * *

"The men of experiment are like the ant, they only collect and use; the reasoners resemble spiders, who make cobwebs out of their own substance. But the bee takes the middle course: it gathers its materials from the flowers of the garden and field, but transforms and digests it by a power of its own.

Not unlike this is the true business of philosophy [science]; for it neither relies solely or chiefly on the powers of the mind, nor does it take the matter which it gathers from natural history and mechanical experiments and lay up in the memory whole, as it finds it, but lays it up in the understanding altered and digested.

Therefore, from a closer and purer league between these two faculties, the experimental and the rational (such as has never been made), much may be hoped".

* * *

"That all things are changed, and that nothing really perishes, and that the sum of matter remains exactly the same, is sufficiently certain".

* * *

"Great discoveries appear simple once they are made".

* * *

"I take it, that all those things are to be held possible and performable, which may be done by some persons, though not by everyone; and which may be done by many together, though not by one alone; and which may be done in the succession of ages, though not in one man's life; and lastly, which may be done by public designation and expense, though not by private means and endeavor".

* * *

"It is not the pleasure of curiosity nor the raising of the spirit, nor victory of wit, nor lucre of profession, nor ambition of honor or fame, nor opportunity

for business, that are the true ends of knowledge. It is a restitution and reinvesting of man to the sovereignty and power which he had in the first state of creation".

* * *

"The technological arts have an ambiguous or double use, and serve as well to promote as to prevent mischief and destruction, so that their virtue almost destroys or unwinds itself. All natural bodies have really two faces, a superior and inferior. He who will not attend to things like these can neither win the knowledge of nature nor govern it".

* * *

"There is no excellent beauty that hath not some strangeness in the proportion".

* * *

"Truth is more likely to emerge from error than from confusion".

* * *

"Crafty men condemn studies, simple men admire them, and wise men use them".

* * *

"Knowledge is power".

* * *

History of Theories of Light I

The science of optics and optical devices embraces a vast body of knowledge accumulated over roughly 5000 *years of the human scene. To view modern optics in full perspective one must trace the road that led us there. The complete story has myriad subplots and characters, heroes, quasi-heroes and an occasional villain or two. Yet from our vantage in time, we can discern* 4 *epochs in the history of optics:*

A. The Beginning — Optics of Reflection and Refraction (3000 BCE–1589 CE*)*

The ancients were more familiar with optics than with any other branch of physics due to the fact that for the knowledge of external things man is indebted to the sense of vision in a far greater degree than to other senses. That light travels in straight lines (i.e. that an object is seen in the direction in which it really lies) must have been realized in very remote times. The antiquity of mirrors points to some acquaintance with the phenomena of reflection. The lens, as an instrument of magnifying object or for concentrating rays to effect combustion, was also known.

The cuneiforms of Sumer describe a highly sophisticated society more than 5000 *years ago. These cuneiforms were written by pressing a stylus of bone or hard reed into a tablet of soft river mud. Some of the cuneiform letters are of the order of a millimeter in size and cannot be read (and written!) without a magnifying glass.* **Henry Austin Layard** *(1885) excavated, amongst the ruins of the palace of King Sennacherib (705–681 BCE) of Assyria, a quartz magnifier: It seemed to be cut and polished to the shape of a plano-convex lens with* $f \approx 10$ cm *and could have been used as a magnifying glass, for both making the inscriptions and reading them.*

Another significant development in optics during this period is the use of metallic hand-held mirrors. Early mirrors were made of polished copper, bronze and later of speculum, a copper alloy rich in tin. Specimens have survived from ancient Egypt: a mirror in perfect condition was found along with some tools nearby the Pyramid of Sesostris II (ca 2000 BCE) in the Nile Valley. Exodus ***38***: 8 *(ca 1200 BCE) recounts how* **Bezalel**, *while preparing the ark and tabernacles, recast "the looking-glasses of the women" into a brass laver (a ceremonial basin).*

The Bible also tells us that ***Joshua*** *(***8***, 18) used a mirror to send a light signal (sun's reflected rays) to his ambush force, to rise and to take the city of Ai.*

Chinese philosophers (Mohists, Ca 479–381 BCE), studied the reflection of light from plane, concave and convex mirrors, and obtained empirical rules connecting the size and position of objects and images with the curvature of the mirror used.

In the first systematic writings, of which we have any definite knowledge, the Greek philosophers: ***Pythagoras***, ***Empedocles***, ***Democritos***, ***Plato*** *and* ***Aristotle*** *speculated (550–350 BCE) over the nature of light and vision, evolving several theories. The aversion of Greek thinkers to detailed experimental inquiry stultified the progress of the science; instead of acquiring facts necessary for formulating scientific laws and correcting hypotheses, the Greeks devoted their intellectual energies to philosophizing on the nature of light.*

In their search for theory the Greeks were mainly concerned with vision: they sought to determine how an object was seen, and to what its color was due. Emission theories, involving the concept that light was a stream of particles, were formulated.

Their hypothesis was that the eyes emanate vision rays *and the returned rays create vision, a principle similar to that of modern day radar or sonar. As a result of the concept of the eye-ray, direction of arrows and the designation of the incident and reflected angles were reversed from what these are today*[23].

The Pythagoreans *assumed that vision and color was caused by the bombardment of the eye by minute particles projected from the surface of the object seen. The* Platonists *subsequently introduced three elements – a stream of particles emitted by the eye which united with the solar rays, and, after the combination had met the stream from the object, returned to the eye and excited vision.*

Democritos *maintained that extremely small particles chip off from the object and go into the viewer's eye and imprinted there by the moisture in the eyes*[24]. ***Aristotle*** *argued against this theory because it could not explain the inability to see in the dark.*

The left-right reversal of the image of a vertical mirror, or the upside-down image of a horizontal mirror, aroused the curiosity of the Greek philosophers, but even Plato could not provide a satisfactory explanation. From a treatise an optics, the catoptrics, *assigned to* ***Euclid*** *(ca 300 BCE) by* ***Proclos***

[23] It took 1400 years before the direction of the arrows was reversed by **Alhazen** (1026).

[24] This is perhaps the origin of the corpuscular theory of light.

and ***Marinos****, we learn that the rectilinear propagation of light, the law of reflection (viz. the equality of the angles of incidence and reflection) were known to the Greeks.* ***Hero*** *of Alexandria attempted to explain both these phenomena by asserting that light traverses the* shortest allowed path *between two points*[25] *(150 BCE).*

The Greeks were also acquainted with the production of images by plane, cylindrical and concave and convex spherical mirrors. Reflections, or catoptrics, was the key-note of the Greeks explanations of most optical phenomena: it is to the reflection of solar rays by the air that Aristotle ascribed twilight, and from his observations of the colors formed by light on spray, he attributed the rainbow to reflections from drops of rain.

A burning-glass (a positive lens) was alluded to by ***Aristophanes*** *in his comic play* The Clouds *(424 BCE). The apparent bending of objects partly immersed in water is mentioned in Plato's* Republic*.* ***Archimedes*** *(250 BCE) used* concave mirrors *as burning-glasses. Certain elementary phenomena of refraction were studied by* ***Cleomedes*** *(50 CE) and later by* ***Claudios Ptolemy*** *of Alexandria (150 CE) who attempted to explain the 'coin-in-a-cup' experiment*[26] *of* ***Ctesibios*** *(50 BCE).*

Ptolemy measured the refractive effects of water and discussed refraction in the atmosphere. He tabulated fairly precise measurements of angles of incidence and refraction for several media and obtained the small-angle approximation to Snell's law, concluding that the ratio of the angles of incidence and refraction were constant. He also discussed the refraction of starlight by the atmosphere but held to the theory that vision is due to rays emitted from the eye. The quantitative law of refraction was unknown (in fact it was not formulated until the beginning of the 17^{th} *century).*

It is clear from the accounts of ***Pliny the Elder*** *(23–79 CE) that the Romans also possessed burning-glasses. Several glass and crystal spheres, which were probably used to start fires, have been found amongst Roman ruins, and a planar convex lens was recovered in Pompeii. The Roman philosopher* ***Seneca*** *(4 BCE–65 CE) pointed out in his book* Naturalium Quaestionum *that a glass globe filled with water could be used for magnifying purposes. It is certainly possible that Roman artisans may have used magnifying glasses to facilitate very fine detailed work.*

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[25] A forerunner of Fermat's principle of least *time* (ca 1638).

[26] The apparent *elevation* of a coin in a basin, by filling the basin with water. Similarly, the Greeks sought to explain the apparent *bending* of the oar at the point where it met the water.

Seneca also observed the analysis of white light into the continuous spectrum of rainbow colors by transmission through prism. *His friend, the Emperor Nero (37–68 CE), may have been the first to use a* monocle, *employing an emerald lens to view bloody gladiator combats in the Coliseum. In Rome, during the first century CE mirrors were made of polished glass, behind which a sheet of silver was placed.*

Aristotle (ca 330 BCE) describes image projection in terms of the camera obscura[27]. *His concept involves a '*darkened box or chamber*' with a small hole in one side though which light is admitted. An inverted image of the scene is projected onto an interior wall, where it can be viewed and traced by an artist. From the opening passage of Euclid's* Optics *(ca 300 BCE), it would appear that the above phenomena of the simple darkened room were used by him to demonstrate the rectilinear propagation of light by the passage of sunbeams or the projection of the images of objects through small openings in windows.*

The first reference to persistence of vision *appears in De Rerum Natura (Book* ***4****, lines 771–810) by the Roman poet and natural philosopher* ***Titus Lucretius Carus*** *(98–55 BCE):*

"...when the first image perishes and a second is then produced in another position, the former seems to have altered its pose. Of course this must be supposed to take place very swiftly: so great is their velocity, so great the store of particles in any single moment of sensation, *to enable the supply to come up."*

Here Lucretius describes frame sequential animation almost 2000 years before the advent of motion pictures.

All through the Dark Ages, optics lay dormant in Europe, but the center of scholarship shifted to the Arab world (where the scientific and philosophical treasures of the past were translated and preserved) and eventually extended at the hands of ***Alhazen*** *(ca 1010–1030* CE*). He elaborated on the law of reflection, (putting the angles of incidence and reflection in the same plane normal to the interface), studied spherical and parabolic mirrors and gave a detailed description of the human eye as an optical instrument.*

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[27] The invention of this instrument has generally been ascribed to **Giovanni Battista della Porta** (ca 1558). However, all he seems really to have done was to popularize it. In southern climes, where during the summer heat it is usual to close the rooms from the glare of the sunshine outside, we may often see depicted on the walls vivid inverted images of outside objects formed by the light reflected from them passing through chinks or small apertures in doors or window-shutters.

By the latter part of the 13^{th} *century, Europe was only beginning to rouse from its intellectual stupor. Alhazen's work was translated into Latin and had a great effect on the writings of* ***Robert Grosseteste****, the Polish mathematician* ***Vitelo*** *(ca* 1230–1275, *Silesia), and the textbook of* ***John Peckham*** *(ca 1230–1292), the archbishop of Canterbury. All of these were influential in rekindling the study of optics.*

Their works were known to ***Roger Bacon*** *(1215–1294), who initiated the idea of using lenses for correcting vision and even hinted at the possibility of combining lenses to form a telescope. After his death optics again languished. Even so, by ca* 1350, *European paintings were depicting monks wearing eyeglasses, and alchemists had come up with a liquid amalgam of tin and mercury that was rubbed onto the back of glass to make mirrors.*

The great Italian artist, architect and scientist, ***Leonardo da Vinci*** *(1452–1519) followed up Alhazen's experiments and developed the pinhole camera. He indulged in the study of color, made analogy between sound and light waves and believed that light is a wave and color is determined by its frequency. In* 1589, *the Italian* ***Giovanni Battista della Porta*** *(1535–1615) published his treatise Magiae Naturalis in which he discussed multiple mirrors and combinations of positive and negative lenses. This work can be viewed as contributing to the theoretical preparation for the invention of the telescope in 1608.*

1596 CE **Ludolph van Ceulen** (1540–1610, Netherlands). A 'digit-hunter', at the University of Leyden, who calculated π to 32 decimal places. The value of π was therefore often named "*Ludolph's number*". His performance was considered so extraordinary, that the numbers were carved on his tombstone (now lost) in St. Peter's churchyard, at Leyden [he used the Archimedean method of in- and circumscribed polygons].

1596–1616 CE **Eliyahu de Luna Montalto** (1560–1616, Italy and France). Distinguished physician and medical researcher. Author of extensive medical writings dealing especially with the mind and the nervous system. Physician at the court of Maria de Medicis and Louis XIII, France.

Montalto was born in Castel Branco, Portugal in a Marrano family under the name Philippe Rodrigues. Studied medicine at the University of Salamanca and moved to Livorno, Italy (1596). He was summoned to the French Court in Paris at a period when Jews had been exiled from France for two

centuries. On his return to Italy he was appointed to the chair of medicine at the University of Pisa, where he published his research in the fields of optics and medicine (1606). He returned formally to Judaism in Venice. In 1611 he was invited back to Paris to serve as the personal physician of the Queen with a special permission from the Pope to practice his own religion. Died of the plague in Tour, France and buried in Amsterdam.

1597–1613 CE **Andreas Libau** (Libavius, 1540–1616, Germany). Physician, alchemist and chemist. Wrote the first important textbooks in chemistry (*Alchemia*, 1597; *Syntagma*, 1611), in which he described a wide range of chemical methods and preparations such as: hydrochloric acid (HCl), sulfuric acid (H_2SO_4), tin tetrachloride ($SnCl_4$, 1605), ammonium sulphate [$(NH_4)_2SO_4$], and others. Wrote medical texts emphasizing the importance of chemistry for medicine (1606). He pointed out in 1597, before Steno, that the salts present in mineral waters could be ascertained by an examination of the shapes of the *crystals* left upon evaporation of the water.

Libau studied medicine at the University of Jena (1586–1591) and became a professor of history and literature there. He then practiced medicine at Rotenburg, serving also as superintendent of schools until 1607. He was among the first to introduce the study of science into the school curriculum.

He was a follower of **Paracelsus**, and as such belongs to the transition period from alchemy to chemistry. He is counted among the pioneers of the independent science of chemistry.

1599–1603 CE **Ulisse Aldrovandi** (1522–1605, Italy). Physician and naturalist. One of the founders of modern zoology. The results of his various researches were embodied in a *magnum opus*, which was designed to include everything that was known about natural history. The first three volumes, comprising his *ornithology*, were published in 1599, and a fourth, treating of insects, appeared in 1602. After his death a number of other volumes were compiled from his manuscript materials, under the editorship of several of his pupils, to whom the task was entrusted by the senate of Bologna.

1600 CE Pestilence and famine stroke Russia. Ca 500,000 people perished.

From Alchemy to Chemistry[28] (1530–1789), *or – the Alchemists died poor*

Alchemy was one of the earliest forms of chemistry. This ancient practice originated amongst the followers of Lao Tsu in China and Pythagoras in Greece (6^{th} century BCE), and combined science, religion, philosophy and magic. Alchemy developed as men applied theories about nature to metalworking, medicine, and other crafts. As the practice of alchemy developed and moved Westward, Taoist ideas about chemicals were combined with Pythagorean number mysticism. Another strand of alchemical tradition came from the Egyptian embalmers.

In China, the early alchemists were searching for the elixir of life[29] *(a substance that would provide long or never-ending life and health). Chinese alchemy was passed on to the Hindus, who were more interested in using alchemical ideas to cure diseases. About 300–400 CE the Alexandrians supposedly used sorcery to convert base metals to gold.*

Eventually the Arabs put together the ideas from the East with the Alexandrian traditions of alchemy that had descended from the Pythagoreans. In this form of alchemy, astrological influences were important; chemical reactions were believed to be determined by the influences of the planets, the shapes of the vessels and numerology, and the elixir of life became mingled with the concept of a philosopher's stone *(an object whose presence would enable to transmute other metals into gold).*

Jabir Ibn Hayyan *(721–815, Baghdad) claimed that all base metals consisted only of brimstone (sulfur) and mercury. To make the metal less coarse, the sulfur had to be driven out. According to the alchemists, gold contained almost no sulfur. Arabian alchemists developed a theory in which different metals could be formed by combining mercury and sulfur in various proportions.*

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[28] The word *chemistry* probably originated in 400 BCE from the Greek word *chemeia*, which designated the art of metal working. At a later time, the Arabs added the prefix *al*. The new word *alchemy* signified the art of chemistry in general.

[29] Some of their accomplishments were remarkable: a woman (known as the *Lady of Tai*) was buried about 185 BCE in a double coffin filled with a brown liquid containing mercuric sulphide (HgS) and pressurized methane. There was no observable deterioration of her flesh when she was exhumed after more than 2000 years.

The alchemists also thought that bodies were made up of 'matter' and 'spirit', and they supposed that in some cases they could isolate the spirit by heating the body and condensing the exuded vapor. Thus they obtained alcohol, or 'spirit of wine', and hydrochloric acid, or 'spirit of salt'. In this way, the alchemists managed to obtain various practical results, including the first strong acids and the distillation of alcohol.

It has been estimated that in the past 2000 years over 100,000 tomes have been written by Western Alchemists. Who were the Alchemists? We know that ***Geber*** *(fl. 1350;* Spain) and ***Avicenna*** *(fl. 1020) were physicians and alchemists. In the Middle Ages,* ***Albertus Magnus*** *(fl. 1250),* ***Thomas Aquinas*** *(fl. 1260) and* ***Raymond Lully*** *(fl. 1280) were adept alchemists.*

Arabs and Moors invaded and conquered most of Spain during the 700*'s. Spanish scholars did not, however, translate Arabic alchemy books into Latin until the* 1100*'s. These translations introduced alchemy to England and the rest of Europe.*

The English philosopher and alchemist ***Roger Bacon*** *(1214–1294) laid the foundation for the experimental method of chemical research. Unlike the early alchemists, Bacon planned his experiments and carefully interpreted his laboratory work.*

During the Renaissance, the West absorbed Arabic alchemy along with more substantial Arabic science. By the 16^{th} *century, alchemy was being practiced mainly in Europe; some alchemists and physicians began to apply their knowledge of chemistry to the treatment of disease.*

Since ancient times, man had known how to prepare and use various drugs. He did so, however, without understanding how the drugs worked. The medical chemistry of the 15^{th} *and* 16^{th} *centuries is called iatrochemistry (from the Greek iatros = physician).*

*Iatrochemists were the first to study the chemical effects of medicines on the body [****Paracelsus****, 1530;* ***Libau****, 1597;* ***Helmont****, 1620]. As scientists learned more about medicine, they gradually lost interest in the impractical theories of alchemy.*

However, the tradition of alchemy persisted well into the 18^{th} *century:* ***Newton*** *(1642–1727) spent much of his later life trying to find the philosopher's stone, and may have gone mad from mercury poisoning caused during his experiments. Other most distinguished* 17^{th} *century scientists,*

G.W. Leibniz *(1646–1716) and* ***Robert Boyle*** *(1627–1691), "the father of modern chemistry", clearly accepted the theory of alchemical transmutation.*[30]

Finally, ***Lavoisier*** *(1743–1794) put together a scientific view of chemistry that effectively abolished the alchemical tradition that had persisted for over two millennia.*

The ancient dream of the alchemists was realized in 1941 CE through the artificial production of several isotopes of gold (Atomic number = 79) from Mercury (Atomic number = 80; Atomic mass = 201) by ***Sherr***, ***Bainbridge*** *and* ***Anderson***, *via a nuclear reaction initiated by fast-neutron bombardment of mercury.*

The 17th century Often is known as the 'age of genius' – and this for at least two reasons. The century *effectively invented far more than its share of scientific instruments: the thermoscope, the telescope, the microscope, the pendulum clock*, are but a few of these. But, more than this, the 'age of genius' also produced more than its just measure of ideas: among them, the *circulation of the blood*, the *wave theory of light*, and the *law of gravitation*. To some extent, it is true, the instruments and ideas had been adumbrated by earlier periods; but it probably is safe to say that in no century, with the possible exception of the 20th, was *the interplay of instruments and ideas more effective that during the 'age of genius'*.

1600–1750 CE The European *Baroque Period* in music. The leading composers are:

- Heinrich Schütz 1585–1672
- Dietrich Buxtehude 1637–1707
- Alessandro Stradella 1642–1682
- Arcangelo Corelli 1653–1713
- Johann Pachelbel 1653–1706

[30] For further reading, see:

- Leicester, H.M., *The Historical Background of Chemistry*, Dover: New York, 1971, 260 pp.
- Partington, J.R., *A Short History of Chemistry*, Dover: New York, 1989, 415 pp.

• Giuseppe Torelli	1658–1709
• Henry Purcell	1659–1695
• Tommaso Vitali	1663–1745
• Francois Couperin	1668–1733
• Thomasso Albinoni	1671–1751
• **Antonio Vivaldi**	**1678–1741**
• Francesco Manfredini	1680–1748
• Jean-Baptiste Loeillet	1680–1730
• **Georg Telemann**	**1681–1767**
• Jean-Philippe Rameau	1683–1764
• Domenico Scarlatti	1685–1757
• **Johann Sebastian Bach**	**1685–1750**
• **Georg Friedrich Handel**	**1685–1759**
• Francesco Geminiani	1687–1762
• Johann Friedrich Fasch	1688–1758
• Carlo Tessarini	1690–1765
• Giuseppe Tartini	1692–1770
• Pietro Locatelli	1695–1764
• Giovanni Batista Sammartini	1701–1775
• Giovanni Batista Pergolesi	1710–1736
• Christoph Willibald von Gluck	1714–1787
• Pietro Nardini	1722–1793

1600 CE **William Gilbert** (1544–1603, England). Physician and scientist. The father of the science of magnetism[31]. Asserted that the earth is a giant magnet, thus explaining for the first time why the compass needle

[31] Gilbert must have been aware of the contributions of **William Borough** (1536–1599, England) and **Robert Norman**. Borough published *A Discourse of the Variation of the Compass, or Magnetical Needle* (1581), based on his observations during several marine expeditions. Norman (1581) described his discovery made some years before (1570) of the *inclination* or *dip*. He devised a form of a dip-circle, and found the value for the inclination in London.

Another fundamental discovery, that of the *secular change of declination*, was made in England by **Henry Gellibrand** (1597–1636), a mathematician and astronomer, professor of mathematics at Gresham College, who described it in his *Discourse Mathematical on the Variation of the Magnetical Needle together with its Admirable Diminution lately discovered* (1635).

Gellibrand also noticed *diurnal* changes in the declination, which he attributed to instrumental uncertainties. However, the reality of this phenomenon was first emphasized by **George Graham** (1675–1751, England), a London instrument maker, in 1724.

seeks the poles. His findings were published in his book: "*De magnete magneticisque corporibus, et de magno magnete tellure physiologia nova*". In his book (1600) Gilbert convincingly demonstrated, with the aid of an enormous body of experimental material, that the magnetic field of the earth is like the field of a uniformly magnetized sphere made of magnetic iron ore[32]. Gilbert's book laid the foundations for the scientific approach to magnetism *in general*, and to terrestrial magnetism in particular. For two centuries following his discovery, nothing of substance that was not either a repetition or a development of what Gilbert had already done, was added to the subject.

Gilbert's work, which embodied the results of many years of research, was distinguished by its strict adherence to the scientific method of investigation by experiment, and by the originality of its material. He explained not only the north-south alignment of the magnetic needle, but also the variation in the dipping (inclination) of the needle. Gilbert's is therefore the first systematic contribution to the science of magnetism.

Gilbert was born at Colchester of an ancient Suffolk family. He entered St. John's College, Cambridge in 1558, and graduated M.D. in 1569. After spending three years in Italy and other parts of Europe, he settled in London, where he practiced as a physician with great success. In 1599 he became president of the college of physicians, and in 1601, court physician to Queen Elizabeth I. On the death of the queen in 1603 he was reappointed by her successor, but died soon thereafter of the plague.

1603–1644 CE **Theodore Turquet de Mayerne** (1573–1655, France and England). Physician, Physiologist and Chemist. One of the great physicians of the Baroque era: added chemical principles to humoral medicine, a greater empiricism to a more rational approach to medicine, and an interventionist therapeutics to a more cautious view of therapy. Thus he was influential in the introduction and support of chemical therapy in medicine, endorsing the use of chemical remedies in his practice.

Turquet was born in Mayerne, near Geneva, the son of a noted *Huguenot* historian and political theorist, **Louis Turquet de Mayerne**. He completed his early schooling in Geneva and took his undergraduate degree at the University of Heidelberg. He received his M.D. in 1597 at the University of Montpellier. For 50 years he served as a royal physician to three kings in France and England (Henri IV, James I, Charles II).

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[32] In explaining terrestrial magnetism Gilbert suggested that the earth was made of magnetized iron, which created the magnetic field; but his proposition was not correct. He himself discovered that iron, at the high temperatures that we now know to exist at the center of the earth, completely loses its magnetic qualities.

Turquet was one of the 17^{th} century most renowned authority on the technical aspects of painting and art: he prepared instructions for varnishes, painting mediums, coating canvases, enamels and pigments for **Peter Paul Rubens**, **Anthony van Dyck** and a host of other well known painters and craftsmen of the Baroque[33].

33 Many of the practices used by Renaissance and Baroque painters were often kept secret.

History of Magnetism I (1100 BCE–1600 CE)

Certain naturally occurring substances (e.g. magnetite Fe_3O_4*, magnetic pyrites* $6FeS \cdot Fe_2S_3$*) posses the property of attracting neighboring particles of iron over considerable distances. Such bodies are called* magnets. *If a steel rod be stroked with such a natural magnet, it also assumes the property of attracting particles of iron. A splinter of magnetite, hanging by a thread, takes up a definite position, resulted in being called* loadstone *or* lodestone.

These curious facts were known to the ancient Greeks *at least as early as 800 BCE. Apart from these two magnetic phenomena, no additional knowledge about magnetism was gained up to the end of the* 15^{th} *century. Upon one of these is based the principle of the mariner's* compass[34]*, which is said to have been known to the* Chinese[35] *as early as 1100 BCE, thought it was not introduced into Europe until more than 2000 years later.*

A passage in De Rerum Natura (VI, 910–915) by the Roman poet ***Lucretius*** *(ca 60 BCE) indicates that in his time the phenomenon of magnetization by induction has been observed. The property of orientation, in virtue of which a freely suspended magnet points approximately to the geographical north and south, is not referred to by any European writer before the* 12^{th} *century (****A. Neckham*** *of Great Britain in 1187 CE).*

The needles of primitive compasses, being made of iron, would require frequent re-magnetization, and a "stone" for the purpose of "touching the needle" was therefore generally included in the navigator's outfit. With the constant practice of this operation, it is hardly possible that the repulsion acting between like poles should have entirely escaped recognition; but though it appears to have been noticed that the loadstone sometimes repelled *iron instead of attracting it, no clear statement of the fundamental law that unlike poles attract while like poles repel was recorded before the publication (1581) of the New Attractive by* ***Robert Norman***.

The foundations of the modern science of magnetism were laid by ***William Gilbert*** *(1600) in his book De Magnete. It contains many references to the exposition of earlier writers from Plato to the author's own age. He admitted therein that the north seeking property of magnetite was brought to Europe from China by* ***Marco Polo***. *Gilbert showed that the earth's magnetic field was equivalent to that of a permanent magnet, lying in a general north-south direction, near the earth's rotational axis.*

[34] From the Latin *cum* = with, *passus* = a step; compass = a measuring instrument.

[35] First mentioned by **Shen Kua** of China in 1088 CE.

No material advance upon the knowledge recorded in Gilbert's book was made until the establishment by ***Coulomb*** *(1785) of the law of magnetic action.*

1603–1614 CE **Santorio Santorio or Sanctorius Sanctorius** (1561–1636, Italy). Physician and physiologist. Pioneer of quantitative experimental medicine. His experimental studies established quantitative metabolic phenomena of body weight (1614). Introduced measurements and quantification into physiology and medicine.

Santorio was born in Justinopolis, Venetian Republic (now Koper, former Yugoslavia) to a noble Venetian family. He studied philosophy and medicine at Padua (1579), where he received his M.D. (1582). Served as a personal physician of a Croatian nobleman (1587–1599) and then set up a medical practice in Venice (1599). Here he became part of the circle of learned men, befriending **Galileo** and other leading figures of the *Scientific Revolution*. Appointed to the chair of theoretical medicine at the University of Padua (1611), where he taught until his retirement (1624).

Santorio is best known for his investigations into *metabolism*: over a period of 30 years he carried out an elaborate series of measurements, described in his *De Statica Medicina*. He placed himself on a platform suspended from an arm of an enormous balance, and weighted both himself and his food, drink, and waste products. He determined that over half of normal weight loss is due to 'insensible perspiration'. He invented instruments to measure humidity, temperature (1611) and pulse rate (1603).

Although in treating his patients Santorio did not stray far from Hippocratic and Galenic practice (based on the notion of a balance of fluids), he differed from the classical authors a great deal in his theory and method of investigation. Rather than relying on authority in the first instance, he argued that one should first rely on sense experience, then on reasoning, and only lastly on authority.

Rather than describing the body and its functions in terms of Aristotelian and Galenic elements and qualities, Santorio argued that the fundamental properties were mathematical ones, such as number, position, and form. The body was like a clock, the working of which was determined by the shapes and positions of its interlocking parts. This was a radical break with traditional medical theory and natural philosophy, in which the discourse was about qualities and essences.

1603 CE **Johann Bayer** (1572–1625, Germany). Amateur astronomer and lawyer. Introduced the method of describing the locations of stars and of naming them with Greek letters and by the constellation they are in; this system continues to be used today. His *Uranometria* (1603) is the first attempt at a complete celestial atlas.

Bayer was born in Rain, Germany.

1604–1619 CE **Hieronymus Fabricius** (Geronimo Fabricio; Girolamo Fabrici, 1537–1619, Italy). Surgeon, anatomist and embryologist. Founder of comparative anatomy.

He was born at Aquapendente. Student of **Gabriele Fallopio** and his successor at Padua (1562–1613). Conducted studies in embryology of various animals and man, published in his *De formato foetu* (1604) and *De formatione ovi et pulli* (1621).

1605–1638 CE **Willem Janszoon Blaeu** (1571–1638, Holland). Map maker and astronomer. One of the leading map makers of the early 17^{th} century. His works include a world map issued in 1605, a three-volume sea atlas [*The Light of Navigation* (1608–1621)], and a series of atlases.

Blaeu was born in Alkmaar and developed his geographical and astronomical skills under the guidance of Tycho Brahe in Denmark. He founded a publishing house (1599), specializing in cartography. His instruments and globes featured unprecedented precision.

1608–1609 CE **Hans Lippershey** (1587–1619) and **Zacharias Jansen** (1588–1630), Dutch spectacle makers from Middleburg, and **James Metius of Alkmaar** invented both the compound microscope and the telescope. **Anton van Leeuwenhoek** (1632–1723, Netherlands, 1668) first used microscopes for scientific research.

1609 CE First regularly published *newspaper* in Germany.

1609–1621 CE **Johannes Kepler** (1571–1630, Germany). Court astrologer and astronomer. The founding father of modern astronomy. By careful observations and years of painstaking calculations, was able to derive the laws of elliptical planetary motion, thus providing evidence for the Copernican system.

With his resolution to submit every physical and astronomical law to the test of experiment and observation, he contributed much to the inauguration of the present scientific age. Kepler dissented from the Aristotelian metaphysics of his day and maintained that the Copernican system was not merely

a convenient hypothesis but a true image of nature, and that it was amenable to verification through quantitative measurements.

Born in Weil-der-Stadt, Württemberg (near Stuttgart), he attended the University of Tübingen and studied for a theological career. There he learned the principles of the Copernican system. In 1594 he was offered a position of teaching mathematics and astronomy at the Lutheran school in Graz. As part of his duties, he prepared astronomical almanacs and furnished astrological "data". But he left Graz rather then undergo compulsory conversion to Roman Catholicism. While he was seeking another post, he formed an association with **Tycho Brahe** which shaped the rest of his life.

Tycho set Kepler to work trying to find a satisfactory theory of planetary motion — one that was compatible with the long series of observations that he had made. Brahe, however, was reluctant to supply Kepler with enough data to enable him to make substantial progress, perhaps because he was afraid of being "scooped" by the young mathematician.

After Tycho's death in 1601, Kepler succeeded him as mathematician to Rudolph II, the Holy Roman Emperor, and obtained possession of the majority of Tycho's records: Their study occupied most of Kepler's time for more than 20 years. In 1604 Kepler observed what is today known as a *supernova explosion*. [In the same year he also suggested that the opposite ends of a straight line meet at infinity and that two parallel lines intersect at infinity!]

Kepler made his most significant discoveries when he tried to find an orbit to fit all Brahe's observations of the planet Mars. Earlier astronomers thought that a planetary orbit was a circle or a combination of circles. But Kepler could not find a circular orbit that would agree with Brahe's observations. He spent *several years* on this problem. At one point he found a combination of circular arcs that agreed with the observations to within 8 arcminutes (quarter of a diameter of a full moon), but he believed that Tycho's observations could not have been in error by even this small amount, and so, with characteristic integrity, he discarded the hypothesis. He then took the bold step of assuming that the orbit of Mars *cannot be circular*, and tried to represent it with an *oval* instead. He soon discovered that the orbit could be fitted well by an *ellipse* (Kepler's First Law).

Kepler found that the eccentricity of the orbit of Mars is only 0.1: the orbit, drawn to scale, would be practically *indistinguishable from a circle*. It is a tribute to Tycho's observations and to Kepler's perseverance, that he was able to determine that the orbit was an ellipse at all. Kepler's achievement in dislodging the 2000 year old belief in circular orbits, is all the more remarkable since he himself was quite partial to perfect heavenly spheres.

In the year 1609, Kepler published his new results in a book '*Astronomia Nova*', on which he worked altogether for six years. Before he saw that the

orbit of Mars could be represented accurately by an ellipse, Kepler had already investigated the manner in which the planet's orbital speed varied. After some calculations, he found that Mars speeds up as it comes closer to the sun and slows down as it pulls away from the sun. Kepler expressed this relation by imagining that the sun and Mars are connected by a straight, elastic line. As Mars travels in its elliptical orbit around the sun, the areas swept out in space by this imaginary line in equal intervals of time are always equal (Kepler's Second Law).

At the time of publication of his book in 1609 Kepler appeared to have demonstrated the validity of his two laws for the case of Mars alone. However, he expressed the opinion that they hold also for the other planets.

Kepler believed in an underlying harmony in nature, and he constantly searched for numerological relations in the celestial realm. This belief was triumphantly vindicated when he found a simple algebraic relation between the length of the semi-major axis of a planet's orbit and its sidereal period: namely that the squares of the sidereal periods of the planets are in direct proportion to the cubes of the semi-major axes of their orbits (Kepler's Third Law[36]).

Kepler published this third law in a second book, "*De Harmonice Mundi*" in 1619.[37] To arrive at this law it was not necessary for him to know the *actual* distances of the planets from the sun, only the distance in units of the earth's distance. [There were very slight discrepancies when the third law was applied to the orbits of Jupiter and Saturn. Decades later, Newton gave an explanation for them, but within the limits of accuracy of the observational data available in 1619, Kepler was justified in considering his formula to be exact.]

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[36] Kepler himself never realized the real importance of his three laws. Indeed, without differential calculus and analytical geometry, these laws show no apparent connection with each other – they are disjointed bits of information which do not make much sense. Once you know the inverse square law of gravitation and Newton's mathematical equations, all this become beautifully self-evident. Thus, Kepler's laws seem to have no particular raison dêtre: of the First he was almost ashamed – it was a departure from the circle sacred to the ancients and there was nothing to recommend it in the eyes of God. The Second Law he regarded as a mere calculating device. The Third he saw as necessary link in the system of harmonies, and nothing more.

[37] For further reading, see:

- Adler, M.J. (ed), *Great Books of the Western World. No. 16. Ptolemy, Copernicus, Kepler*, William Benton, Publisher, The University of Chicago, 1952, 1085 pp.

Much of the rest of "*De Harmonice Mundi*" deals with Kepler's attempts to associate numerical relations in the solar system with the regular Platonic Solids[38] and with music. He tried to derive notes of music played by the planets as they move harmoniously in their orbits (!). The earth, for example, plays the notes mi, fa, mi, which he took to symbolize the "miseria" (misery), "fames" (famine), "miseria" of our planet.

Buried amongst the musical notes was a curious little relationship: "It seems that the squares of the periods of revolution (T) of any two planets are as the cubes of their mean distance from the Sun (r)."

	Year (T)	T squared	Orbit (r)	r cubed
Mercury	0.2408	0.0580	0.388	0.0584
Venus	0.6152	0.3785	0.724	0.3795
Earth	1.0000	1.0000	1.000	1.0000
Mars	1.881	3.5378	1.524	3.5396
Jupiter	11.862	140.71	5.200	140.61
Saturn	29.457	867.72	9.510	860.09

This was to become – even though he didn't know it himself – Kepler's Third Law. It is the key to the orderliness of the solar system, for it indicates in what way the motions of the five planets are mathematically interdependent.

The book containing it was universally ignored. Three days after the completion of *The Harmony of the Worlds*, the Thirty Years War broke out.

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38 Kepler attempted here to bare the ultimate secret of the universe in an all-embracing synthesis of geometry, musics, astrology, astronomy and epistemology. It was the first attempt of this kind since Plato, and it is the last to our day. After Kepler, science is divorced from religion, religion from art and matter from mind.

According to Kepler, the existence of just six planets (with the five intervals between them) matching the five Platonic Solids, was not by chance – but a divine arrangement: into the orbit (or sphere) of *Saturn* he inscribed a *cube*; and into the *cube* another sphere, which was that of *Jupiter*. Inscribed in that was the *tetrahedron* and inscribed in it was the sphere of *Mars*. Between the spheres of *Mars* and *Earth* came the *dodecahedron*; between *Earth* and *Venus* the *icosahedron*; between *Venus* and *Mercury* the *octahedron*.

In the Third Law, Kepler saw the pinnacle of all his achievements: here at last was the connection between characteristic *distances and times* associated with the solar system – the ultimate harmony of the spatial architecture of the Platonic Solids and the temporal musical scale of the planetary spheres.

It is indeed surprising to perceive in his work copious signs of superstition and a keen devotion to astrology. Neo-Platonic and religious conceptions are even more evident than in Copernicus. Still under the spell of apriorism, he was anxious to interpret the universe as motivated by mathematico-aesthetic numerical harmony and exhibiting a surpassing simplicity and unity.

In 1618, 1620, and 1621, Kepler published his text "*Epitome Astronomiae Copernicanae*". Here, he stated that his first two laws had been tested and found valid for the other planets besides Mars, and for the moon. Also, he reported that the third law applies to the motions of the four newly discovered satellites of Jupiter as well as to the motions of the planets about the sun.

In 1623, Kepler concluded work on his last book, the "*Tabulae Rudolphinae*", which consisted of tables and rules for determining the positions of the planets and a catalogue of star positions, mostly based on the data of Brahe. This book ranked for a century as the best aid to astronomy. The printing of this book was delayed [by the *30 year war* (1618–1648) which raged at that time in Europe] and was finalized only in 1627.

Kepler also studied optics and designed a telescope that he probably built but never used. He discovered the inverse-square law of the decrease in the brightness of a source of light, for he saw instinctively that light from a faint source spreads out spherically and that the brightness of the source therefore varies inversely as the square of the observer's distance from it. Kepler also investigated the *refraction of light* and showed that Ptolemy's approximate law of refraction (i.e. the proportionality of the angles of refraction and incidence) holds only for small angles of incidence. However, he did not discover the correct law of refraction[39].

At that time, the insolvent imperial exchequer owed Kepler some 12,000 florins, for which **Wallenstein** assumed full responsibility. But Wallenstein's promises to Kepler were not kept. In lieu of the sums due, he offered him a professorship at Rostock, which Kepler declined. An expedition to Ratisbon, undertaken for the purpose of presenting his case to the diet, terminated his life: shaken by the journey, which he had performed across Europe entirely on horseback in the autumn of 1630, he came down with fever and died at Ratisbon, on the 15^{th} of November 1630 in the 59^{th} year of his life. By his first wife (ca 1611) he had five children, and by his second wife — seven children. Of these only two, a son and a daughter, reached maturity. In 1615 his mother was charged with witchcraft; it was only due to his indefatigable

[39] Kepler used his own approximation $i = \frac{kr\cos r}{k\cos r-(k-1)}$, where i is the *angle of incidence* (w.r.t. the normal) and r is the corresponding *angle of refraction*; k is a fixed number for any pair of media

efforts that she was acquitted, after having suffered 13 month's imprisonment under imminent threat of torture.

Kepler was buried in a cemetery outside the town of Ratisbon. The cemetery was destroyed during the 30 years war and his bones were scattered. There remained, however, the epitaph that he had prepared for himself:

"*Mensus eram coelos, nunc terrae metior umbras,*
Mens coelestis erat, corporis umbra iacet".

Worldview VIII: Johannes Kepler

* * *

"Ubi materia, ibi geometria"

* * *

"Expectet ille suum lectoremper annos centum; si Deus ipse perannorum sena millia contemplatorem praestolatus est."

"(It may well wait a century for a reader, as God has waited six thousand years for an observer.)"

(Harmonice Mundi, 1619)

* * *

"I measured the skies, now the shadows I measured. Sky-bound was my mind, earth-bound the body rests."

* * *

"When the storm rages and the state is threatened by shipwreck, we can do nothing more noble than to lower the anchor of our peaceful studies into the ground of eternity."

(1629)

* * *

"I have the answer, the orbit of the planet is a perfect ellipse."

(1609)

* * *

"God always geometrizes"

* * *

"The universe was stamped with the adornment of harmonic proportions"

* * *

"I undertake to prove that God, in creating the universe and regulating the order of the cosmos, had in view the five regular bodies *of geometry as known since the days of Pythagoras and Plato, and that he has fixed according to those dimensions, the number of heavens, their proportions, and the relations of their movements."*

* * *

Others on Kepler

* * *

"Johannes Kepler set out to discover India and found America. It is an event repeated over and again in the quest for knowledge. But the result is indifferent to the motive. A fact, once discovered, leads an existence of its own, and enters into relations with other facts of which their discoverers have never dreamed. Apollonios of Perga discovered the laws of the useless curves which emerge when a plane intersects a cone at various angles: these curves proved, centuries later, to represent the paths followed by planets, comets, rockets, and satellites."

(Arthur Koestler "The Watershed", 1960)

* * *

How did Kepler derive his three 'laws'?[40] (1609–1619)

The manner in which Kepler used the empirical astronomical data available to him (consisting of antiquity's accumulated lore plus Tycho Brahe's observations) is an instructive case study of how Science's knowledge of the laws of nature is actually abstracted from observations and experience. It was done through a sequence of interactive*,* iterative *and* convergent *interplays between empirical investigations on the one hand, and theoretical speculation*

[40] This article was written by Dr. Shahar Ben-Menahem.

and modeling on the other. It led to Kepler's three laws of planetary motion, and then to Newton's theories of mechanics and universal gravitation.

In order to deduce his 'laws', Kepler had to determine the distances of the planets from the sun and show that the orbits are not circles, but ellipses. Before we see how he accomplished this feat, let us regress momentarily to ***Copernicus*** *(1543) who went back to the doctrine of* **Aristarchos** *(270 BCE) and put the sun at the center of the whole planetary system, including the earth as a planet. Having no telescopic information, he stuck to the idealistic belief that each planet moves in the most perfect plane.*

The hypotheses which Copernicus adopted may be summarized under four headings:

1. *The apparent diurnal rotation of the celestial sphere is due to the complete rotation of the earth about its polar axis in a period of 24 hours.*
2. *The moon revolves around the earth in a period of* $27\frac{1}{3}$ *days.*
3. *The earth and the planets revolve in circular orbits about the sun in the same direction as the earth's diurnal motion.*
4. *The orbits of Mercury and Venus lie between the sun and that of the earth, while the orbits of Mars, Jupiter, and Saturn, lie beyond the earth's orbit.*

The tracks of the planets lie close to the ecliptic plane. So it is better to calculate their positions in celestial longitude and latitude as Copernicus did. For the purpose of grasping the principles employed in tracing out their orbits it will be sufficient to employ right ascension to measure their angular displacement. This is equivalent to projecting their movements onto the plane of the celestial equator.

Once one accepts the Copernican system for the solar system, the simplest set of assumptions is that each planet (earth included) describes a closed orbit around the sun — and furthermore, that these orbits are circular (centered at the sun), and that each planet moves around its orbit at a uniform angular speed. These assumptions are 'theoretical' in the sense that the only data that were actually 'measured' from any single observatory on earth (that is to say, excluding 'triangulation' measurements using some terrestrial distance as baseline) were angular position (right-ascensions and declinations) at which the planets and sun appear, in reference to the celestial sphere anchored to the fixed stars. Note that all the above-mentioned theoretical assumptions turned out in the end to be not quite accurate — but they did play an initial role in interpreting the "pure", apparent-angular-position data.

Kepler's first task was to calculate the planets' sidereal periods. True, Copernicus and his predecessors calculated it from the synodic period (known to the ancients); the latter is the time elapsed between two successive occasions when Mars (or any other planet), the sun and the earth occupy the same relative positions. It is done by noting when Mars is in opposition, i.e., when it is on the meridian at midnight, and counting the number of days which intervene before its next midnight meridional crossing (780 earth days on the average).

But the Copernican formula was based on the assumption of constant angular speed (circular orbit). Kepler, however, found very soon from Tycho's observations that the assumption of a circular Martian orbit is in conflict with the data. He therefore realized that strictly speaking, there is no such thing as a precisely-defined synodic period. In other words, because of the non-uniform planetary speeds, the times of conjunction and opposition do not occur with exact regularity and one can speak only of a mean synodic period.

So Kepler was led to determine the planets sidereal periods in a better way, one that does not make use of the (now discredited) 'theory' of uniform angular speeds.

The new, better method involved scanning the Tables for a planets' apparent angular positions at a sequence of dates separated by an integral number of earth years; one might term this the "strobing out" method — the well-known periodicity of the earth's own sidereal motion (namely, one earth-year) is removed from the compounded motion by viewing the accumulated data through a "stroboscope", as it were, having a period of one earth year. This method, when applied to the planetary data that had accumulated over the centuries, gave Kepler accurate values for each planet's sidereal period.

Having eliminated his dependence upon the uniform-revolutions assumption, Kepler found that the assumption that each planet's orbit is a sun-centered perfect circle, was also wanting. By laborious fitting of Tycho's data[41] , he found that models using circles simply would not work — even if he shifted the sun away from their centers. Thus, even when a given such model seemed to work for Mars (the planet boasting the most eccentric of the planetary orbits) based on R.A. (Right Ascension) data, it failed when "declinations" (due to the differing orbital planes of Earth and Mars) were taken into consideration. In other words, neither circular orbits nor uniform

[41] Kepler's various fitting efforts were rendered all the more tedious by his lack of proper, statistics-based best-fit procedures — which were only developed later, starting with the work of Gauss

angular speeds yielded theoretical models which could be reconciled with Tycho's data (to the level of accuracy to which Kepler believed Tycho's purely empirical results held true).

Note that in the case of inferior *planets (i.e. Mercury and Venus), the assumption of sun-centered circular orbits immediately allows the extraction from Tycho's (or even the ancients') data of a reasonable value for these two planets' orbital radii – in units of earth's radius. This can be done by measuring the elongation angle (maximal apparent angular separation between the planet and the sun) for each inferior planet, and then utilizing simple trigonometry to compute the ratio of the respective planets' orbital radii to that of earth. But the* superior *planets (Mars and outwards) have no elongations as viewed from earth; and in any case Kepler could no longer rely on the sun-centered-circles model – as explained above.*

Once Kepler was convinced that a circular Martian orbit about the sun would not do, he had to obtain a real picture, based on Brahe's data which he trusted. However, this was not easy since he only had observation of the apparent path of Mars from a moving earth. *The true distances were unknown, only angles were measured, and those gave a foreshortened compound of Mars' orbital motion and the earth's. So Kepler decided to attack the earth's orbit first by a method that had the hallmark of genius.*

To use Tycho's data to extract the correct shapes and sizes of the planetary orbits and the rates at which the planets move along these orbits, Kepler applied the 'strobing out' method in reverse!! *Namely, by picking out of Tycho's tables the apparent angles of a given planet at many observation times* spaced by integral numbers of that planet's sidereal period *— and assuming, as for the earth, that the given planet's orbit is a closed curve — Kepler was able to use apparent angular positions of a* single position *along the planet's orbit, as viewed by many earth positions, to determine the* distance *of that particular position of the planet from the sun in units of the average earth-sun distance (the so-called "Astronomical Unit" – A.U. for short) — via a simple geometrical construction.*

In fact, if one believes that the earth's orbit itself is *a sun-centered circle, then it suffices to employ* two *earth positions along its year-long orbit to properly triangulate each planet; Kepler then could (and did) repeat this procedure for many different positions of each planet along its orbit, thereby determining the detailed shapes and sizes of all planetary orbits.*

However, the earth's orbit – although fairly close to being a sun-centered circle – does have some *eccentricity; which is why the "strobed triangulation" procedure just described, needs to be over-determined. With enough different earth positions per given position of the planet being investigated, one can*

compensate for the earth's orbital eccentricity. As a result of this investigation, Kepler found that the planets, earth included, moved in ellipses with the sun at one focus (Kepler's First Law).

After he had worked out the geometry of planetary orbits, Kepler proceeded to investigate the detailed motion *of each planet around its orbit — finding his* second law, *governing the way in which a planet's varying distance from the sun modulates its (non-uniform!) angular speed along its orbit, as subtended at the sun.*

Finally, Kepler asked himself whether there is any systematic relationship between the sizes *of these orbits and their respective* sidereal periods; *after various attempts, he found such a simple rule – the celebrated Kepler's Third Law.*

Note that there are many other effects, not mentioned above, which "contaminate" Kepler's interpretation of the "pure data" with unwarranted assumptions: to mention just two, there is the earth's precession (caused by tidal torques upon the earth's equatorial bulge, and resulting in the famed "precession of the equinoxes" thanks to which we are said to be entering "the Age of Aquarius"), and perturbation of planetary orbits due to inter-planetary gravitational attractions.

The tidal-precession effect causes an additional apparent motion of the fixed stars, over and above the familiar diurnal motion — and this additional rotation is about an axis perpendicular to the ecliptic plane, and thus at an angle to the rotation axis of the earth. This effect certainly introduced complication in principle *for Kepler's program – but fortunately, the precession is very slow.*

As for the inter-planetary perturbations – those, too, are small; and once the Keplerian picture (as completed by Newton's new physics) clarified the basic *dynamics of the solar system, these perturbations were used by subsequent scientists to work out such details as three-body dynamics (Lagrange points, etc.) and to successfully predict new, previously unobserved, planets from purely theoretical calculations (the planets Neptune and Pluto were discovered with the aid of successive application of this technique). We thus see that the grand "iterative, interactive interplay of theory and experiment" continues to spin and converge long after Kepler, and each new "twist in the plot" demonstrates anew the fundamental robustness of this never-ending iteration.*

Kepler's three laws of planetary motion – augmented by Galileo's observation of the systematics of the Jovian moons' motion about Jupiter – allowed ***Newton*** *to arrive at his laws of mechanics and universal gravitation.*

Late-17th-century triangulation measurements (via a terrestrial baseline) of distances from earth to the nearest planets (Mars and Venus) by Cassini et al., fixed the absolute distance scale *of Kepler's solar system model – thus allowing the determination of its basic scale, the A.U. (Astronomical Unit), in terms of terrestrial units such as kilometers (although the metric system was yet to be invented).*

The Kepler Problem

The motion of an isolated system of two masses, moving under the sole influence of their mutual gravitation, is known as the two body problem *or the* Kepler problem. *The motion is governed by a single ODE equation of the second order:*

$$\frac{d^2\mathbf{r}_{12}}{dt^2} = -\big[G(m_1+m_2)/r_{12}^3\big]\mathbf{r}_{12}, \tag{1}$$

where $\mathbf{r}_{12}(t)$ is the relative *vectorial distance between the mass m_1 and the mass m_2 at time t. In this form the problem is represented in terms of the separation $\mathbf{r}_{12}$, which can be determined* directly. *The* force *between the masses is*

$$\mathbf{F}_{12} = -[GM\mu/r_{12}^3]\mathbf{r}_{12},$$

where $M = m_1 + m_2$ is the total mass and $\mu = \frac{m_1 m_2}{m_1+m_2}$ is the reduced mass. *Thus the orbit of each mass about the other is equivalent to the orbit of a mass μ about a mass M that is fixed in an unaccelerated, unrotating (inertial) frame.*

The exact solution of the above equation of motion can be written as a time-eliminated polar equation of the conic section curve:

$$r_{12}(\theta) = \Big[\frac{MG}{h^2} + \frac{1}{h}\Big\{2E + \frac{M^2G^2}{h^2}\Big\}^{1/2}\cos\theta\Big]^{-1},$$

*where θ is the angle at the focus of the conic between the radius vector $\mathbf{r}_{12}$ and the major axis ('*true anomaly*'), and (E, h) are the two constants of motion,*

namely the total energy and the orbital angular momentum, both per unit reduced mass (h is also twice the area swept out by the radius vector per unit time).

Explicitly

$$E = -\frac{MG}{2a}, \quad 1 - e^2 = \frac{h^2}{MGa}, \quad P = 2\pi a\sqrt{\frac{a}{MG}},$$

where a is the semi-major axis, e is the eccentricity of the orbit and P is the orbital period.

If (a, P) for a binary system can be evaluated by direct astronomical observations, and if the motion of one of the two masses (which could be a star, planet, comet, moon, etc.) w.r.t. the common center of mass is known, the individual masses (m_1, m_2) of the pair can also be determined.

The equation of motion can be further exploited to obtain useful relations: from the area-rate constant

$$r^2\frac{d\theta}{dt} = h = \sqrt{GMa(1-e^2)}$$

and the energy constant

$$\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right),$$

one obtains (by eliminating $\frac{d\theta}{dt}$)

$$\frac{dr}{dt} = \frac{na}{r}\sqrt{a^2e^2 - (a-r)^2},$$

where $n = \frac{2\pi}{P}$. Defining the eccentric anomaly E via

$$a - r = ae\cos E,$$

a straightforward integration of the above first-order differential equation for $r(t)$ yields the Kepler equation

$$n(t - T) = E - e\sin E,$$

where T is an integration constant.

The geometric interpretation of E is clear from its defining equation:

$$r = a(1 - e\cos E).$$

Construct an auxiliary circle *in the orbital plane such that its diameter coincides with the major axis of the orbital ellipse, and their centers coincide.*

From a point $S(r,\theta)$ on the ellipse draw a normal to the major axis and extend it until it meets the circle at S'. The angle subtended at the circles' center between the major axis and S' is E.

Then $M = n(t-T)$ is the angle which would have been described by a fictitious point moving on the auxiliary circle with mean angular velocity such that it revolves along the circle (and the ellipse) with period P. The angle M is known as the mean anomaly. *The entity $(t-T)$ is the* epoch *relative to T, the time of the perihelian passage.*

Kepler noticed that given M, one must solve a transcendental equation *for E, namely*

$$E - e\sin E - M = 0.$$

Once $E(e;M)$ is known, the orbit is calculated from $r = a(1-e\cos E)$ and $\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}$, yielding $R(t,\theta(t))$. During 1609–1819, more than 100 methods of solving Kepler's equation had been proposed, the most elegant being that of ***Bessel*** *(1819).*

Eq. (1) is equivalent to a system of 6 scalar ODE's of the first order in 6 unknown functions, namely the components of the relative separation of the masses and of their relative velocity vector at any given time. This is then a well-posed problem which needs for its complete solution 6 *constants of integration[42]. These constants are, for example, the relative positions and velocities of the masses at any given fiducial time.*

A difficulty arises from the fact that the observations which are made from the moving earth give only the direction of the line of sight to the object seen by the observer and furnish no direct information regarding its distance or line-of-sight velocity component. The position of the body in space is therefore not given and of course, its components of velocity are not determined. It thus becomes necessary to secure additional observations *at other times (or use Dopler measurements to extract line-of-sight speeds via spectral means). It*

[42] The six arbitrary constants of integration can be represented by six independent functions (orbital elements) of these constants, which are direct and indirect observable parameters of the orbit:

a = major semi-axis, which defines the size of the orbit, its energy and its period;

e = the eccentricity, which defines the shape of the orbit;

Ω, i = two angles which together define the position of the plane of the orbit relative to the plane of the ecliptic (longitude of ascending node and inclination to the plane of the ecliptic);

ω (or π) = an angle defining orientation of orbit in the plane of the orbit;

T = time of perihelion passage, defining, with the other elements, the position of the body in its orbit at any time.

is clear that the problem of finding the position of the body and the elements of its orbit from such data present some difficulties.

Clearly, six independent entities must be found by observations in order that the elements could be determined. A single complete geometrical observation gives two quantities: the angular coordinates of the body. Therefore 3 *complete observations are just sufficient to determine the orbit.*

It required the combined genius of **Euler** *(1744),* **Lagrange** *(1778–1783),* **Laplace** *(1780) and* **Gauss** *(1809) to perfect precise and elegant computational tools for determining the orbital elements of planets and comets from observations by earthbound spectators.*

The Advent of Optical Instruments

The growth of maritime commerce was reinforced by the introduction of new technical inventions which emerged in a different context from the world's everyday work.

One of these was the invention of spectacles*. Although devices of one kind or another for magnifying objects are of considerable antiquity, there does not seem to have been any general use of them in everyday life till the close of the Middle Ages.*

The introduction of spectacles *at about* 1300 *in Florence involved no theoretical discovery about phenomena of which the Alexandrian and Arab astronomers were not fully conversant [Ptolemy, 150; Alhazen, 1026]. It is therefore more reasonable to suppose that introduction of paper, the invention of printing and* the use of books *in the* 15^{th} *century, stimulated the demand for eye glasses. The trade increased during the* 16^{th} *century, especially in Italy and in Southern Germany. By* 1600 *opticians were to be found in most of the larger towns on the continent.*

Two other inventions, which are signposts in the history of science, came as quite fortuitous by-products of the new industry: the telescope (1608) and the microscope (1609).

On Galileo's visit to Venice in May 1609, *he heard that an instrument for making objects appear nearer and larger had been invented. Returning to Padua, he made his first telescope by fixing a convex lens in one end of a leaden tube and a concave lens in the other end. Then he made a better one, went to Venice, and presented the instrument to the Doge Leonardo Donato. His first telescope magnified* 3 *diameters. He soon made others which magnified* 8 *diameters and finally one that magnified* 33 *diameters. Kepler devised an alternative form using a convex eyepiece.*

The three years which followed the invention of the telescope by Lippershey, Jansen and Metius, were eventful. Kepler's account of the motion of Mars appeared in 1609. *His telescope was constructed in* 1611. *Eight years later he was able to announce his complete vindication of the fundamental doctrine of Copernicus and his epoch-making laws of the solar system.*

Meanwhile, Galileo had observed the motion of the sun's spots and had seen the moons of Jupiter. Galileo's discoveries were important partly because it deprived the geocentric view of the universe of the inherent plausibility it enjoyed before people realized that there were other worlds with satellites circling about them.

The Inquisition rightly judged the psychological effect of the new realization that our own small world is not a unique one. Thus the tract on the moons of Jupiter became one of the most decisive battle fields between science and the priestly superstition.

The telescope had a threefold significance for the age of the Great Navigators. The determination of longitude for westerly sailing had become a technical issue of cardinal importance, and on this account astronomy retained its place as the queen of the sciences till the end of the 18^{th} *century. At a time when the only method of determining longitude was based on the use of celestial signals (eclipses and conjunctions), such signals were events of vital significance for the world's work, and the discovery of Jupiter's moons brought a new battery of celestial signals to the aid of seafaring and scientific geography. More directly, the telescope was of value to the mariner as a "spy glass".*

A less obvious use is related to one of the pivotal inventions in the history of mankind. The age of the Great Navigators was a period of revolutionary and imperialist wars in which success depended on exploiting the new technique of artillery. The demands of marksmanship called for accurate devices for surveying and sighting distant objects. Galileo was not slow to recognize the

possibilities of the telescope for navigation. Indeed, he offered his invention consecutively to the Catholic Emperor and to the opposing Protestants in letters adapted to the convictions of either parties.

The design of better telescopes immediately created two needs: high magnification led inevitably to a more precise statement of the law of refraction by ***Kepler****,* ***Snell*** *(1618) and* ***Descartes*** *(1637). The need to eliminate the colored fringe which blurs the outline of the image obtained with simple lenses, led* ***Newton*** *to the study of the spectrum (1665).*

The invention of the telescope is the culmination of a chain of events that spread over a period of 2000 *years from* ***Euclid*** *to* ***Galileo*** *[Euclid composed a work on the geometrical principles of reflection and* ***Archimedes*** *is credited with constructing concave mirrors for use as burning-glasses].*

1611 CE KING JAMES VERSION OF THE PROTESTANT BIBLE: In 1604, King James I of England authorized a committee of 54 scholars to prepare a revision of earlier English translations of the Bible. The new version appeared in 1611 and became known as the *King James*, or *Authorized* Version. The beauty and grace of the translation established it as one of the great treasures of the English language and Western Culture in general. A revised version by the Church of England (1870) failed to compete with the King James Version.

In the Middle Ages the Bible was brought to the people indirectly through the miracle plays and directly through the translations supervised by **John Wyclif** (c. 1330–1384). In the sixteenth century came **William Tyndale** (c. 1494–1536), whose ambition was thus expressed to a well-known divine of his day: "If God spare me life, I will cause the boy that driveth the plow to know more of the Scriptures than you do." Tyndale suffered martyrdom for his work, but his translation of the New Testament enabled his successor, **Miles Coverdale** (ca 1488–1569), to complete it. By 1540 religious dissensions were somewhat quieted down, and this "Great Bible", as it was called, was established in all the churches.

The scholars of the King James Version made considerable use of Tyndale's vigorous phrases, and we owe more to Tyndale than to any other one man.

1611 CE **Marco Antonio de Dominis** (1560–1624, Italy). Natural philosopher, mathematician and theologian. First to put forward an explanation of the rainbow which, with all its faults, was superior to any other published theory over 300 years before him.

Dominis was born of a noble Venetian family in the Island of Arbe, off the coast of Dalmatia. For some time he was employed as professor of mathematics at Padua and professor of rhetoric and philosophy at Brescia. He rose to the rank of archbishop of Spalato (1600). In his endeavors to reform the Church he got involved in quarrel between the papacy and Venice. He crossed to England (1616) and converted to Anglicanism, becoming the dean of Windsor (1619). His attacks on the papacy (1617–1618) aggravated the Church and he was enticed back to Rome by the promise of pardon and rich preferment. But he was doomed to bitter disappointment: he was thrown into the Inquisition's prison and died there. Even this did not end his miseries. By order of the inquisition his body was taken from the coffin, dragged through the streets of Rome, and publicly burnt in the Campo di Fiore.

1614–1617 CE **John Napier** (1550–1617, Scotland). Mathematician, inventor of *logarithms* and the man who first *used* the decimal point in the arithmetic of decimal fractions. In the absence of any exponential notation or concept of bases (let alone any knowledge about '*e*') this self-taught man labored 20 years to develop a geometrical scheme that simulated natural logarithms. In 1624, **Henry Briggs** (1561–1637, England) published tables [as did **Johannes Kepler**] of logarithms to base 10. Briggs introduced the word 'mantissa', which is a late Latin term of Etruscan origin meaning an "addition" or "appendix". The Swiss **Jobst Bürgi** (1552–1632), using an algebraic approach, conceived and constructed a table of logarithms independently of Napier in 1620.

One of the anomalies in the history of mathematics is the fact that logarithms were discovered before exponents were in use (1637). Another fact which stands out in connection with this invention is the well known motto, that necessity is the mother of invention. Indeed, the rapid development of astronomy, trade, navigation, engineering and warfare made ever increasing demands on the speed and accuracy of computations. These demands were met successively by the *adoption* of three remarkable inventions: The Hindu-Arabic notation (ca 1500), decimal fractions (1592) and logarithms (1614).

The nations of antiquity experimented for thousands of years with numerical notations before they developed the so-called '*Arabic notation*'. In the simple expedient of the zero which was introduced by the Hindus, mathematics received one of its most powerful stimuli. One would suppose that once the 'Arabic notation' was thoroughly understood, decimal fractions would occur as an obvious extension of it. But simple as decimal fractions may appear to us, the invention of them is not the result of one mind or even of one age. They came into use by a slow and imperceptible process. The first mathematicians associated with their history did not perceive their true nature and importance, and failed to invent a suitable notation.

The idea of decimal fractions made its first appearance in methods for approximating the square roots of numbers, but the first systematic treatment of decimal fractions is due to **Simon Stevin**, who in his *La Disme* (1585) described the advantages of decimal fractions and decimal division in systems of weights and measures. Stevin applied the new fractions to all operations of ordinary arithmetic, but he lacked a suitable notation. In place of our decimal point, he used a zero.

It has not been agreed yet to whom the first introduction of the *decimal point* or comma should be ascribed. However, if a requirement is made that the point or comma was with the candidates not merely a general symbol to indicate separation, but that the symbol has actually been used in operations including multiplication or division of decimal fractions, then it would seem that the honor falls to **John Napier**, who exhibited such use in his *Rabdologiae* (1617). Napier's decimal point did not meet with immediate adoption. It was only in the first quarter of the 18^{th} century that the decimal point achieved a complete and final victory.

By the beginning of the 17^{th} century the victory of the Arabic system of numeration — for both calculation and recording — was complete in most of Europe. As a result the abacus went out of use in the countries west of Russia. It was a long time, however, before even the basic processes of calculation became either commonly understood or widely practiced[43]. The blockage was cleared by two inventions (one quite minor and the other of the very first importance) which effectively reduced all arithmetical calculations to addition and subtraction. Both were due to the same man — John Napier.

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[43] On 4 July 1662, Samuel Pepys, then in charge of the Contract Division of the Admiralty, wrote in his diary:

"*Up by five o'clock, and after my journal put in order to my office about my business... By and by comes Mr. Cooper, of whom I intend to learn mathematiques, and do begin with him today, he being a very able man. After an hour being with him at arithmetique* (*my first attempt being to learn the multiplication table*)*; then we parted till tomorrow*".

Pepys was one of the best educated men of his time. He was a senior Civil Servant, he had been to Cambridge, and later in life he became president of the Royal Society and a friend of such men as **Isaac Newton** and **Christopher Wren**. Yet the poor man had to struggle with multiplication tables at an early hour in the morning! (He probably could add and subtract reasonably well; it was *multiplication*, and still more *division*, of large numbers, that required the skill of a professional mathematician in his day.)

John Napier[44] was born at the family estate of Merchiston Castle near Edinburgh and was the 8^{th} Baron of Merchiston. His father was only 16 years of age when John was born. In 1563, the year his mother died, he matriculated at St. Salvator's College, St. Andrews. After that, his stay at the university was short, and only the groundwork of his education was laid there. To complete his education he studied at the University of Paris, and visited Italy and Germany. He returned home in 1571 and a year later married Elizabeth Stirling, who died in 1579, leaving him a son and a daughter. A few years afterwards he married again, having by his second wife five sons and five daughters.

During 1588–1614 Napier expended much of his energies in the political and religious controversies of his day. He was violently anti-Catholic and championed the causes of John Knox and James I. In 1593, he published a bitter and widely-read attack on the Church of Rome entitled *A Plaine Discovery of the whole Revelation of Saint John*, in which he endeavored to prove that the Pope was the Antichrist and that the Creator proposed to end the world in the years between 1688–1700. The book run through 21 editions, at least ten of them during the author's lifetime, and Napier sincerely believed that his reputation with posterity would rest upon this book. He also wrote prophetically of various infernal war engines and of "*devices of slaying under water*", accompanying his writings with plans and diagrams. Some of his war chariots are remarkably like a modern tank. It is no wonder that Napier's ingenuity and imagination led some to believe he was mentally unbalanced and other to regard him as a dealer in the black art.

As a relaxation from his political and religious polemics, Napier amused himself with the study of mathematics and science. In 1614 appeared the work which in the history of British science can be placed as second only to Newton's *Principia*: *Mirifici Logarithmorum*[45] *Cannonis Descriptio* ("A Description of the Marvelous Rule of Logarithms"), containing 57 pages of explanatory text and 90 pages of tables. It introduced logarithms and simplified the representation of decimal fractions.

The fundamental idea of relating an arithmetic and a geometric series is physically represented by Napier through the motion of two points on separate parallel straight lines, one point moving with uniform velocity and the other

[44] The family name was originally Lennox. When one of its members distinguished himself in battle the King of Scotland changed his name to Napier, to honor his valor, saying: "*You have Na-Peer*" (i.e. no equal).

[45] The compound of two Greek words: *Logos* (ratio) and *Arithmos* (number).

with accelerated velocity[46]. [The idea originated with him in 1594: John Craig, physician to James VI of Scotland, called on him and told him that on his visit to the astronomical observatory of **Tycho Brahe** in 1590, the latter showed him a marvelous mathematical device through which a product of two numbers is converted into a sum[47].]

The publication in 1614 of the system of logarithms was greeted with prompt recognition, and among the most enthusiastic admirers was Henry Briggs, the first Savilian professor of geometry at Oxford. In 1615, he left his studies in London to do homage to Napier at his home in Scotland. There they discussed possible modifications in the method of logarithms. They agreed that powers of ten should be used, that the logarithm of one should be zero and that the logarithm of ten should be one.

Previous to Napier's publication of his *Cannonis Descriptio* England had taken a minor part in the advance of science, and there is no British author of the time except Napier whose name can be placed in the same rank as those of **Copernicus**, **Tycho Brahe**, **Kepler**, **Galileo**, or **Stevinus**. Scotland had produced nothing, and was perhaps the last country in Europe from which a great mathematical discovery would have been expected.

Napier lived not only in a wild country, which was lawless and unsettled during most of his life, but also in a credulous and superstitious age. Like Kepler and all his contemporaries, he believed in astrology. Such was the state of society in the midst of which logarithms had their birth.

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[46] We can arrive at the definition of the Napierian logarithm with the aid of the Newtonian calculus (which was unknown to Napier): A point C moves on a segment $AB = a$ from A to B such that its velocity is always proportional to $x = CB$, i.e. $\frac{dx}{dt}|_C = -x$, $x(0) = a$. A second point F moves uniformly on a segment DE from D to E such that $y = DF$, $\frac{dy}{dt} = a$. The two points start at the same time $t = 0$. Since $y = at$ and $t = \log_e \frac{a}{x}$, we have the Napierian logarithm $= y = a \log_{\frac{1}{e}} \left(\frac{x}{a}\right)$. Napier chose $a = 10^7$ and called $y = DF$ the logarithm of $x = CB$. It is evident from this formula that Napier's logarithms are *not* the same as natural logarithms. The notion of a *base* never suggested itself to him because it is not applicable to his system.

[47] Such as $\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$.

Calculating Devices

Calculating with the aid of machines or devices began well before the use of electricity. The most important 'hardware' inventions in this category were:

ca 500 BCE *The bead-and-wire abacus, used for adding and subtracting large numbers. Invented probably in ancient Egypt.*

ca 1500 CE *The quadrant, an astronomical calculation tool, widely used in Europe.*

1502 CE *The first watch – an analogue time-keeper.*

Many of the applied fields in which numerical calculations are important, such as astronomy, navigation, trade, engineering, and war, made ever increasing demands that computations be performed more quickly and accurately. These increasing demands were met successfully by three remarkable 'software' inventions:

- *Hindu-Arabic notation (including 'zero')*
- *Decimal fractions (Stevin, 1585)*
- *Logarithms*

Logarithms were invented independently by ***Napier*** *(1594, 1614) and* ***Bürgi*** *(1600) and developed further by* ***Briggs*** *(1615, 1624). Their big advantage is the replacement of multiplication with simple addition, thus saving calculator's time by a large margin. It enabled European mathematics to break away from slow ancient calculating systems and procedures.*

To multiply a number a by another number b we write

$$a = \epsilon^x, \quad b = \epsilon^y$$

where ϵ is an arbitrary base (usually chosen as $\epsilon = 10$ or $\epsilon = e$). Then

$$ab = \epsilon^{x+y}.$$

The number x is the logarithm of a to base ϵ and the number y is the logarithm of b to base ϵ. We write

$$x = \log_\epsilon a; \qquad y = \log_\epsilon b$$

It then follows that

$$\log ab = x + y$$

and multiplication is reduced to addition.

One of the anomalies in the history of mathematics is the fact that logarithms were discovered before exponents were in use (1637).

The basic idea of logarithms was noted by ***Stifel*** *(1544). He observed that the terms of the geometric progression $\{1, r, r^2, r^3, \dots\}$ correspond to the terms in the arithmetic progression $\{0, 1, 2, 3, \dots\}$ formed by the exponents. Multiplication of two terms in the geometric progression yield a term whose exponent is the sum of the corresponding terms in the arithmetic progression. This observation had already been made earlier by* ***Chuquet*** *(1484). Stifel extended this connection between the two progressions to negative and fractional exponents. Thus the division of r^2 by r^3 yield r^{-1}, which corresponds to the term -1 in the extended arithmetic progression.*

Though the definition of logarithms as exponents of the powers that represent the numbers in a fixed base became the common approach, they were not defined as exponents in the early 17^{th} century because fractional and irrational exponents were not in use. By the end of the century a number of mathematicians recognized that logarithms could be so defined, but the first systematic exposition of this approach was made by ***Euler*** *(1728).*

During 1617–1674 mathematicians in England, France and Germany invented and developed mechanical devices and machines to speed up the execution of arithmetic processes; It all started with Napier himself who invented a mechanical numbering device called "Napier Bones". It was made of horn, bone, or ivory. This device evolved into the logarithmic slide-rule by ***Edmund Gunter*** *(1620) and* ***William Oughtred*** *(1622).*

Wilhelm Schickard *(1592–1685) of Tuebingen, Germany, made a 'calculating clock' (1623). This mechanical machine was capable of adding and subtracting up to 6 digit numbers, and warned of an overflow by ringing a bell. The machine and its plans were lost and forgotten in the war that was going on, at that time, then rediscovered in 1935, only to be lost in war again, and then finally re-rediscovered in 1956 by the same man! The machine was reconstructed in 1960, and found to be workable. (Schickard was a friend of the astronomer Kepler.)*

Pascal *(1642) built one of the first calculating machines that handled addition by carrying from units to tens, tens to hundreds, etc.* ***Samuel***

Moreland *(1625–1695, England) produced both adding and multiplication machines (1668–72).* **Leibniz** *saw the Pascal machine in Paris and then invented (1671–4) a machine which could carry out multiplication and division. It could multiply numbers up to 5 and 12 digits to give a 16 digit operand. The machine was later lost in an attic until 1879.*

Until 1940 machines of this kind were simply mechanical devices that performed only arithmetic and had no influence on the course of mathematics.

1615–1616 CE **Willem Corneliszoon Schouten** (1580–1625, The Netherlands). Navigator and explorer. First to traverse the *Drake Passage* (1615); discovered *Cape Horn* (1616).

1616–1629 CE **Joseph Solomon Delmedigo** (1591–1655), known as the *Yashar of Candia*. Mathematician, astronomer, philosopher, linguist, and physician. The greatest secular Jewish savant of the late Renaissance era, who exerted great influence on the thinking of **Baruch Spinoza**. He was a keen critic of medieval philosophy of nature and carried the ideas of the scientific revolution into the Near East and Central and Eastern Europe.

Delmedigo peregrinated from his native town of Candia, Crete, to Padua, Italy, where in 1606 he became a pupil and disciple of **Galileo Galilei**. Following his studies of mathematics, astronomy and philosophy, he took on medicine under **Hieronymus Fabricius** (1537–1619). After graduating in 1613, he spent most of his life in travels and short sojourns in Egypt, Turkey, Poland, Lithuania, Bohemia, Hamburg, Amsterdam, Frankfurt, Worms and Prague, where he died. He earned his living as a physician and served during 1620–1623 as court physician to Prince Radzivil of Poland.

He wrote some 50 books on mathematics, mechanics, astronomy, medicine and philosophy, but published only a few — since he had to be careful lest the ecclesiastical and secular authorities be offended by his ideas. He stressed the need for experiments in aviation and for the construction of aircraft to collect data for *weather prediction*(!).

1618–1648 CE *The Thirty Years War*; Germany, the most populous province of the Holy Roman Empire became a playground for the invading armies of Spain, Denmark, Sweden, and France. Seven to eight million people (about one third of the total population) were killed. At the time of the *Peace of Westphalia* (1648) the empire remained politically fragmented,

divided into 300 autonomous, sovereign states, most of them very small and weak. Historians have summed up their feelings about this conflict as follows:

"Morally subversive, economically destructive, socially degrading, confused in its causes, devious in its cause, futile in its results — an outstanding example in European history of meaningless conflicts".

1620 CE **Pierre Gassendi** (1592–1655, France). Mathematician, philosopher and scientist. A Catholic priest who taught mathematics at College Royal in Paris. He rejected the dogmatic teaching of Aristotelian science and proclaimed his adherence to the Epicurean belief in the atomistic structure of matter. Like Bacon he urged the importance of experimental research, and formulated correctly the law of inertia in 1636. Helped his friend **Mersenne** to measure the speed of sound in air. Although he added little to the stock of human knowledge, he holds an honorable place in the history of science.

1620–1624 CE **Cornelius Jacobszoon van Drebbel** (1572–1633, Netherlands). Engraver, alchemist, instrument-maker and an inventor far ahead of this time. Built the first navigable submarine that could carry a number of people. It cruised 5 m below the surface of the Thames in London, on several occasions.

Drebbel was born in Alkmaar, the son of a well-to-do farmer. He had no university education and as a young man apprenticed to the famous engraver and alchemist **Hendrik Goltzius** (1558–1617), who taught him some chemical ideas and processes. Drebbel then devoted himself to engraving but soon turned to mechanical inventions and instrument making. About 1604 he went to the court of King James I in England, who became his patron. In 1610 Drebbel visited the court of Emperor Rudolf II in Prague, at the Emperor's invitation.

He lingered a decade and instructed the son of Archduke Ferdinand of Bohemia who would later become Holy Roman Emperor. At the beginning of the Thirty Years' War, Ferdinand V's forces imprisoned Drebbel and took all his possessions, for he was affluent at this time. Through the intervention of Prince Henry, Drebbel was set free to return to England in 1613.

During the next several years he lived mostly in London. About 1620 he began to devote himself to the manufacture of microscopes and to the construction of a submarine. For the next several years he was employed by the British Navy, partly in connection with the submarine, but mostly to make explosive devices with which to attack other ships. During 1626 to 1628, he advised the military on how to relieve the French Huguenots under siege at La Rochelle. From 1629 until his death in 1633 he was extremely poor and earned his living by keeping an alehouse.

His most phenomenal work was definitely the submarine. In 1620, he made the first "rudimentary" submarine. Drebbel constructed his vessel while working for the British Navy. They never used it, but tested it many times. He had a wooden row boat; it had a wooden hull wrapped tightly in waterproofed leather. His row boat was the first to answer the question of air replenishment underwater. Air tubes with floats went to the surface to provide the craft with oxygen. Oars went through the hull at leather gaskets. Twelve oarsmen and some other passengers were on board. The trip at the Thames River took three hours. The secret of the craft was probably the production of oxygen from saltpeter by a process discovered by Drebbel already in 1608.

Drebbel also invented thermostats, a thermoscope and a microscope with two sets of convex lenses. He made compound microscopes as early as 1619. He also made telescopes and constructed a camera obscura with a lens in the aperture. A lunar crater is named after him.

1620–1644 CE **Johann Baptista van Helmont** (1579–1644, The Netherlands). Chemist and physician. The first to understand that there are gases other then atmospheric air, and one of the first to apply chemical principles to physiological processes. He was a forerunner of the iatrochemical school, and rendered an important service to the art of medicine by applying chemical methods to the preparation of drugs. He invented the word *gas*[48] to describe substances that are like air (1620). He even isolated several such gases, including oxides of carbon (CO_2, CO) nitrogen and sulfur (he studied gases released by burning charcoal and fermenting wine). Helmont maintained that gases were substances differing fundamentally one from the other, and from air and condensible vapors.

Helmont was born at Brussels, a member of a noble family. He was educated at Louvain, and after ranging restlessly from one science to another and finding satisfaction in none, turned to medicine, taking his doctor's degree in 1599. The next few years he spent in traveling through Switzerland, Italy, France and England. He settled in 1609 at Vilvorde, where he occupied himself with chemical experiments and medical practice until his death.

Helmont presents curious contradictions, characteristic of chemists of his age: On the one hand he was a disciple of **Paracelsus** (a mystic with strong leanings to the supernatural, an alchemist who believed that with a small piece of the philosopher's stone he could transmute 2000 times as much mercury into gold); on the other hand he was touched with the new learning that was

[48] This he derived from the Greek *chaos*, meaning space. In this way, the word described the ability of a gas to fill any amount of space. Before his time, and indeed for some time after, gases were thought to be different forms of the *element* of air, or air mingled with some impurities.

producing men like Harvey, Galileo and Bacon — a careful observer of nature and an exact experimenter, who in some cases realized that matter can neither be created nor destroyed.

The Rise of the New World, I

The Pilgrims[49] *(1620)*

In 1002, *the Viking Leif Ericsson led an expedition across the north Atlantic to the shores of North America, where colonies persisted for many years.*

Intrepid mariners from half a dozen European nations had explored America's coastal waters before England planted its first colonies on the eastern shore. There were earlier visits by Portuguese, Italian and Spanish seamen: armored Spanish Conquistadors filed north from Mexico to explore the southwest in the mid 1500*'s, and tonsured Franciscan friars from Spain established missions in Florida, Georgia and California late in the* 16^{th} *century.*

But it was England that founded the first permanent *colonies in the early* 1600*'s, and the British, with their* staying power*, who outlasted all their colonial rivals and built the thriving North American empire that eventually became the United States.*

The motives that brought millions of Europeans to America were mixed, but most of the immigrants hoped to find wealth and a new start in life, or religious and political sanctuary.

On April 26, 1607, *three shiploads of* 140 *English adventurers, lead by* ***John Smith*** *(1580–1631), anchored in the James River near the mouth of Chesapeake Bay. In the words of Sir Walter Rayleigh, they came "to seek new worlds for gold, for praise, for glory". They found far more hardship than gold or glory, and many of them died of disease and malnutrition, but they did establish Jamestown, the first permanent English settlement in the New World. They began growing Tobacco (1612), an Indian staple, which*

[49] To dig deeper, see:

- Johnson, Paul, *A History of the American People*, Harper Collins, 1997, 1088 pp.

came into vogue in Europe and became the economic mainstay of Virginia and much of the Colonial South.

In 1620 *the small ship '**Mayflower**' sailed from Plymouth in England with* 102 *passengers bound for religious freedom in the New World. Most of them were Puritans who had run afoul of the religious laws of Britain. Some had been in exile in the Netherlands.*

The expedition reached Cape Cod Bay in Massachusetts after a 65*-day voyage, and finally landed on part of the rocky shore which had been given the name Plymouth a few years earlier. These early settlers, forerunners of the colonists who were to form the independent United States* 150 *years later, are generally known as the* pilgrim fathers, *a term first used in* 1799. *They settled the first permanent colony of Europeans in New England.*

As Jamestown, Plymouth, and other British settlements that soon lined the Atlantic coast, grew and prospered, their restless inhabitants gradually worked their way inland to establish new outposts. England began almost two centuries of struggle with her Colonial rivals.

1618 CE **Willebrod van Roijen Snell** (or **Snel**) (1591–1626, Netherlands). Dutch astronomer and mathematician. Rediscovered the law of refraction of light[50]. In 1637 **Descartes** derived this law from more basic principles. In 1657 **Fermat** derived Snell's law from his principle of least time.

When a ray of light passes from one homogeneous medium into another it undergoes a change of direction, and is said to be *refracted*. The acute angles made by the two parts of the ray with the normal to the surface of separation of the two media at the point of incidence are called the angles of incidence and refraction. The complete relation between the two directions is given by the following laws:

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[50] The design of better telescopes created a need for a precise statement of the law of refraction. This explains why the best physicists and mathematicians of the 17^{th} century were preoccupied with this problem. Indeed, **Kepler** (1611), **Snell** (1618), Descartes (1638), **Fermat** (1657), **Newton** (1665) and **Huygens** (1678), contributed to the physics of light propagation through inhomogeneous media.

- The two parts of the ray are in the same plane with the normal and on opposite sides of it.
- The ratio of the sine of the angle of incidence i to the sine of the angle of refraction r is a constant, K, depending on the media, and on the nature of light.

Thus, Snell found empirically (1618) that $\sin i = K \sin r$, where K is a fixed number for any pair of media. That the ratio varies with the nature of light was proved by Newton. The ratio K is known as the *relative index of refraction* of the two media[51]. Clearly, for a ray going in the reverse direction $n_{ir} = \frac{1}{n_{ri}}$.

The first attempts to deduce a law of refraction go back to **Claudius Ptolemy** of Alexandria (150 BCE). His measurements of the angles of refraction of water and glass have come down to us and probably represent the most ancient physical experiments recorded historically.

Clearly, the relation $n_{ri} = \frac{v_i}{v_r}$ could be verified experimentally only in the 19th century, when suitable techniques for measuring velocities of light in material media were developed. In contradiction with Snell law $v_r \sin i = v_i \sin r$, particle kinematics yields, via the law of conservation of linear momentum $v_i \sin i = v_r \sin r$, where i is the angle of incidence and r is the angle of refraction. Thus, in the hands of Huygens, Snell's law provided evidence for the wave theory of light.

In 1617 Snell developed a method of determining distances by trigonometric triangulation. He also devised an efficient method for the evaluation of π to 35 decimal places.

Snell was born in Leyden. In 1613 he succeeded his father **Rudolph Snell** (1546–1613) as a professor of mathematics at the University of Leyden. It is not known just how Snell discovered the law of refraction. When the author died he left his manuscript unedited. This manuscript, which may have been available to Descartes, apparently was last seen by Huygens, and it now appears to be lost.

1622–1632 CE **William Oughtred** (1575–1660, England). Mathematician. Invented the rectilinear slide rule in 1622 soon after the invention

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[51] If we write $K = n_{ri}$, it is shown that

$$\frac{\sin i}{\sin r} = n_{ri} = \frac{v_i}{v_r} = \frac{n_r}{n_i}, \qquad \text{or} \qquad n_i \sin i = n_r \sin r,$$

where v_i is the velocity of light in the medium of incidence and v_r is the velocity of light in the second medium. Here n_r and n_i are known as the *absolute indices of refraction* of each of the media (i.e. relative to vacuum).

of logarithms (1614). Introduced the multiplication sign (×) into algebra in 1631. In 1632, he invented the circular slide rule.

Oughtred was born at Eton, and educated there and at King's College, Cambridge, of which he became a fellow. He left the University about 1603 to become a priest, and at about 1628 he was appointed by the earl of Arundel to instruct his son in mathematics. He corresponded with some of the most eminent scholars of his time on mathematical subjects. In 1631 he wrote a short compact treatise on arithmetic and algebra, *Calvis mathematicae*, in which he employed new mathematical symbols.

1623 CE **Wilhelm Schickard** (1592–1635, Germany). Scholar and inventor. Built a practical calculating machine which was used by **Kepler**. He invented many machines like one to calculate astronomical dates and one for the Hebrew grammar.

Schickard was professor for biblical languages at Tübingen University. His research was broad and included astronomy, mathematics and surveying. He died of the plague.

1625–1640 CE **Hugo Grotius** (**Huig de Groot**, 1583–1645; The Netherlands). Political philosopher, humanist and statesman. Considered a founder of international law. He wrote '*The Law of War and Peace*' (*De Jure Belli ac Pacis*, 1625), which had influenced Spinoza's political philosophy.

Grotius recognized that the corruption and decline of Papal jurisdiction, and the birth of the modern state, together give rise to an urgent need to a form of legality that would transcend the writ of any particular sovereign: all law must stem either from free association of people, or from the higher law – the law of nature – which applies to all men, and all nations, in every circumstance of life. The law of nature is eternal and immutable; to discern it we have but to employ our reason, which leads us to the perception of right and wrong, as it leads us to the truths of logic and mathematics.

Grotius was by no means an isolated thinker; the ideas which he expressed were current in The Netherlands and were to elicit an equal interest in England.

Grotius was born in Delft and graduated from the University of Leiden at 15. He became chief magistrate of Roterdam (1613). Condemned to life imprisonment (1619) for opposing strict Calvinism, but with the aid of his wife, escaped prison in a trunk of books (1621); Lived in Paris (1621–1631); Swedish ambassador in Paris (1634–1644).

1625–1642 CE **Zacutus Lusitanus** (Avraham Zacuto II, 1576–1642). Physician and medical writer in Amsterdam. One of the most celebrated

physicians in Europe during the first half of the 17^{th} century. Studied medicine and philosophy at the Universities of Salamanca (Spain) and Coimbra (Portugal). Fled the Inquisition (1625) to Amsterdam, where he returned openly to his Jewish faith. Published a 12 volumes encyclopedia on the history of medicine: *De medicorum principium historia* (1642).

1626 CE **Peter Minuit** (1580–1638). Paid the equivalent of 24 dollars for the Manhattan Island, currently worth more than 100 billion dollars.

1628–1651 CE **William Harvey** (1578–1657, England). Physician. Discovered how blood circulates in the human body (1628), and established the foundations of modern embryology (1651).

Harvey's book *An Anatomical Treatise on the Motion of the Heart and Blood in Animals*, is considered the most important single volume in the history of physiology. In it Harvey showed that the heart, by repeated contractions, produces a continuous stream of blood throughout the body which continually returns to its source. It is amazing how such a fundamental fact escaped all the savants of antiquity[52] and had to await discovery until the 17^{th} century. Even so, Harvey's theory was severely attacked by followers of Galen[53] in spite of the fact that he based his ideas on firsthand observation

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[52] The Greek physician **Erasistratos** came very close to recognizing the circulation of the blood (ca 280 BCE). A Cairo physician, **Ibn al-Nafis** (1210–1288), who came from Damascus, pointed out that the dividing wall of the heart, the *septum*, was solid, and quite devoid of pores permitting the passage of blood, which Galen has postulated. Thus, he argued, the blood must flow from the right to the left ventricle of the heart through the lungs. In this way Ibn al-Nafis arrived at the theory of the lesser circulation of blood. His discovery, however, did not pass into the mainstream of science as his work did not come to light until 1924.

[53] The first idea of this discovery occurred to him not later than 1616 but he did not publish it until 1628 in a little book dealing with the motion of the heart and blood. One is rather surprised to find that this book did not make more stir; neither did it arouse much opposition, at least in England. In France the opposition to the new theory was considerable, but even there, and bitter as it was, it did not last long. More happy in this than many other forerunners, Harvey was granted a taste of victory before his death in 1657. By 1673 his cause was definitely won, even in France, and the people who had been his contemporaries could witness the complete supremacy of the new doctrine.

Until the time of Harvey, the prevalent conception was that promulgated by Galen. According to him, the blood was produced in the liver from the materials furnished by our food and was then transported to the right half of the heart. Some of it passed into the left half, where it was imbued with new properties, and became fit to nourish the whole body. To use Galenic language, the blood of the

and experiment. Harvey lived to see his discovery widely accepted, although full credit only came after his death.

Harvey was born at Folkestone. He studied medicine at Padua under **Fabricius** and became a doctor of medicine in 1602. He returned to London and practiced medicine. Harvey became a member of the Royal College of Physicians in 1607, and later served as physician to James I and Charles I.

In one point only was his demonstration of the circulation incomplete: Harvey could not discover the capillary channels by which the blood passes from the arteries to the veins. This gap in the circulation was bridged in 1661 by the physiologist **Marcello Malpighi** (1628–1694, Italy), who saw in the lungs of a frog, by the newly invented *microscope*, how the blood passes from the one set of vessels to the other. Harvey saw all that could be seen by the unaided eye in his observations on living animals.

Harvey speculated that humans and others mammals must *reproduce through the joining of an egg and sperm.* No other theory made sense It was 200 years before a mammalian egg was finally observed, but Harvey's theory was so compelling and so well thought out that the world assumed he was right long before the discovery was finally made.

Harvey remained a physician at St. Bartholomew's until 1643. He maintained his college lectureship until 1656, the year before his death, missing by a moment the dismantling under Cromwell of the monarchy that had supported his research throughout his life.

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right heart was endowed with "natural spirits", that of the left heart with "vital spirits". The latter blood was thus essentially different from the former. They did not circulate in the body, but both moved in a ceaseless ebb and flow, each in its own domain. But how did the blood pass from the right to the left ventricle? To explain the impossible, Galen had been obliged to assume that it passed through innumerable *invisible* pores in the solid wall which divides the right heart from the left. Nobody ever detected these pores for they are not simply invisible but nonexistent. Yet Galen, supreme pontiff of Greek medicine, and nine centuries later Avicenna, the infallible medical pope of the Middle Ages, had spoken *ex cathedra* with such indisputable authority that this gratuitous assumption was generally taken for gospel.

Even a man like Leonardo da Vinci, endowed with so much genius and originality, and who had himself dissected a large number of bodies and examined very minutely many a heart, even he was subjugated by this intangible dogma. This is the more pathetic in that Leonardo was certainly on the scent of the true explanation, but the invisible holes were too sacred to be touched, and nothing but this prejudice caused his failure to discover and to proclaim the circulation of the blood.

The Cult of the Virtuoso

Throughout the 17th century, the majority of university mathematicians continued in the restricted tradition of scholasticism, and the main impetus for mathematical advance came from the Renaissance humanist reaction against the universities.

The most fruitful and original research was carried out by gifted amateurs, who were sometimes called virtuosi*, as being endowed with a special, individual genius. This tendency to single people out as intellectual heroes fostered a spirit of* competitive individualism*, rather than of co-operative research — an attitude which probably encouraged the development of new ideas, but which tended to recede as mathematics became more and more technical.*

The competitive spirit gave rise to considerable jealousies as to priority over discovery of new theorems and methods. One manifestation of this was the custom of setting challenge-problems. Often the challenger had already solved the problem himself, and wanted to publicize his individual achievement. The emphasis on inventive genius encouraged greater interest in ideas themselves rather than in their detailed elaboration.

With the advent of navigation maps and the Renaissance of algebra, the time was ripe for the algebraization of geometry*. It began with the concept of* coordinate system *in the framework of 'analytic geometry'. It was invented, nearly simultaneously and independently, by Fermat and Descartes.*

1629–1654 CE **Pierre de Fermat**[54] (1601–1665, France). One of the greatest mathematicians of all times. Accomplished Toulouse Jurist and a universalist, who cultivated poetry, Greek philosophy, law and philology, and devoted to mathematics only the leisure of a laborious life. His father was a prosperous leather merchant and his mother came from a family of high social standing. He obtained his law degree from the University of Orleans in 1631,

[54] For further reading, see:

- Mahoney, M.S., *The Mathematical Career of Pierre de Fermat*, Princeton University Press: Princeton, NJ, 1973.

and in that year was appointed to a position in the high court of Toulouse and became entitled to include the honorific ‘de’ in his name.

He began his serious mathematical studies in 1629 when he discovered independently, and ahead of **Descartes**, ‘analytic geometry’.[55] This included: general equations of the straight line, the circle (centered at the origin), the ellipse, the parabola and the rectangular hyperbola. Fermat’s analytic geometry appears to be as general as that of Descartes, but is more complete and systematic, and corresponds much more closely to modern day analytic geometry.

In 1638 he communicated to Descartes his method of drawing tangents to plane curves. Fermat made numerous contributions to the development of differential and integral calculus: in particular he introduced the notion of “difference quotient” which he used to define the *derivative*, and used it in the study of problems of minima and maxima. The French, including **Lagrange**, claim Fermat as the true originator of the calculus.

Along with **Pascal** he is regarded as the founder of the theory of probability (1654). In physics, Fermat discovered in 1657 the ‘*principle of least time*’, valid for the propagation of light in material media. It is also known as the principle of shortest optical path. [The optical path is determined by the integral $\int_{r_2}^{r_1} n\,ds$, where n is the refraction index, which may change from point to point.]

However, the greatness of Fermat rests mainly in his contribution to number theory, and for that he is known as the “father of modern number theory”. Some of his discoveries are:

(1) *Fermat’s little theorem* (1640) [if p is a prime number and if a is an integer, then $a^p \equiv a(\text{mod } p)$. In particular, if p does not divide a then $a^{p-1} \equiv 1(\text{mod } p)$. This was known to the Chinese for $a = 2$.]

(2) Fermat’s method of factorization.

(3) Fermat’s method of infinite descent.

(4) Structure of perfect numbers.

(5) Every prime of the form $4m + 1$ is the sum of two squares in a unique way.

(6) Every positive integer is expressible as a sum of 4 squares of integers.

[55] **Marino Ghetaldi** [1566–1626, Dalmatia (now Croatia)] made early applications of algebra to geometry (1603).

(7) *Fermat's conjecture* ('Last Theorem'[56]): The equation $a^n + b^n = c^n$ has no solution in positive integers if $n > 2$ (1637)[57].

A general proof[58] has been attempted by Euler, Legendre, Gauss, Abel, Dirichlet, Cauchy, Kummer and many others over the past four centuries. **Fermat** himself proved it for $n = 3$ and 4 (1659), **Euler** for $n = 3$ and 4 (independently, 1738), **Legendre** and **Dirichlet** for $n = 5$ (1828–1830), **Lamé** for $n = 7$ (1839) and **Kummer** for $n < 100$ except for $n = 37, 59, 67$ (1859). By 1978, the conjecture was known to be true for all integer exponents up to 150,000 and by 1993 for all exponents less or equal to 4,000,000. A large part of *algebraic number theory* originated through attempts to prove Fermat's conjecture. Thus, in spite of the great frustration that this problem caused 15 generations of mathematicians, it turned out to be a blessing in disguise.

David Hilbert (1862–1943), when asked once why he did not attempt to prove Fermat's conjecture, replied: "*Why should I kill the goose that lays the golden egg*?"

Fermat firmly believed that $f(n) = 2^{2^n} + 1$ would yield primes for all values of n, but he was very wrong. Only 5 primes have been discovered which conform to this formula: $f(0) = 3$, $f(1) = 5$, $f(2) = 17$, $f(3) = 257$ and $f(4) = 65,537$, but already $f(5) = 4,294,967,297 = 641 \times 6,700,417$. The compositeness of some *Fermat numbers* has been established, but no further primes have been discovered among them.

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[56] To dig deeper, see:

- Stewart, I. and D. Tall, *Algebraic Number Theory and Fermat's Last Theorem*, A.K. Peters, 2002, 313 pp.
- Van der Poorten, A., *Fermat's Last Theorem*, Wiley, 1996, 222 pp.

[57] For $n = 2$, the solution of the Diophantine equation $x^2 + y^2 = z^2$ proceeds through the factorization $(x + y\sqrt{-1})(x - y\sqrt{-1}) = z^2$. Putting $(x + y\sqrt{-1}) = (u + v\sqrt{-1})^2 \equiv (u^2 - v^2) + 2uv\sqrt{-1}$, we find $x = u^2 - v^2$, $y = 2uv$, $z = u^2 + v^2$, which indeed yields the Pythagorean triplets.

[58] In 1909 **A. Wieferich** proved that $a^n + b^n = c^n$ is impossible for n an odd prime not dividing abc with n^2 not dividing $2^{n-1} - 1$ (the second condition holds for all $n < 3 \times 10^9$ except 1093 and 3511). In 1922 **L. Mordell** showed that the Fermat conjecture holds with finitely many exceptions for any $n \geq 3$ provided the *Mordell conjecture* is true. In 1983 **G. Faltings** proved the *Mordell conjecture*. In 1987 **D.R. Heath-Brown** proved the impossibility of $a^n + b^n = c^n$ for "*almost all*" n. Finally, **Andrew John Wiles** (b. 1953, England) proved Fermat's Last Theorem in *Modular elliptic curves and Fermat's Last Theorem* which appeared in the *Annals of Mathematics* in 1995.

To sum up: Fermat and Pascal share the invention (1654) of the mathematical theory of probability, Fermat alone founded the theory of numbers, Fermat and Descartes share the invention of analytic geometry and Fermat is a harbinger of the differential and variational calculus.

The influence of most of his works upon his contemporaries seems to have been slight. The impact of his discoveries in number theory were just about non-existent. It might have been greater had he agreed to publish his findings, but he shunned this aspect of communication. He began to be appreciated only after his death. His influence on later generations led to the Renaissance of modern number theory.[59]

Fermat and the Theory of Numbers

I. Fermat Numbers and their Associates

In search of an algebraic expression that would yield primes only, Fermat conjectured (1640) that $F_n = 2^{2^n} + 1$ *is prime for all values of* n*. This is true for* $n = 0, 1, 2, 3, 4$*, yielding the series of primes 3, 5, 17, 257, 65537 respectively. But in 1732* ***Euler***[60] *showed that already* F_5 *in composite, and*

[59] For further reading, see:

- Mahoney, M.S., *The Mathematical Career of Pierre de Fermat*, Princeton, 1994.
- Singh, S.L., *Fermat's last Theorem*, London, 1997.
- Bell, E.T., *Men of Mathematics*, Simon and Schuster: New York, 1937, 592 pp.

[60] It is suspected that **Fermat** was led to his conjecture that all numbers $F_n = 2^{2^n}+1$, $(n = 0, 1, 2, 3, ...)$ are primes by the *Chinese theorem*, since he could prove that F_n divides $2^{F_n} - 2$, by induction. During Fermat's time it was thought that the Chinese theorem is true, for it was not known then that it breaks down for $n = 341$.

We must not rush to condemn Fermat for his blunder. Since F_5 has 10 digits, in

during the 276 years that followed, no one was able to find even one additional prime number in the series beyond F_4. It is perhaps more probable that the number of primes F_n is finite.[61]

All numbers of the form $2^{2^n}+1$, whether prime or composite, are called Fermat numbers. They obey the simple recursion $F_{n+1}-2 = F_n(F_n-2)$ which leads to the interesting product

$$F_n - 2 = F_0 F_1 F_2 \cdots F_{n-1}.$$

In other words, $F_n - 2$ is divisible by all lower Fermat numbers:

$$F_{n-k}|(F_n-2), \quad 1 < k \leq n.$$

On March 30, 1796, the Fermat numbers, until then largely a numerical curiosity, were raised from dormancy and took on a new beauty, linking number theory with a classical problem of Greek geometry.

*On that day, the young **Gauss** showed that a circle can be divided into n equal parts using ruler and compass alone, if n was a Fermat number. In other words: if F_n is prime, then a regular polygon of n sides can be inscribed in a circle by Euclidean methods. The Greek themselves knew how to construct regular n-sided polygon for n = 3, 4, 5, 6, 8, 10, 12, 15, 16 but progress in this problem had eluded mathematician ever since.*[62]

The most important properties of the Fermat numbers are:

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order to test its primality, it would be necessary to have tables of primes up to 100,000, which was unavailable to him. He could, of course, derive and use some criterion for a number to be a factor of a Fermat number, but this he failed to do. Euler, on the other hand, *knew* that $5\cdot 2^7+1 = 641$ was a *potential* factor of F_5 and he could do the necessary calculations *in his head* (!) without the need of table or calculators.

61 **Hardy** (1938) suggested, by considerations of probability, that since the corresponding number of primes from 1 through x $\pi(x) \sim \frac{x}{\ln x}$, the probability, that a number is prime is $\frac{1}{\ln n}$. Therefore, the total a priori expectation of Fermat primes is at most

$$\sum\left\{\frac{1}{\ln(2^{2^n}+1)}\right\} < \frac{2}{\ln 2}\sum 2^{-n} < \frac{2}{\ln 2}.$$

This argument assumes that there are no special reasons why a Fermat number should be likely to be a prime. But the fact that no two Fermat numbers have a common divisor greater than 1 and the fact that 2^n+1 is composite if n is not a power of 2, suggest that such special reasons may exist.

62 It is easy to construct a regular 85-gon, using constructions for the 5-gon and 17-gon, and since angles can be bisected, one can construct regular 170-gons, 340-gon and more generally regular polygons for which the number of sides is

- *No two Fermat numbers have a common divisor greater than 1*

  For suppose that F_n and F_{n+k}, where k >0, are two Fermat numbers, and that

  $$m|F_n, \quad m|F_{n+k}.$$

  If $x = 2^{2^n}$, we have

  $$\frac{F_{n+k}-2}{F_n} = \frac{2^{2^{n+k}}-1}{2^{2^n}+1} = \frac{x^{2^k}-1}{x+1} = x^{2^k-1} - x^{2^k-2} + ... - 1,$$

  and so $F_n|F_{n+k}-2$. Hence

  $$m|F_{n+k}, \quad m|(F_{n+k}-2);$$

  and therefore $m|2$. Since F_n is odd, $m = 1$, which proves the theorem.

- ***Fermat** (1640) showed that for $2^n + 1$ to be prime, n must be a power of 2, i.e. $n = 2^m$. Equivalently n has no odd factors, for if n has an odd factor t, then $2^n + 1$ has $(2^{n/t} + 1)$ as a factor. Therefore $2^n + 1$ is composite, if n is not a power of 2. The inverse statement is false, as we know that F_m is composite for many values of m >4. In general, for any $a^n + 1$ to be prime, a must be even and $n = 2^m$.*

- ***Euler** (1739) showed that for $n \geq 2$ a prime divisor of F_n is necessarily of the form $p = k2^{n+2}+1$. For assume $p|2^{2^n}+1$. Then $2^{2^n} = -1(mod\ p)$, and upon squaring each side, $2^{2^{n+1}} = 1(mod\ p)$. On the other hand, by Fermat's Little Theorem we know that $2^{p-1} = 1(mod\ p)$. The two relations are compatible iff $p - 1 = r2^{n+1}$ for some r. Further investigation shows that r must be even and so $p = k2^{n+2} + 1$. Indeed, Euler found that $p = 5 \times 2^7 + 1 = 641$ divides F_5. Note that for n = 2, 3, 4*

$n = 2^k \times F_l \times F_m \times ...$, where k = 0,1,2,... and the Fermat numbers are distinct primes.

Let us take $k = 0$ l, m = 0, 1, 2, 3, 4. Then, these polygons with an odd number of sides are built from the first five Fermat numbers $2^1 + 1$, $2^2 + 1$, $2^4 + 1$, $2^8 + 1$, $2^{16} + 1$. If we multiply $1 = 2^1 - 1$ into the cumulative products we obtain $2^2 - 1$, $2^4 - 1$, $2^8 - 1$, $2^{16} - 1$, $2^{32} - 1$, the latter being the product of the first five Fermat numbers

$$2^{32} - 1 = 3 \times 5 \times 17 \times 257 \times 65357 = 4\ 294\ 967\ 295$$

It's quite probable that there are no more such odd polygons, because it seems likely that

$$3, 5, 17, 257 \text{ and } 65537$$

are the only prime Fermat numbers.

the values of p must be identical with the Fermat number themselves, implying k = 1, 2^3, 2^{10} respectively.

Thus every divisor of F_n occurs in the arithmetic progression

$$1, \quad 2^{n+2}+1, \quad 2\cdot 2^{n+2}+1, \quad 3\cdot 2^{n+2}+1, ...$$

For given n, then, we can work out terms of this progression and check to see if any is a divisor of F_n. For n = 5 we obtain the sequence 1, 129, 257, 385, 513, 641, 769, ... A great time-saver is provided by the observation that, for any number, the least divisor greater than 1 must be a prime number.

Consequently, in the investigation of F_5, we need not even bother with the composite 129. Since 257 is prime it needs to be tried, but it does not divide. Again, 385 and 513 are composite, so they can be passed over. This brings us to the prime 641, which actually divides F_5.

This procedure is based upon the work of Edward Lucas , who published it in 1877. However, Euler knew almost as much a century and a half earlier. In 1739 one of his publications contained the result that every prime divisor of F_n is of the form $2^{n+1}k+1$. (Lucas' improvement amounts only to showing that k must be even.) Presumably he knew this in 1732 and used it to find the divisor 641. For F_5 we have $k\cdot 2^{n+1} = 2^6k+1 = 64k+1$, and for k = 10 we obtain the factor 641.

Hence with the help of only two divisions we can ascertain that 641 is the smallest prime divisor of the number F_5.

- *F_n divides $2^{F_n}-2$; this is demonstrated in two steps: first it is shown by induction that, for positive integers, $2^n \geq n+1$. This implies that 2^{n+1} divides 2^{2^n}, i.e. for some k, we have $2^{2^n} = k\cdot 2^{n+1}$. Consequently*

$$\begin{aligned}
2^{F_n}-2 &= 2^{2^{2^n}+1}-2 = 2[2^{2^{2^n}}-1] = 2[2^{(2^{n+1}k)}-1] \\
&= 2[(2^{(2^{n+1})})^k - 1^k] = 2[(2^{(2^{n+1})}-1)(\cdots)] \\
&= 2[((2^{2^n})^2-1^2)(\cdots)] = 2[(2^{(2^n)}+1)(2^{(2^n)}-1)(\cdots)] \\
&= 2[(F_n)(2^{(2^n)}-1)(\cdots)].
\end{aligned}$$

It is suspected that this relation led Fermat to his conjecture that all numbers F_n (n = 1, 2, ...) are primes. During Fermat's times it was thought that the so-called Chinese theorem is true, namely the theorem asserting that if an integer $m > 1$ satisfies the relation $m|2^m-2$, then m is a prime (it was checked for first several hundred integers). This breaks down, however, for $m = 341 = 11\cdot 31$, which was not then known.

Table 3.1: PRIME FACTORS OF $2^n + 1$, $n \le 128$

	n			n	
				30	*5 · 5 · 13 · 41 · 61 · 1321*
F_0	***1***	***3***		31	*3 · 715827883*
F_1	***2***	***5***	F_5	32	*641 · 6700417*
	3	*3 · 3*		33	*3 · 3 · 67 · 683 · 20857*
F_2	***4***	***17***		34	*5 · 137 · 953 · 26317*
	5	*3 · 11*		35	*3 · 11 · 43 · 281 · 86171*
	6	*5 · 13*		36	*17 · 241 · 433 · 38737*
	7	*3 · 43*		37	*3 · 1777 · 25781083*
F_3	***8***	***257***		38	*5 · 229 · 457 · 525313*
	9	*3 · 3 · 3 · 19*		39	*3 · 3 · 2731 · 22366891*
	10	*5 · 5 · 41*		40	*257 · 4278255361*
	11	*3 · 683*		41	*3 · 83 · 8831418697*
	12	*17 · 241*		42	*5 · 13 · 29 · 113 · 1429 ·14449*
	13	*3 · 2731*		43	*3 · 2932031007403*
	14	*5 · 29 · 113*		44	*17 · 353 · 2931542417*
	15	*3 · 3 · 11 · 331*		45	*3 · 3 · 3 · 11 · 19 · 331 ·18837001*
F_4	***16***	***65537***		46	*5 · 277 · 1013 · 1657 · 30269*
	17	*3 · 43691*		47	*3 · 283 · 165768537521*
	18	*5 · 13 · 37 · 109*		48	*193 · 65537 · 22253377*
	19	*3 · 174763*		49	*3 · 43 · 4363953127297*
	20	*17 · 61681*		50	*5 · 5 · 5 · 41 · 101 · 8101 ·268501*
	21	*3 · 3 · 43 · 5419*		51	*3 · 3 · 307 · 2857 · 6529 · 43691*
	22	*5 · 397 · 2113*		52	*17 · 858001 · 308761441*
	23	*3 ·2796203*		53	*3 · 107 · 28059810762433*
	24	*97 · 257 · 673*		54	*5 · 13 · 37 · 109 · 246241 · 279073*
	25	*3 · 11 · 251·4051*		55	*3 · 11 · 11 · 683 · 2971 · 48912491*
	26	*5 · 53 · 157 · 1613*		56	*257 · 5153 ·54410972897*
	27	*3 · 3 · 3 · 3 · 19 · 87211*		57	*3 · 3 · 571 · 174763 ·160465489*
	28	*17 · 15790321*		58	*5 · 107367629 · 536903681*
	29	*3 · 59 · 3033169*		59	*3 · 2833 · 37171 · 1824726041*

Table 3.1: (Cont.)

	n		n	
	60	*17 · 241 · 61681 · 4562284561*	80	*65537 · 414721 · 44479210368001*
	61	*3 · 768614336404564651*	81	*3 · 3 · 3 · 3 · 3 · 19 · 163 · 87211 · 135433 · 272010961*
	62	*5 · 5581 · 8681 · 49477 · 384773*	82	*5 · 10169 · 181549 · 12112549 · 43249589*
	63	*3 · 3 · 3 · 19 · 43 · 5419 · 77158673929*	83	*3 · 499 · 1163 · 2657 · 155377 · 13455809771*
F_6	64	*274177 · 67280421310721*	84	*17 · 241 · 3361 · 15790321 · 88959882481*
	65	*3 · 11 · 131 · 2731 · 409891 · 7623851*	85	*3 · 11 · 43691 · 26831423036065352611*
	66	*5 · 13 · 397 · 2113 · 312709 · 4327489*	86	*5 · 173 · 101653 · 500177 · 1759217765581*
	67	*3 · 7327657 ·6713103182899*	87	*3 · 3 · 59 · 3033169 · 96076791871613611*
	68	*17 · 17 · 354689 ·2879347902817*	88	*257 · 229153 · 119782433 · 43872038849*
	69	*3 · 3 · 139 · 2796203 · 168749965921*	89	*3 · 179 · 62020897 · 18584774046020617*
	70	*5 · 5 · 29 · 41 · 113 · 7416361 · 47392381*	90	*5 · 5 · 13 · 37 · 41 · 61 · 109 · 181 · 1321 · 54001 · 29247661*
	71	*3 · 5640964 3 · 13952598148481*	91	*3 · 43 · 2731 · 224771 · 1210483 · 25829691707*
	72	*97 · 257 · 577 · 673 · 487824887233*	92	*17 · 291280009243618888211558641*
	73	*3 · 1753 · 1795918038741070627*	93	*3 · 3 · 529510939 · 715827883 · 2903110321*
	74	*5 · 149 · 593 · 184481113 · 231769777*	94	*5 · 3761 · 7484047069 · 140737471578113*
	75	*3 · 3 · 11 · 251 · 331 · 4051 · 1133836730401*	95	*3 · 11 · 2281 · 174763 · 3011347479614249131*
	76	*17 · 1217 · 148961 · 24517014940753*	96	*641 · 6700417 · 18446744069414584321*
	77	*3 · 43 · 617 · 683 · 78233 · 35532364099*	97	*3 · 971 · 1553 · 31817 · 1100876018364883721*
	78	*5 · 13 · 13 · 53 · 157 · 313 · 1249 · 1613 · 3121 · 21841*	98	*5 · 29 · 113 · 197 · 19707683773 · 4981857697937*
	79	*3 · 201487636602438195784363*	99	*3 · 3 · 3 · 19 · 67 · 683 · 5347 · 20857 · 242099935645987*

Table 3.1: (Cont.)

n			n	
100	17 · 401 · 61681 · 340801 · 2787601 · 3173389601		115	3 · 11 · 691 · 2796203 · 1884103651 · 345767385170491
101	3 · 845100400152152934331135470251		116	17 · 59393 · 82280195167144119832390568177
102	5 · 13 · 137 · 409 · 953 · 3061 · 13669 · 26317 · 1326700741		117	3 · 3 · 3 · 19 · 2731 · 22366891 · 5302306226370307681801
103	3 · 415141630193 · 8142767081771726171		118	5 · 1181 · 3541 · 157649 · 174877 · 5521693 · 104399276341
104	257 · 78919881726271091143763623681		119	3 · 43 · 43691 · 823679683 · 143162553165560959297
105	3 · 3 · 11 · 43 · 211 · 281 · 331 · 5419 · 86171 · 664441 · 1564921		120	97 · 257 · 673 · 394783681 · 4278255361 · 46908728641
106	5 · 15358129 · 586477649 · 1801439824104653		121	3 · 683 · 117371 · 11054184582797800455736061107
107	3 · 643 · 84115747449047881488635567801		122	5 · 733 · 1709 · 3456749 · 368140581013 · 667055378149
108	17 · 241 · 433 · 38737 · 33975937 · 138991501037953		123	3 · 3 · 83 · 739 · 165313 · 8831418697 · 13194317913029593
109	3 · 104124649 · 2077756847362348863128179		124	17 · 290657 · 3770202641 · 1141629180401976895873
110	5 · 5 · 41 · 397 · 2113 · 415878438361 · 3630105520141		125	3 · 11 · 251 · 4051 · 229668251 · 5519485418336288303251
111	3 · 3 · 1777 · 3331 · 17539 · 25781083 · 107775231312019		126	5 · 13 · 29 · 37 · 109 · 113 · 1429 · 14449 · 40388473189 · 118750098349
112	449 · 2689 · 65537 · 183076097 · 358429848460993		127	3 · 56713727820156410577229101238628035243
113	3 · 227 · 48817 · 636190001 · 491003369344660409	F_7	128	59649589127497217 · 5704689200685129054721
114	5 · 13 · 229 · 457 · 131101 · 160969 · 525313 · 275415303169			

Table 3.2: Prime Factors of Fermat Numbers $F_n = 2^{2^n} + 1,\ n \leq 19$

		F_n			
n	2^n	Digits	Factors in integer form	Discoverer	Factors in power form
0	1	1	3	Fermat, 1640	
1	2	1	5		
2	4	2	17		
3	8	3	257		
4	16	5	65537		
5	32	10	$641 \cdot 6,700,417 = 4,294,976,297 = 62,264^2 + 20,449^2 = (143)^4$	Euler, 1732	$(a_5 \cdot 2^7 + 1)(b_5 \cdot 2^7 + 1)$
6	64	20	$274,177 \cdot 67,280,421,310,721$	Landry, Lasseur, 1880	$(a_6 \cdot 2^8 + 1)(b_6 \cdot 2^8 + 1)$
7	128	39	$59{,}649{,}589{,}127{,}497{,}217 \cdot 5{,}704{,}689{,}200{,}685{,}129{,}054{,}721 = 340{,}282{,}366{,}920{,}938{,}463{,}463{,}374{,}607{,}431{,}768{,}211{,}457$	Morrison, Brillhart, 1974	$(a_7 \cdot 2^9 + 1)(b_7 \cdot 2^9 + 1)$
8	256	78	$1{,}238{,}926{,}361{,}552{,}897 \cdot 93{,}461{,}639{,}715{,}357{,}977{,}769{,}163{,}558{,}199{,}606{,}896{,}584{,}051{,}237{,}541{,}638{,}188{,}580{,}280{,}321$	Brent, Pollard, 1981	$(a_8 \cdot 2^{11} + 1)(b_8 \cdot 2^{11} + 1)$

$a_5 = 5$ $\qquad b_5 = 3 \cdot 17,449$

$a_6 = 3^2 \cdot 7 \cdot 17$ $\qquad b_6 = 5 \cdot 52,562,829,149$

$a_7 = 116,503,103,764,643$ $\qquad b_7 = 5 \cdot 228,394,219,017,628,537$

$a_8 = 604,944,512,477$ $\qquad b_8 =$ prime number with 59 digits

Table 3.2: (Cont.)

$$
\begin{aligned}
F_9 &= 2424833 \cdot \\
&\quad 7455602825647884208337395736200454918783366342657 \\
&\quad \cdot\, p_{99} \\
F_{10} &= 45592577 \cdot \\
&\quad 6487031809.4659775785220018543264560743076778192897 \\
&\quad \cdot p_{252} \\
F_{11} &= 319489 \cdot 974849 \cdot 167988556341760475137 \cdot \\
&\quad 3560841906445833920513 \cdot p_{564} \\
F_{12} &= .114689 \cdot 26017793 \cdot 63766529 \cdot 190274191361 \cdot \\
&\quad 1256132134125569 \cdot c_{1187} \\
F_{13} &= 2710954639361 \cdot 2663848877152141313 \cdot \\
&\quad 3603109844542291969 \cdot \\
&\quad 319546020820551643220672513 \cdot c_{2391} \\
F_{14} &= c_{4933} \\
F_{15} &= 1214251009232704250386841 7 \cdot \\
&\quad 168768817029516972383024127016961 \cdot c_{9808} \\
F_{16} &= 825753601 \cdot c_{19720}, \quad F_{17} = 31065037602817 \cdot c_{39444} \\
F_{18} &= 13631489 \cdot c_{78906} \\
F_{19} &= 70525124609 \cdot 646730219521 \cdot c_{157804}
\end{aligned}
$$

where the numbers written out in full are primes, and p_N or c_N denotes an N-digit prime or composite number.

II. Fermat's Little Theorem

In a letter to **Bernard Frenicle de Bessy** *dated Oct. 18 1640, Fermat stated without proof one of the most important theorems in the theory of numbers*[63]*:*

"If p is prime and $p \neq a$, then $a^p \equiv a \ (mod\ p)$". This can also be written as $p \mid a(a^{p-1} - 1)$. So if we add the condition $p \not| a$, p must divide $a^{p-1} - 1$:

$$a^{p-1} = 1 \ (mod\ p).$$

63 It was not until 1736 that Euler made public a proof of the theorem though it is known that a similar proof was contained in a manuscript of **Leibniz** (1683), unpublished at the time. **De Bessy** (1605–1675) was an official at the French mint and amateur mathematician, well-known for his unusual ability in numerical computations.

A proof by induction in a is immediate: The theorem is certainly true for $a = 1$ since 1=1 (mod p). Now, suppose it is true that $a^p - a$ is divisible by p for some $a = b$; then it follows that it is true for $a = b + 1$. Indeed, by the binomial expansion:

$$(b+1)^p - (b+1) = \{b^p + 1 + \text{terms divisible by } p\} - (b+1) \tag{1}$$
$$= (b^p - b) + Np, \text{ say.} \tag{2}$$

*But $p|(b^p - b)$ on the strength of the induction assumption, and so $(b+1)^p = (b+1)(mod\ p)$ proves the theorem. Another variant of the same proof is due to **Leibniz**: for two arbitrary integers A, and B we have*

$$(A+B)^p = A^p + \binom{p}{1} A^{p-1}B + \cdots + B^p,$$

so

$$(A+B)^p \equiv (A^p + B^p)\ (mod\ p);$$

Again

$$(A+B+C)^p \equiv (A+B)^p + C^p \equiv (A^p + B^p + C^p)\ (mod\ p),$$

and so in general

$$(A+B+C+\cdots+K)^p \equiv (A^p + B^p + \cdots + K^p)\ (mod\ p).$$

It suffices to take $A = B = \cdots = K = 1$ and denote their number by a to get again $a^p \equiv a$ (mod p).

The theorem may be described as "little" in comparison with Fermat's more famous theorems, but his "small" result is truly remarkable because there is nothing analogous to it in the classic theory of polynomial equations. A similar, modern proof of this theorem uses group theory.

Applications

- *Prove that if n is prime, then n divides*

  $$S = 1^{n-1} + 2^{n-1} + \cdots + (n-1)^{n-1} + 1.$$

  By FLT for $p = n$, $a^{n-1} \equiv 1\ (mod\ n)$ for $1 \leq a < p$. Thus $S = 0\ (mod\ n)$ and the theorem is proved.

  We do not know any composite *number satisfying this relation. It has been conjectured that there is no such composite number $n < 10^{1000}$.*

- *Prove that $S = 1^n + 2^n + 3^n + 4^n$ is divisible by 5 iff n is not divisible by 4.*

  By FLT $a^4 \equiv 1 \ (mod\ 5)$ for a = 1, 2, 3, 4. Therefore $a^{4k} = 1 \ (mod\ 5)$, where k is an integer. Let $n = 4k + r$, where r = 0, 1, 2, or 3. Thus $a^n = a^{4k}a^r \equiv a^r \ (mod\ 5)$. Consequently

  $$S = 1^n + 2^n + 3^n + 4^n \equiv (1^r + 2^r + 3^r + 4^r) \ (mod\ 5).$$

  It follows that

  $$\begin{aligned} S &\equiv 0 \ (mod\ 5) \quad \text{if } r = 1, 2, 3 \\ S &\equiv 4 \ (mod\ 5) \quad \text{if } r = 0 \end{aligned}$$

- *Verify that $97^{104} - 1$ is divisible by $105 = 3 \cdot 5 \cdot 7$.*

  $$\begin{aligned} &97 \equiv 1 \ (mod\ 3) \therefore 97^{104} \equiv 1^{104} \equiv 1 (mod\ 3) \\ &97 \equiv 2 \ (mod\ 5) \therefore 97^2 \equiv 4 \ (mod\ 5) \equiv -1 (mod\ 5) \\ &97^4 \equiv 1 \ (mod\ 5) \therefore (97^4)^{26} = 97^{104} \equiv 1 \ (mod\ 5) \\ &97 \equiv -1 \ (mod\ 7) \therefore 97^{104} \equiv 1 \ (mod\ 7) \end{aligned}$$

  Since 3, 5, 7 have no factor in common we have $97^{104} \equiv 1 \ (mod\ 105)$.

- *By FLT $10^{p-1} - 1$ is divisible by p if p is not a factor of 10, i.e. if $p \neq 2$ and $p \neq 5$. Thus $10^6 - 1 = 7k$ or $\frac{1}{7} = \frac{k}{10^6-1}$, where $k = 142857$. This implies:*

  $$\frac{1}{7} = \frac{k}{10^6}\frac{1}{1 - 10^{-6}} = \frac{k}{10^6} + \frac{k}{10^{12}} + \cdots + \frac{k}{10^{6m}} + \cdots$$

  This is the basis for decimal expansion of fractions, suggesting that any rational number is always periodic. Note that the period-length may be less than $(p-1)$ as for example in $\frac{1}{3} = 0.\overline{3}$ or as in $\frac{1}{13} = .\overline{076923}$, because in these cases p divides $(10^{\frac{p-1}{2}} - 1)$.

- *Show that $n^{13} - n$ is always divisible by 2730:*

  $$\begin{aligned} f(n) = n^{13} - n \quad &= n(n^{12} - 1) = n(n^6 + 1)(n^6 - 1) = (n^6 + 1)(n^7 - n) \\ &= n[(n^3)^4 - 1] \\ &= n(n+1)(n-1)g(n) \end{aligned}$$

  But

  $$\begin{array}{rll} n^{13} - n & \text{is divisible by 13} & \\ n^7 - n & \text{is divisible by 7} & \\ n(n+1)(n-1) & \text{is divisible by 6} & \\ n \text{ or } (n^3)^4 - 1 & \text{is divisible by 5} & \therefore 5 \cdot 6 \cdot 7 \cdot 13 = 2730 \text{ divides } n^{13} - n \end{array}$$

- *If x, y, z are integers such that $x^2 + y^2 = z^2$, then $xyz = 0$ (mod 60).*

  The general integer Pythagorean triplet is: $x = 2kab$, $y = k(a^2 - b^2)$, $z = k(a^2 + b^2)$ so $xyz = 2k^3ab(a^4 - b^4)$. Either a, b or $a^2 - b^2$ is even $\therefore xyz = 0$ (mod 4). Either a, b or $a^2 - b^2$ is a multiple of 3, since by Fermat's theorem $3 \not| a$, $3 \not| b$ imply $a^2 - b^2 = (1-1)$ (mod 3) $= 0$ (mod 3) $\therefore xyz = 0$ (mod 3). Similarly by Fermat's theorem $a^4 - b^4 = 0$ (mod 5) if neither a nor b are divisible by 5. Thus $xyz = 0$ (mod $3 \cdot 4 \cdot 5$) $=$ 0 (mod 60).

- *Prove that 19 divides $2^{2^{6k+2}} + 3$ for k = 0, 1, 2.*

  We have $2^6 = 64 \equiv 1$ (mod 9), hence for k = 0, 1, 2, ... we also have $2^{6k} \equiv 1$ (mod 9). Therefore $2^{6k+2} \equiv 2^2$ (mod 9), and since both sides are even, we get $2^{6k+2} \equiv 2^2$ (mod 18). It follows that $2^{6k+2} = 18t + 2^2$, where t is an integer ≥ 0. However, by Fermat's theorem, $2^{18} \equiv 1$ (mod 19), and therefore $2^{18t} \equiv 1$ (mod 19) for t = 0, 1, 2, ... Thus $2^{2^{6k+2}} = 2^{18t+4} \equiv 2^4$ (mod 19); it follows that $2^{2^{6k+2}} + 3 \equiv 2^4 + 3 \equiv 0$ (mod 19), which was to be proved.

- *Prove that 13 divides $2^{70} + 3^{70}$*

  By Fermat's Theorem we have $2^{12} \equiv 1$ (mod 13); hence $2^{60} \equiv 1$ (mod 13), and since $2^5 \equiv 6$ (mod 13), which implies $2^{10} \equiv -3$ (mod 13), we get $2^{70} \equiv -3$ (mod 13). On the other hand, $3^3 \equiv 1$ (mod 13), hence $3^{69} \equiv 1$ (mod 13) and $3^{70} \equiv 3$ (mod 13). Therefore $2^{70} + 3^{70} \equiv 0$ (mod 13), or $13 | 2^{70} + 3^{70}$, which was to be proved.

- *Prove that $11 \cdot 31 \cdot 61$ divides $20^{15} - 1$*

  Obviously, it suffices to show that each of the primes 11, 31, and 61 divides $20^{15} - 1$. We have $2^5 \equiv -1$ (mod 11), and $10 \equiv -1$ (mod 11), hence $10^5 \equiv -1$ (mod 11), which implies $20^5 \equiv 1$ (mod 11), and $20^{15} \equiv 1$ (mod 11). Thus $11 | 20^{15} - 1$. Next, we have $20 \equiv -11$ (mod 31), hence $20^2 \equiv 121 \equiv -3$ (mod 31). Therefore $20^3 \equiv (-11)(-3) \equiv 33 \equiv 2$ (mod 31), which implies $20^{15} \equiv 2^5 \equiv 1$ (mod 31). Thus, $31 | 20^{15} - 1$. Finally, we have $3^4 \equiv 20$ (mod 61), which implies $20^{15} \equiv 3^{60} \equiv 1$ (mod 61) (by Fermat's theorem); thus $61 | 20^{15} - 1$ as well.

The Old Chinese Theorem

As early as 500 BCE the Chinese were aware of one divisibility fact included in Fermat's Theorem, for their manuscripts asserted that $2^p - 2$ is divisible by p when p is prime. Thus $2^{11} - 2 = 2046$ is divisible by 11, which

can readily be checked, and $2^{9941} - 2$ *is divisible by the prime* 9941*, a fact which no one would care to verify "by hand".*

But Fermat's theorem implies an infinite number of other divisibility statements. For example, $3^{9941} - 3$, $4^{9941} - 4$, $5^{9941} - 5, \ldots,$ $9940^{9941} - 9940$ *must all be divisible by* 9941*, and* $2^{65537} - 2$, $3^{65537} - 3, \ldots,$ $65536^{65537} - 65536$ *are all divisible by the Fermat's prime* 65537*.*

Although $2^n - 2$ *must be divisible by* n *if* n *is a prime number, the early Chinese (and even, much later, Leibniz himself) erred in conjecturing that the converse statement would be true. They believed that if* $2^n - n$ *is divisible by* n*, then* n *would, of necessity, be prime, so that the divisibility property could then be used as a test of primality.*

The conjecture was discovered to be false only in 1819, when it was shown that $2^{341} - 2$ *is exactly divisible by* $341 = 11 \cdot 31$*, a* composite *number. (Subsequently it was found that* $2^n - 2$ *is divisible by* n *for an infinite number of other composite values of* n*.)*

To see this we just use the binomial theorem through which it is shown that $(a - b)$ *divides* $a^k - b^k$*. Since* $(2^{10} - 1) = 1023 = 3 \cdot 341$ *we can write*

$$
\begin{aligned}
2^{341} - 2 \quad &= [(2^{31})^{11} - 2^{11}] + [2^{11} - 2] \\
&= (2^{31} - 2)M_1 + (2^{11} - 2) \\
&= 2\{(2^{10})^3 - 1\}M_1 + 2(2^{10} - 1) \\
&= 2(2^{10} - 1)M_1 M_2 + 2(2^{10} - 1) \\
&= (2^{10} - 1)J = 341Q
\end{aligned}
$$

M_1, M_2, J, Q *integers.*

Another way of showing this is that $2^{340} - 1 \equiv 0 \ (mod\ 341)$*. Indeed,*

$$2^{10} \equiv 1 \ (mod\ 11); \qquad 2^{10} \equiv 1 \ (mod\ 31)$$

$$\therefore (2^{10})^{34} \equiv 1 \ (mod\ 11); \qquad (2^{10})^{34} \equiv 1 \ (mod\ 31)$$

This means that 11 and 31 each divide $2^{340} - 1$*. But then, since* $(11, 31) = 1$ *so does their product 341.*

A composite number n *which divides* $2^n - 2$ *is a* pseudoprime: *pseudoprimes can also be* even; **D.H. Lehmer** *discovered (1950) the pseudoprime* $161,038 = 2 \cdot 73 \cdot 1103$ *yielding*

$$2^{161,038} - 2 = 2(2^{161,037} - 1); \qquad 161,037 = 3^2 \cdot 29 \cdot 617$$

$$2^{161,037} - 1 = (2^9)^{29 \cdot 617} - 1^{29 \cdot 617} = (2^9 - 1)(\cdots) = 7 \cdot 73(\cdots)$$

Similarly,

$$2^{161,037} - 1 = (2^{29})^{9 \cdot 617} - 1^{9 \cdot 617} = (2^{29} - 1)(\cdots) = 233 \cdot 1103 \cdot 2089(\cdots)$$

Since 73 and 1103 are both primes, dividing $2^{161,037} - 1$, *it follows that 161038 is an even pseudoprime.*

A composite number n *which divides* $3^n - 3$, *or* $4^n - 4$, *or etc* $\ldots$, *strikes us as sharing in the property of pseudoprimality. A composite number* n *which divides* $2^n - 2$, *and* $3^n - 3$, *and* $4^n - 4$, *and* $\ldots$, *and* $a^n - a$, *and* $\ldots$, *for every integer* a, *even the negative integers, is certainly the ultimate in this regard, and is called an* absolute pseudoprime.

The smallest one is 561. That is to say, 561 is a composite number and $a^{561} - a$ *is divisible by 561 no matter what integer is substituted for* a. *This follows directly from Fermat's Little Theorem: the prime decomposition of 561 is* $3 \cdot 11 \cdot 17$. *We need to show that* $a^{561} - a$ *is divisible by each of these primes. We have*

$$a^{561} - a = a(a^{560} - 1) = a[(a^{10})^{56} - 1^{56}] = a[(a^{10} - 1)(\cdots)]$$
$$= (a^{11} - a)(\cdots).$$

But $a^{11} - a$ *is divisible by 11, by Fermat's theorem, because 11 is a prime number. Thus 11 divides* $a^{561} - a$. *Similarly 3 and 17 are also shown to be divisors.*

A few other absolute pseudoprimes are

$2821 = 7 \cdot 13 \cdot 31$ $\quad 4991 = 7 \cdot 23 \cdot 31$ $\quad 10585 = 5 \cdot 29 \cdot 73$ $\quad 15841 = 7 \cdot 31 \cdot 73$

$29341 = 13 \cdot 37 \cdot 61;$ $\quad 5 \cdot 17 \cdot 29 \cdot 113 \cdot 337 \cdot 673 \cdot 2689.$

It is unknown whether or not there exists an infinity of absolute pseudoprimes.

The Isogonic Center

In 1643 ***Fermat*** *posed the following problem to the Italian mathematicians* ***Evangelista Torricelli*** *(a pupil of Galileo, 1608–1647) and* ***Francesco Cavalieri*** *(1598–1647):*

To find a point P of the plane, the sum of whose distances from the vertices of a given triangle ABC is the smallest possible. Torricelli's solution was published posthumously in 1659 by his pupil ***Viviani***[64] *(1622–1703). A simple non-calculus solution, published in* 1929, *(for the case where each angle in the triangle is less than 120°) is this: Let P be a point inside the triangle ABC. Rotate the triangle APC by 60° about A in a direction away from the opposite vertex B and denote its new position in the plane by $AP'C'$. Clearly, the sum of distances $\overline{AP}+\overline{BP}+\overline{CP}$ is now equal to the sum of the segments $\overline{C'P'}+\overline{P'P}+\overline{PB}$, which in general will constitute a continuous broken line. Since the end points of this line are fixed (the position of C' is independent of P!), its length will be minimal if P and P' are on $\overline{C'B}$.*

This implies that the sought point P is such that the sides of the triangle are seen from P at equal angles of 120°. The construction of P is simple: build on each side of the triangle a new equilateral triangle and connect the new vertices to the corresponding opposite vertices of the original triangle. The three lines will meet at P. This solution is undoubtedly one of the most beautiful ones in the entire Euclidean geometry. The point P is known as the Fermat point.

It is of interest to mention that the above solution is the amalgam of the contributions of four mathematicians during 1643–1846: The first, ***Torricelli****, knew that the circumcircles of the outward, equilateral triangles on the sides*

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[64] **Vincenzo Viviani** was an assistant to both Galileo and Torricelli. His primary interests lay in geometry, hydraulics and mechanics. He discovered the geometrical theorem (named after him): For a point P inside an equilateral triangle ABC, the sum of the perpendiculars a, b, c from P to the sides is equal to the altitude h.

Viviani studied with the Jesuits in Florence. His years with Galileo took the place of a university education, and he was Galileo's companion and pupil during the final two years of his master's life.

In 1660, together with Borelli, Viviani measured the velocity of sound by timing the difference between the flash and the sound of a cannon. They obtained a value of 350 $\frac{\mathrm{m}}{\mathrm{sec}}$, which is considerably better than the previous value of 478 $\frac{\mathrm{m}}{\mathrm{sec}}$ obtained by Gassendi (the currently accepted value is 331.29 $\frac{\mathrm{m}}{\mathrm{sec}}$ at 0°C).

of the triangle ABC intersect at P. The second, **Cavalieri**, *found that each side of the triangle ABC is seen from P at an angle of $120°$. The third,* **Thomas Simpson** *(1710–1761), realized in 1750, that the lines joining the outer vertices of the triangle ABC intersect at P. In 1834,* **Heinen** *noted that if one of the interior angles (say B) is greater or equal to $120°$, then the shortest pathway linking A, B, and C consists of the segments AB and BC.*

In 1846, **Eduard Fasbender** *discovered the following* maximum property *of P associated with its minimum property, if P lies in the interior of ABC: The least value of the sum of distances $\overline{AP} + \overline{BP} + \overline{CP}$ in the triangle ABC is equal to the maximum of the altitudes of all equilateral triangles circumscribing the triangle ABC.*

The point P is now known as the isogonic center *of the triangle. It was the first notable point of the triangle to be discovered in times more recent than that of Greek mathematics. A thorough analysis of the problem and its generalization to an arbitrary number of point in any number of dimensions, was given (1843) by the geometer* **Jacob Steiner** *(1796–1863) and is known therefore as the* Steiner problem.

The Steiner figure can be obtained in a soap-film experiment: *To this end one takes two glass plates kept parallel by three perpendicular pins of equal length. If the configuration is immersed in a soap bath and taken out again, one obtains a system of three soap films perpendicular to each of the plates. These soap laminae touch each plate in three segments that yield the shortest pathway linking the three pins at either plate.*

As noted, for two or three points the minimal pathway is uniquely determined. For four or more points, however, we must generally expect more than one minimal pathway. *We must even distinguish between* stationary *and* stable *pathways. The stable pathways yield either absolute, or merely relative, minima.*

The generalization of the Steiner problem to n points in a plane does not lead to interesting results. To find a really significant extension we must abandon the search for a single point P. Instead we look for the "street network" of shortest total length! Thus, if we choose four points that are vertices of a square, then we obtain two different but congruent minimal pathways. If we stretch the square into a rectangle, then we obtain two minimal pathways of different length, one of which is an absolute and the other a relative minimum.

Mathematically expressed, the problem is: Given n points $A_1, A_2, \ldots, A_n$, to find a connected system of straight line segments of shortest total length such that any two of the given points can be joined by a polygon consisting of segments of the system. This problem is known today as the Steiner

problem[65]*, and its solution has eluded the fastest computers and the sharpest mathematical minds.*

The Steiner problem cannot be solved by simply drawing lines between the given points, but it can be solved by adding new ones called Steiner points*, that serve as junctions in a shortest network.*

To determine the location and number of Steiner points, mathematicians and computer scientists have developed algorithms. Yet, even the best of these procedures running on the fastest computers cannot provide a solution for a large set of given points because the time it would take to solve such a problem is impractically long. Furthermore, the Steiner problem belongs to a class of problems for which many computer scientists now believe an efficient algorithm may never be found.

However, approximate *solutions to the shortest-network problem are computed routinely for numerous applications, among them designing integrated circuits, determining the evolution tree of a group of organisms and minimizing materials used for networks of telephone lines, pipelines and roadways.*

About 200 years after Fermat, when calculus was well established, an analytic solution for the triangle was given: Let (a_1, b_1), (a_2, b_2), (a_3, b_3) *be respectively the coordinates of the vertices* A, B, C, *referred to a system of rectangular coordinates. The function whose minimum is sought is*

$$z(x,y) = \left[(x-a_1)^2 + (y-b_1)^2\right]^{1/2} + \left[(x-a_2)^2 + (y-b_2)^2\right]^{1/2}$$
$$+\left[(x-a_3)^2 + (y-b_3)^2\right]^{1/2}.$$

From the relations $\frac{\partial z}{\partial x} = 0$, $\frac{\partial z}{\partial y} = 0$, *one obtains two algebraic equations*

$$\frac{x-a_1}{a} + \frac{x-a_2}{b} = -\frac{x-a_3}{c}, \quad \frac{y-b_1}{a} + \frac{y-b_2}{b} = -\frac{y-b_3}{c},$$

where $PA = a$, $PB = b$, $PC = c$.

Then, squaring and adding, we find the condition

$$1 + 2\left[\frac{(x-a_1)}{a}\frac{(x-a_2)}{b} + \frac{(y-b_1)}{a}\frac{(y-b_2)}{b}\right] = 0.$$

The geometrical interpretation of this result is straightforward: denoting by α *and* β *the cosines of the angles which the direction* PA *makes with the*

[65] *The Shortest-Network Problem*, M.W. Bern and R.L. Graham, Scientific American, January 1985.

axes x and y, respectively, and by α' and β' the cosines of the angles which PB makes with the same axes, we may write this last condition in the form

$$1 + 2(\alpha\alpha' + \beta\beta') = 0,$$

or by denoting the angle APB by ω,

$$2\cos\omega + 1 = 0.$$

This condition expresses the fact that the segment AB subtends an angle of 120° at the point P. For the same reason, each of the angles BPC and CPA must be 120°.

The sum $PA + PB + PC$ is less than the sum of any two sides of the triangle:

$$AB + AC = \sqrt{a^2 + b^2 + ab} + \sqrt{a^2 + c^2 + ac} > \left(b + \frac{a}{2}\right) + \left(c + \frac{a}{2}\right).$$

Hence $AB + AC > a + b + c$ and P therefore actually corresponds to a minimum. When one of the angles of the triangle is equal or greater to 120°, the minimum must be given by the vertex of the obtuse angle.

1626–1629 CE **Albert Girard** (1595–1632, Netherlands). Mathematician. First to accept and use negative roots of equations in the solutions of geometrical problems. Conjectured that an algebraic equation of degree n has n roots, some of which may be non-real (the *fundamental theorem of algebra*).

First to show how to express the sums of the powers of the roots in terms of the equation's coefficients. First to publish (1629) the equation

$$A = \pi r^2\left(\frac{s}{180} - 1\right),$$

relating the area A of a geodesic triangle on a sphere of radius r to the sum of angles s (in degrees) of that triangle.

Published a treatise on trigonometry (1626) containing the first use of the abbreviations *sin, cos, tan.*

1630–1668 CE **Jan Amos Komensky (Comenius)** (1592–1670, Moravia, Poland and Holland). Pioneer of modern education; educational

reformer and philosopher, promoter of scientific societies. His ultimate aim was universal peace.

He recognized that the necessary steps preliminary to the attainment of this goal involved the unification of rival Christian denominations, fundamental reforms in education and new approach to natural science.

It was largely the result of his initiative that scientific societies promoting research were founded throughout Europe during the 17^{th} century. He insisted that education should be free, universally available, and compulsory for every child, that automatic memorization should be replaced by teaching words with perceptual objects, and that the sensual faculties of school children be taken into consideration.

Comenius stands on a transitional figure in the area of science – half-way between the medieval Aristotelianism and modern empiricism. He believed that independent study and observation offered greater intellectual rewards than did constant reliance upon Aristotle or Pliny. His textbooks, translated into 17 languages were used in the early years of Harvard University, and throughout the 17^{th} century schools of Europe, Asia and the New England.

His principal works were: *Gate of Languages Unlocked* (1631); *The Way of Light* (1642); *Patterns of Universal Knowledge* (1651); *The Great Didactic* (1657); *Visible World* (1658); The last was the first textbook in which pictures were as important as text.

Central to his philosophy is the proliferation of *truth*, which, being one and universal, carries a chance for world's peace. Men should be educated trilaterally to spiritual life, secular moral life and faithful religious life. Hence the three aims of education: enlightenment, virtues and God-fearing.

Comenius developed a new philosophy of education. He favored broad general education, rather than the narrow training of his day. His curriculum consisted of: singing, languages, economy, politics, world history, science, geography, arts and handicrafts.

He suggested four stages of education, each of 6 years:

(1) 0–6 "mother school" in the family;

(2) grammar school 6–12, emphasizing the development of imagination, memory and the basic skills;

(3) *Latin school*, 12–18, for the development of the intellect;

(4) *Universities and traveling*, 18–24, to consolidate the will and endeavor to harmonize the various domains of education.

First to advocate teaching of science in schools. Urged the establishment of more schools and universities. Developed new method of teaching languages and issued the first children picture book.

He was born in Comna in Moravia of poor parents and studied at Heidelberg. Fled the Thirty Years' War to Poland (1621), settling with a group of Bohemian Brethren at Leszno. Invited to England (1641–1642) and Sweden (1642–1648) to advise on school educational reforms.

Twice during his lifetime, Komensky lost all his property and manuscripts: in 1621, during the Spanish invasion and the prosecution of the Protestants in Moravia, and again in 1655 when the Poles burned Lissa during the Swedish-Polish War.

1630–1632 CE **John Rey** (1582–1645, France). Metallurgist. One of the earliest scientists to put forward a mechanical theory of chemical change. It has been known for some time that metals increased in weight when they were heated in air and formed a calx. To explain the phenomenon, Rey suggested that air had weight, and that it was taken up by metals on heating. He did not think of the process as a chemical combination of air with the metal but as a *mechanical mixing*, like dry sand taking up water and becoming heavier.

In 1632 Rey improved the thermoscope of **Galilei** (1596) and **Sanctorius** (1611) when he used liquid instead of air to measure temperature changes, that is, the thermoscope had fluid at the bottom and air at the top, more closely resembling modern-day thermometers.

1630 CE Venice and surrounding Italy devastated by plague. 500,000 died. By 1632, the disease reached France, killing ca 100,000 more.

1634–1643 CE **Gilles Personier de Roberval** (1602–1675, France). Geometer and physicist whose extensive correspondence served as a medium for the intercommunication of mathematical ideas. Developed some pre-calculus methods of integration of some trigonometric functions and drawing tangents to plane curves. Asserted that *gravitation* is an inherent property of matter throughout the universe and that the counter balancing force allowing bodies to remain separated is the resistance of the intervening *ether*.

He was consistently tardy in disclosing his discoveries. This has been explained by the fact that for 40 years he held the professorial chair of Ramus at the Collège Royale. This chair automatically became vacant every three years, to be filled by open competition in mathematical contests in which the questions were set by the outgoing incumbent.

1634–1647 CE **Adam Olearius** (b. Oehlschlaeger; 1599–1671, Germany). Geographer, traveler, mathematician and scholar. His travels in Persia and Russia[66] (1634–1639) were described in his book (1647): "*Voyages of the Ambassadors Sent by Frederic, Duke of Holstein, to the Great Duke of Muscovy and the King of Persia.*"

Olearius' accounts of his travels became one of the major early descriptions of Russia by a European. He was the first to introduce Western Europe to Persian culture.

1635 CE **Francesco Bonaventura Cavalieri** (1598–1647, Italy). Mathematician. Advanced certain rules that constituted valuable tools in the computation of areas and volumes. Also produced explicit formulae which showed how to integrate a class of functions. These methods are essentially those of the definite integral and anticipated the development of the calculus later in the century. Cavalieri used his method to evaluate correctly the area of the ellipse and the volume of the sphere. The methods of Cavalieri were later extended by **Torricelli** (1645), **Fermat** (1654), **Pascal** (1654), **Barrow** (1662) and others.

Cavalieri was born in Milan, studied under **Galileo**, and served as a professor of mathematics at the University of Bologna from 1629 until his premature death at the age of 49. His treatise *Geometria indivisibilibus* (1635) is devoted to the pre-calculus *method of indivisibles* that can be traced back to **Democritos** (ca 410 BCE) and **Archimedes**. It is likely that the attempts at integration made by **Kepler** directly motivated Cavalieri.

1636–1641 CE **Jeremiah Horrocks** (1619–1641, England). Astronomer. First to apply Kepler's laws to the actual motion of the moon. This was later used by **Newton** to forge his synthesis of Kepler laws of the motion of heavenly bodies and Galileo's laws of falling bodies and projectiles. Horrocks clearly perceived the significant analogy between terrestrial gravity and the force exerted in the solar system.

Horrocks was born at Toxteth Park, near Liverpool. His family was poor and he pursued his self-education amidst innumerable difficulties. He entered Emmanuel College, Cambridge, in 1632 and his university career lasted three

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66 During the early 17^{th} century, northern European merchants saw Russia as a land through which secure trade routes might be opened to Persia and points east — without danger from or taxation by the Turks, and unknown to Italy, Spain and Portugal. Adam Olearius was appointed secretary to an embassy from the Duke of Holstein to Muscovy and Persia which sought to make that Duchy an entrepot for overland silk trade.

years. On its termination he became a tutor at Toxteth, devoting to astronomical observations his brief intervals of leisure.

In 1639 he applied himself to the revision of the Rudolphine Tables (published by **Kepler** in 1627), and in the progress of this task became convinced that a transit of Venus[67], overlooked by Kepler, would nevertheless occur on the 24^{th} of November 1639.

He indeed observed it, while a curate at Hoole, near Preston. This transit of Venus is remarkable as the first ever observed (that of 1631 predicted by Kepler, having been invisible in Western Europe). Through this observation he was able to introduce some important corrections into the elements of the planet's orbit and obtain a good estimate of its apparent diameter.

Before he was twenty, Horrocks made an important contribution to lunar theory, by showing that the moon's apparent irregularities could be completely accounted for by supposing it to move in an ellipse with a variable eccentricity and a rotating major axis of which the earth occupies one focus. These precise conditions were afterwards demonstrated by Newton to follow necessarily from the law of gravitation.

Jeremiah Horrocks died when only in his twenty-second year.

1636–1644 CE **Girard Desargues** (1593–1662, France). Mathematician, engineer and architect. The most original contributor to projective geometry in the 17^{th} century. A geometer of profoundly original ideas, sustained at the same time by a good spatial intuition, precise knowledge of the great classic works and exceptional familiarity with the whole range of contemporary techniques.

In 1639 he distributed in Paris a twelve-page booklet under the heading (translated): "Proposed Draft on an Attempt to Deal with the Cases of Meeting of a Cone with a Plane". After presenting his rules of practical perspective, Desargues outlines a program dominated by two basic themes: the concern to rationalize and unify the diverse preexisting graphical techniques and the purely geometric study of perspective.

In this book he developed topics to be found in modern courses in projective geometry, such as: harmonic ranges, homology, poles and polars, perspectives and involution. The book included Desargues' well-known (today) 'two-triangle theorem'. In spite of this, the treatise was ignored, forgotten and lost until 1845, when **Michel Charles** (1793–1880) found a manuscript

[67] Transits of Venus (when the planet is passing between the earth and the sun) are among the rarest of astronomical phenomena; many astronomers cannot possibly see one during their lifetimes. Since 1639 there have been transits in 1761, 1874 and 1882. The next pair will occur in 2004 and 2012.

copy, and since that time the work has been regarded as one of the classics in synthetic projective geometry.

Desargues was born in Lyons, one of the nine children of a collector of the tithes on ecclesiastical revenues in the diocese of Lyons. He apparently was educated as an engineer (and architect), for there is evidence for his presence in Paris in 1626 in connection with a certain engineering project. In 1630 he evidently became friendly with several of the leading mathematicians in Paris: **Mersenne**, **Gassendi** and **Roberval**.

After the publication of his booklet in 1636 he won the esteem and respect of **Descartes**, and young **Pascal**, both members of Mersenne's*, Academie Parisienne*. Throughout the period 1636–1644, many attacks were launched against Desargues' work by second-rate mathematicians, which may have caused his scientific and polemic activity to decline; he then embarked on his new career as an architect (1645–1657).

He returned to Paris from Lyons in 1657. In Paris, the authors of the period attribute to Desargues, besides a few houses and mansions, several staircases whose complex structure and spectacular character attest to the exactitude of his graphical stonecutting procedures. It also seems that he collaborated, for the realization of certain effects of architectural perspective, with the famous painter **Phillipe de Champagne** (1602–1674). In the region of Lyons, Desargues' architectural creations were likewise quite numerous.

Desargues' main accomplishment as an engineer, was a system for raising water that he installed near Paris, at the Château of Beaulieu, based on the use (until then unknown) of epicycloidal wheels (described and drawn by **Huygens** in 1671).

In 1660 Desargues was again active in the scientific life of Paris, attending meetings at Montmor's Academy. He was heard of last on the meeting of 9 November 1660, at which Huygens heard him present a report on a geometrical problem.

Descartes was probably the source of both the inspiration and demise of his book, since geometers at that time were totally absorbed in Cartesian geometry to the exclusion of any new idea in the field. However, in the early 19^{th} century, the mathematical community was once again willing and capable to digest novel, nonorthodox geometrical ideas.

1637 CE **René du Perron Descartes** (1596–1650, France). Distinguished mathematician, scientist and philosopher. Published his work "*Discours de la méthode pour bien conduire sa raison et chercher la vérité dans les sciences*", the third and last appendix of which, "La Géométrie" contains a sufficiently complete (although somewhat confusing) presentation of the mathematical theory that since then has been called analytic geometry. This

discovery of the utility of coordinate systems in relating geometry and algebra opened up an entire new science by enabling the investigation of geometric objects by algebraic means.

Descartes revised exponential notation for integral exponents almost to its 20^{th} century form. He was first to allow powers higher than the third.

He formalized the classical law of inertia, in the form given in Newton's '*Principia*' 40 years later, but he went further to suggest the conservation of linear momentum. Unlike Newton, who used his theory of gravitation to explain how the orbital motions of the planets and satellites can be maintained, but *not* how they have *originated*, Descartes assumed that originally the world has been filled with matter distributed uniformly. He then sketched out a qualitative theory of successive formation of the sun and the planets.

In his book *La Dioptrique* (1637) he was first to publish the now familiar law of refraction in terms of sines. Although the result was given already by **W. Snell**[68] in 1621, Descartes went beyond Snell and derived the law from a new assumption on the nature of light. He considered it to be essentially a pressure transmitted through a perfectly elastic medium (the '*aether*'), which fills all space. In his view, light was *a stream of tiny particles* and the laws of reflection and refraction were explained by using particle kinematics[69]. [In contradistinction, **Fermat** rederived the law of reflection from his own *principle of least time*, which departed from **Hero**'s shortest-path statement.]

In an appendix to *Discours de la Méthode* (1637), Descartes discovered (using the law of refraction) the key to the rainbow problem — the reason for the clustering of rays about the angle 42° in the primary bow. He discovered the effective ray through patient observations and laborious calculations (the Newtonian calculus arrived only in 1671).

While Francis Bacon's empiricism influenced science and philosophy in England, Descartes left a profound mark on the thinking of scientists in Europe for the past 300 years, due to two of his ideas: the first was his conviction that the universe (including man's body but excluding his mind) is a mathematically intelligible machine, that could be deduced from a few simple principles, and eventually even by a single overreaching mathematical theorem. This view was the basis of the later cosmological theories of **Kant**, and **Laplace**.

The second is his program of *total geometrization of physics* via the concept of 'dimension'. This idea began to be realized in the new physics of the 20^{th} century, especially in Einstein's GTR, and in quantum mechanics. Thus,

[68] **Huygens** believed that Descartes had seen Snell's manuscript on refraction.

[69] See (51) p. 1006.

although his generalizations in astronomy, physics and anatomy were often premature and his passion for system-building went beyond his capacity to check by experiment, he remains one of the founders of modern scientific thinking.

Descartes rejected Aristotelian teleology which stated that all natural events are purposeful. He emphasized the use of reason and abstract deductive logic as the chief tool of philosophical inquiry. He greatly influenced later natural philosophers, especially **Berkeley**.

Descartes was the first to term the mathematical rules that others had discovered *"the Laws of Nature"*. God rules the universe through these eternal and unchangeable laws, he maintained. These laws were not mere descriptions of nature, but the very 'legislation' of nature: Descartes' God was the great Lawgiver. Experiment was to be used, as with the Platonists, to illustrate laws that were mathematically deduced from first principles.

Descartes was born at La Haye, in Touraine, midway between Tours and Poitiers. From 1604 to 1612 he studied at a Jesuit school. During the winter of 1612 he took lessons in horsemanship and fencing; and then started, as his own master, to taste the pleasures of Parisian life. Here he renewed an early friendship with **Marin Mersenne**. In 1614, however, he abandoned social life and shut himself up for nearly two years in a secluded house of the Faubourg St. Germain in order to study mathematics.

In may 1617 Descartes set out for The Netherlands and took service in the army of Prince Maurice of Orange. After spending two years in Holland as a soldier in a period of peace, he volunteered in 1619 into the Bavarian service. In 1621 he quit the imperial service and returned to France. Money from an inheritance and from patrons enabled him to devote most of his life to study.

He visited Switzerland and Italy, and lived in Paris before settling in Holland in 1628. Except for short visits to France to settle family affairs, a visit to England in 1630 and an excursion to Denmark (1634), he led a quiet, scholarly life in The Netherlands until 1649, and there most of his philosophical works were written.

During his residence in Holland he lived at 13 different places, and changed his abode 24 times. In the choice of these spots, two motives seem to have influenced him — the neighborhood of a university or college, and the amenities of the situation. His residence in the Netherlands fell in the most prosperous and brilliant days of the Dutch state. Abroad, its navigators monopolized world commerce and explored unknown seas; at home the Dutch school of painting reached its pinnacle in Rembrandt (1607–1669).

In 1649 he accepted an invitation from Queen Christina to visit Sweden. The young queen wanted Descartes to draw up a code for a proposed academy

of sciences, and to give her an hour of philosophic instruction every morning at five in her draughty chambers. However, he fell victim to the inflammation of the lungs, and died soon thereafter in Stockholm.

Descartes' new ideas were slow to gain the recognition they deserved. In his appendices of 1637, he hit upon three capital advances yet not one of them was integrated into scientific thought for several decades. The fault lay in part with Descartes himself. In the case of each of the appendices — *La Dioptrique* and *La Geometrie*, as well as *Les Meteores* — the author was primarily boasting of the efficacy of his methodology. He was not explaining, with a meticulous care required in new situations, the value of these contributions to science. He did not explore them further, nor did he determine their implications and their relationship to other phenomena. He did not surround them with an aura of proselytizing enthusiasm. In fact he promptly lost interest in analytic geometry, the law of refraction, and the rainbow.

Descartes never married. In person he was small, having a large head, protruding brow, prominent nose, and eyes wide apart, with black hair coming down almost to his eyebrows. His voice was feeble. He usually dressed in black, with unobtrusive propriety.

In all his travels he only studied the phenomena of nature and human life. He was a spectator, rather than an actor, on the world stage. He entered into the army, merely because the position gave a vantage-ground from which to make his observations. He took no part in the political interests which these contests involved.

The contempt of aesthetics and erudition is characteristic of the Cartesian system; to him all the heritage of the past seemed but elegant trivia. The science of Descartes was physics in all its branches, but especially as applied to physiology. Science, he said, may be compared to a tree; metaphysics in the root, physics in the trunk, and the three chief branches are mechanics, medicine and morals — the three applications of our knowledge to the outside world, to the human body, and to the conduct of life.

Who Invented Analytic Geometry?

"Everything has been thought of already. The problem is — thinking of it again".

Johann Wolfgang von Goethe (1749–1832)

By definition, analytical geometry is concerned with the representation of geometrical figures and their relations by algebraic *equations. This essentially means that a problem in geometry is transformed into a corresponding one in algebra, the algebraic problem solved, and finally the algebraic solution is interpreted in geometrical terms. It follows that, before analytic geometry could assume its highly practical form, it had to await the development of algebraic procedures and symbolism. These decisive contributions were only made in the* 17^{th} *century by* ***René Descartes*** *(1596–1650) and* ***Pierre Fermat*** *(1601–1665). Not until after the impetus given to the subject by these two men, do we find analytic geometry in a form with which we are familiar.*

Nevertheless, one of the basic ingredients of analytic geometry, namely the concept of fixing the position of a point by means of suitable reference frame, was employed in the ancient world by the Egyptian and the Roman surveyors and by the Greek map-makers. And, if analytic geometry implies not only the use of coordinates but also the geometric interpretation of relations among coordinates, then Greek priority is favored by the fact that ***Apollonios*** *derived the bulk of his geometry of* conic sections *from the geometrical equivalents of certain algebraic equations of these curves, an idea that seems to have originated with* ***Menaechmos*** *about 350 BCE!*

All these results must have been known to Fermat and Descartes, who were both deeply versed in the classical literature of mathematics. At any rate, they certainly could not have escaped reading ***Oresme****.*

1640–1662 CE **Blaise Pascal** (1623–1662, France). Mathematician, theologian, physicist and philosopher who made great contributions to science through his studies in hydrostatics and the mathematical theory of probability.

During the 16^{th} and 17^{th} centuries a great deal of the leisure of the European aristocracy was occupied with games of chance and gambling in general.

This class did not include among their number any mathematicians capable of handling the problems that naturally suggested themselves. Thus it happened that from time to time problems of chance were passed on to the mathematicians of the period. We know for example that **Galileo** (1564–1642) had his attention directed by an Italian nobleman to a problem in dice.

Pascal was drawn into probability theory as a result of problems that arose in gambling houses. At the time, a gambling die game was in vogue which had been played for at least a hundred years and which persists to the present day: the "house" offers to bet even money that a player will throw at least one six in four throws of a single die. [This game is mildly favorable to the "house" since, on the average it wins $\left[1-\left(\frac{5}{6}\right)^4\right]$ to $\left(\frac{5}{6}\right)^4$, i.e., in a ratio $\frac{671}{625}$].

A distinguished Frenchman, **Antoine Gombauld Chevalier de Méré**, **Sieur de Baussay** (1610–1685) was bothered by a number of practical problems concerning the game, one of which is called 'the problem of points' (the division problem): "How should the prize money be divided among the contestants if for some reason it proved necessary to call off the game before it is completed and when the contestants have only partial scores?"

A second problem was: "If the player roles a *pair* of dice, will it be favorable to the 'house' to bet that the player will throw at least one double six in 24 throws of the pair?". Méré consulted Pascal, whom he knew, and Pascal proved that the odds were slightly *against* the house if it wagered on 24 throws, but were slightly *favorable* for 25 throws.

For the solution of the first problem, Pascal introduced the important idea that the amount of the prize any contestant deserved, in a partial game, should depend on the *probability* that this particular player would win the game, were it carried through to its conclusion. And Pascal worked out in detail, for several examples, how the probability of winning could be calculated from a knowledge of the nature of the game and the partial score of each contestant.

Pascal wrote about these matters to Fermat, who had a great reputation as a mathematician and who was in addition a distinguished justice at Toulouse. The resulting exchange of letters went further in working out the mathematics of some games of chance, and became known in the learned society of the day. This episode can properly be regarded as the advent of a new branch of mathematics.

At the time when the theory of probability started at the hands of Pascal and Fermat, they were the most distinguished mathematicians in Europe. [**Descartes** died in 1650. **Newton** (b. 1642) and **Leibniz** (b. 1646) were as yet unknown. **Huygens** (b. 1629) could not, at this time, be placed on the level of Pascal and Fermat.]

It might have been anticipated that a subject of such interest in itself and discussed by the two most distinguished mathematicians of the time, would have attracted rapid and general attention; but such does not appear to have been the case. The two great men themselves seem to have been indifferent to any extensive publication of their investigations. *It was sufficient for each of them to gain the approbation of the other*.

The invention of the calculus by Newton and Leibniz soon offered mathematicians a subject of absorbing interest, and the theory of probability advanced but little during the half century which followed the dates of the correspondence between Pascal and Fermat (1654). In 1658, Pascal published several treatises which established his work as a forerunner of both differential and integral calculus.

In 1648 Pascal formulated the basic laws of equilibrium for fluids (published posthumously in 1663), stating that pressure in a fluid is transmitted equally in all directions, and that the height of the mercury column in a barometer is balanced by the pressure of air. He suggested that the barometer be used to determine altitudes, and further used measurements of the barometric pressure made at the summit of *Mount Puy de Dôme* to estimate the total weight of the atmosphere (his value $= 3.7 \times 10^{18}$ kg).

Pascal was a son of a nobleman. A prodigy of sorts, he had already published an essay on conic sections by the age of 16 in which he discovered and proved '*Pascal's Theorem*'[70]. He also invented one of the early calculating machines that could add and subtract (1642).

In his *Traité du triangle arithmétique* (1654), Pascal united the algebraic and combinatorial theories by showing that the elements of the arithmetic triangle (known as the "*Pascal Triangle*"[71] could be interpreted in two ways: as the coefficient of $a^{n-k}b^k$ in $(a+b)^n$ and as the numbers of combinations of n things taken k at a time. In effect, he showed that $(a+b)^n$ is the *generating function* for the numbers of combinations. As an application, he founded the mathematical theory of probability by solving the problem of division of

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[70] If a hexagon is inscribed in a conic, then the points of intersection of the three pairs of opposite sides are collinear, and conversely (1640).

[71] In this triangle, the k^{th} element $\binom{n}{k}$ of the n^{th} row is the sum $\binom{n-1}{k-1} + \binom{n-1}{k}$ of the two elements above it in the $(n-1)^{th}$ row, as follows from the formula $(a+b)^n = (a+b)^{n-1}a + (a+b)^{n-1}b$. The triangle appeared to the depth of six in **Yang Hui** (1261) and to a depth of eight in **Zhu Shijie** (1303). Yang Hui attributes the triangle to **Jia Xian**, who lived in the 11^{th} century.

The numbers $\binom{n}{k}$ appear as the number of combinations of n things taken k at a time in the writing of **Levi ben Gershon** (1321), who gave the formula $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.

stakes[72], and as a method of proof he used mathematical induction for the first time in a really conscious and unequivocal way.

Late in 1654, he became dissatisfied with experimentation and withdraw from science and the world for a life of religious meditations. He turned to the study of man and his spiritual problems and produced a religion-oriented philosophy that concerned itself primarily with the relation of man to God through faith.[73]

[72] Suppose that a game between players I and II has to be called off with n players remaining, k of which I has to win in order to carry off the stakes. Assuming that I has an even chance of winning each play, the ratio of his chance of winning the stakes to that of II's winning is

$$\left[\binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{k}\right] \Big/ \left[\binom{n}{k-1} + \binom{n}{k-2} + \cdots + \binom{n}{0}\right].$$

[73] For further reading, see:

- Steinmann, J., *Pascal*, Harcourt, Brace and World: New York, 1966, 304 pp.

Worldview IX: Blaise Pascal

* * *

"If God does not exist, one will lose nothing by believing in him, while if he does *exist, one will lose everything by not believing. Hesitate not, then, to wager that he exists".*

* * *

"We arrive at truth, not by reason only, but also by the heart".

* * *

"Nature is an infinite sphere of which the center is everywhere and the circumference nowhere".

* * *

"Contradiction is not a sign of falsity, nor the lack of contradiction a sign of truth".

* * *

"Man is equally incapable of seeing the nothingness from which he emerges and the infinity in which he is engulfed".

* * *

"What is a man in nature? Nothing in relation to the infinite, all in relation to nothing, a mean between nothing and everything".

* * *

"The more intelligent one is, the more men of originality one finds. Ordinary people find no difference between men".

* * *

"[I feel] engulfed in the infinite immensity of spaces whereof I know nothing, and which know nothing of me. The eternal silence of these infinite spaces alarms me".

* * *

"Reason is the slow and tortuous method by which these who do not know the truth discover it. The heart has its own reason which reason does not know".

* * *

"One can have three principal objects in the study of truth: to discover it when one searches for it, to prove it when one possesses it and to distinguish it from falsity when one examines it".

* * *

"By space, the universe encompasses and swallows me up like an atom; by thought I comprehend the world".

(1657)

* * *

Choice and Chance[74] — *The Mathematics of Counting*[75] *and Gambling* (1654–1855)

[74] For further reading, see:

- Levy, H. and L. Roth, *Elements of Probability*, Oxford University Press: London, 1951, 200 pp.
- Parzen, E., *Modern Probability Theory and its Applications*, John Wiley & Sons: New York, 1960, 464 pp.
- Aczel, A.D., *Chance*, Thunder's Mouth Press: New York, 2004, 161 pp.
- Freund, J.E., *Introduction to Probability*, Dover: New York, 1993, 247 pp.
- Mosteller, F., *Fifty Challenging Probability Problems*, Dover Publications: New York, 1965, 88 pp.
- Rozanov, Y.A., *Probability Theory*, Dover Publications: New York, 1969, 148 pp.
- Bates, G.E., *Probability*, Addison-Wesley, 1965, 58 pp.
- Withworth, W.A., *Choice and Chance, I-II*, G.E. Strechert and Co.: New York, 1945.
- Vilenkin, N.Ya., *Combinatorics*, Academic Press: New York, 1971, 296 pp.
- Ball, W.W.R., *Mathematical Recreations and Essays*, Macmillan and Company: London, 1944, 418 pp.

[75] Anthropologists have found that tribes with limited number vocabularies ("one", "two", and "many") had elaborate ways of counting on their fingers, toes, and other parts of their anatomy in a specified order and entirely in their heads. Most primitive counting systems were based on 5, 10, or 20 (*vigesimal*) for the reason that the human animal has 5 fingers on one hand, 10 on both, and 20 fingers and toes. The ancient Chinese, Egyptians, Greeks and Romans used a base of 10. Babylonians, however, used the *sexagesimal* (base 60) which they adopted from the Sumerians, and with that they achieved a remarkably advanced mathematics. The 20–base system was used by the Mayans (together with zero and positional notation) in one of the most advanced of the ancient number systems.

The ancient Greeks and the Romans had an elaborate hand symbolism which they used for counting from one to numbers in the thousands. So did the ancient Chinese and other Oriental cultures. ***Luca Pacioli*** (1494) illustrated the Italian finger symbolism common in the Medieval and Renaissance periods. Moreover, counting symbolism soon developed into *finger arithmetic* for multiplication. This was called for since few people in the Middle Ages and the Renaissance learned the multiplication table beyond 5×5 or had access to an abacus. A variety of simple methods were in use for obtaining the products of numbers from 6 through 10 using both hands.

The Greeks and the Romans were familiar with some mathematics associated with the game of dice. ***Plato*** *in his Laws (Book 12) cited* 3 *and* 18 *as the most difficult sums to roll with three dice. Indeed, they are the only sums that can be made in only one way (1-1-1 and 6-6-6). Since there are* $6^3 = 216$ *equally probable ways of rolling three dice, the probability of making either a* 3 *or an* 18 *is* $\frac{1}{216}$. *The Greeks called 6-6-6 "Aphrodite" and 1-1-1 "the dog".*

There are many references to these and other dicing terms in Greek and Latin literature. The Roman Emperor ***Claudius*** *even wrote a book called* How to Win at Dice *indicating the great popularity of the game among the upper classes (in Greece too). Apart from this, there is no evidence of any* theory *of combinations among the ancients.*

The Latin writers, having little interest in any phase of mathematics except the practical, paid almost no attention to the theory of combinations. The leading exception was ***Anicus Boethius*** *(475–534, Italy) who gave* $\frac{1}{2}n(n-1)$ *as the number of combinations of* n *things taken two at a time.*

The Hindus seem to have given the matter no attention until ***Bhaskara*** *(c. 1150) gave the rules for the permutations of* n *objects taken* k *at a time, with and without repetition, and the number of combinations of* n *objects taken* k *at a time without repetition.*

At about the same time similar results were obtained independently in China and South-Western Europe: in Spain, the great Jewish savant ***Abraham Ibn-Ezra*** *seemed to have been aware (c. 1140) of the rule for finding the combination of* n *objects taken* k *at a time [he knew that* $\binom{7}{2} = \binom{7}{5}$; $\binom{7}{3} = \binom{7}{4}$; $\binom{7}{6} = \binom{7}{1}$*].*

Levi ben Gershon *(1321) in his* Maasei Choscheb *(Work of the Computer), carried the subject considerably farther. He gave the formula* $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ *for the number of combinations of* n *objects, taken* k *at a time, together with the fact that there are* $n!$ *permutations of* n *elements*[76].

[76] In this treatment of permutations and combinations Levi ben Gershon comes very close to using *mathematical induction*, if not actually inventing it [Rabinovitch, N.L., *Arch. Hist. Ex. Sci.* **6** (1969) 237–248].

Early in the Christian Era there developed a close relation between mathematics and the mystic philosophy of the Hebrews known as *Kabbalah*. This led to the belief in the mysticism of arrangements and hence to the study of permutations and combinations. The movement seems to have begun in the anonymous *Sefer Yetzira* (Book of Creation), composed probably between the 3^{rd} and 6^{th} centuries CE in Israel. It seemed to have attracted the attention of the Arabic and Hebrew writers of the Middle Ages in connection with astronomy. They considered it w.r.t. the conjunction of planets, seeking to find the number of ways in which Saturn could be combined with each of the other planets in particular, and, in

The subject of permutations had a feeble beginning in China in the 12^{th} century, but most of the relevant literature was lost.

***Oresme** (ca 1360) wrote a work in which he gave the sum of numbers representing the combinations of* 6 *objects taken* $1, 2, 3, 4$ *and* 5 *at a time. He also gave* $\binom{6}{2} = 15$, $\binom{6}{3} = 20$, *etc., in rhetorical form.*

*First evidence of permutations in print is found in **Pacioli**'s Suma (1494), where he showed how to find the number of permutations of any number of persons sitting at a table. **Tartaglia** (1523) seems first to have applied the theory of the throwing of a dice. In a book Pardes Rimmonim (Orchard of Pomegranates, 1548) the Jewish Kabbalist **Moshe Cordovero** (1522–1570, Israel) gave an interesting treatment of permutations and combinations and showed some knowledge of the general laws governing them.*

*At about the same time **Joannes Buteo** (1492–1572, France) discussed (1559) the question of the number of possible throws with* 4 *dice. The first writer to publish the general rule that* $\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$ *was **Pierre Herigone** (1634).*

***Pascal** (1654) united the algebraic and combinatorial theorems by showing that* $(a+b)^n$ *is a generating function for the number of combinations of* n *objects, taken* k *at a time*[77]*. As an application, he founded the mathematical theory of probability, and as a method of proof he used mathematical induction for the first time in a conscious and unequivocal way.*

*An interest in logic led **Leibniz** to write the essay Dissertatio de arte combinatoria (1666). His aim was "a general method in which all truths of reason would be reduced to a kind of calculation". Leibniz foresaw that permutations and combinations would be involved, but he did not make enough progress to interest* 17^{th} *century mathematicians in the project.*

*The true pioneer of combinatorial analysis was **Abraham de Moivre**, who first published in Phil. Trans. (1697) the form of the general coefficient in the expansion of* $(a + bx + cx^2 + dx^3 + \cdots)$ *raised to any power. His work on probability would naturally lead him to consider questions of this nature.*

In 1730 *he introduced the powerful method of generating functions (for the Fibonacci numbers). This method has been of great importance in combinatorics, probability and number theory.*

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general, the number of combinations of the known planets taken two at a time, three at a time, and so on.

77 The credit for discovering the *Pascal triangle* goes to the Chinese mathematicians **Yang Hui** (1261) and **Zhu Shijie** (1303). This is not the only instance of a mathematical concept being named after a rediscoverer rather than a discoverer, but Pascal deserves here credit for more than just rediscovery.

Mathematicians of the 18^{th} *century applied the algebra of permutations and combinations to solve a host of arithmetical and geometrical problems, some of which had immediate application to probability theory. Although it is generally agreed that the doctrine of probability has been founded by Pascal and Fermat, the need for the theory arose already with regard to throwing of dice and other gambling questions. In the mathematical work Suma (1494) by* ***Pacioli****, two gamblers are playing for a stake which is to go to the one who first wins* n *points, but the play is interrupted when the first has* p *points and the second* q *points. It is required to know how to divide the stakes. The general problem also appears in the works of* ***Cardan*** *(1539) and of* ***Tartaglia*** *(1556).*

The first printed work on the subject was a tract of ***Huygens*** *(1657). There also appeared (1708) an essay on the subject by* ***Pierre-Rémond de Montmort*** *(1678–1719). However, the first book devoted entirely to the theory of probability was Ars Conjectandi (1713) by* ***Jakob Bernoulli****. The second book on the subject was* ***De Moivre****'s Doctrine of Chances (1718). One of the best known works on the theory of probability is* **Laplace***'s Théorie analytique des probabilites (1812). In this is given his proof of the method of least squares.*

The application of the theory to mortality tables started with ***John Graunt****'s book Natural and Political Observations (London, 1662). The first tables of great importance, however, were those of* ***Edmund Halley*** *(1663) in his memoir on Degrees of Mortality of Mankind, in which he made a careful study of annuities. Although a life-insurance policy is known to have been underwritten in London in* 1583*, it was not until* 1699 *that a well-organized company was established for this purpose.*

A few typical examples of problems of historical and aesthetical value are given below:

- The Bernoulli-Euler problem of misaddressed letters

Someone writes n *letters and writes the corresponding addresses on* n *envelopes. How many different ways are there of placing all the letters in the wrong envelopes?*

This problem was first considered by ***Nicholas Bernoulli*** *(1687–1759), the nephew of Jakob and Johann Bernoulli. Later* ***Euler*** *became interested in the problem, which he solved independently of Bernoulli. This problem is particularly interesting because of its ingenious solution:*

Let u_n *be the sought number of ways. Pick a certain letter; by definition it is in a wrong envelope. Then there are two possibilities:*

(I) The letter that matches the wrong envelope was placed in an envelope that matches the originally selected letter. This cross-derangement can occur in $(n-1)$ *ways. The remaining* $(n-2)$ *letters can be misaddressed in* u_{n-2} *ways. The total number of derangements of this type is therefore* $(n-1)u_{n-2}$.

(II) While the originally selected letter is placed in a wrong envelope, the matching envelope for that letter does not *host the letter matching the wrong envelope.* Pretending *that the latter letter-envelope pair is matched, the number of configurations of type II is found to be* u_{n-1} *per choice of wrong envelope, i.e.* $(n-1)u_{n-1}$. *Altogether, adding the counts for cases I and II:*

$$u_n = (n-1)(u_{n-1} + u_{n-2}).$$

This difference equation is solved by

$$u_n = n!\left[\frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right].$$

Montfort[78] *(1713) solved this problem (le problème de recountres) by effectively using what is known today as* The principles of inclusion and exclusion, *which may have been known to the Bernoullis. It states:*

If of N *objects,* $N(a)$ *have a property* a, $N(b)$ *property* $b, \ldots, N(ab)$ *both* a *and* $b, \ldots, N(abc)$ a, b, *and* c, *and so on, the number* $N(a'b'c')$ *with* none *of these properties is given by*

$$\begin{aligned} N(a'b'c'\ldots) = N - N(a) - N(b) - \cdots \\ +N(ab) + N(ac) + \cdots \\ -N(abc) - \cdots \\ +\cdots \end{aligned}$$

The proof by mathematical induction is simple once it is noted that the formula $N(a') = N - N(a)$ *can be applied to any collection of properties which is suitably defined.*

The principle of inclusion and exclusion is an important combinatorial tool.

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[78] **Pierre de Montfort** (1678–1715, France). Mathematician. Made a systematic study of games of chance and contributed to combinatorics.

- EULER'S PROBLEM OF POLYGON DIVISION

In how many ways can a plane convex polygon of n sides be divided into triangles by non-intersecting diagonals inside the polygon?

Euler *posed this problem (1751) to* ***Christian Goldbach****. He then communicated it to* ***Johann Andreas von Segner*** *(1704–1777, Germany), disclosing the first seven division numbers E_n*

n	3	4	5	6	7	8	9
E_n	1	2	5	14	42	132	429

Segner was able to derive a recursion relation for E_n

$$E_n = E_2E_{n-1} + E_3E_{n-2} + \cdots + E_{n-1}E_2$$

where $E_2 \equiv 1$. His solution matched Euler's own result

$$E_n = \frac{2^{n-2}(2n-5)!!}{(n-1)!}.$$

- STEINER'S PROBLEM *(1826)*

n lines are drawn in the Euclidean plane in such a way that no 3 are concurrent and no 2 are parallel. What is the maximal number of regions formed?

Let P_n denote this number. An additional line will cut all previous lines, creating $(n+1)$ new regions. Therefore $P_{n+1} = P_n + (n+1)$. This equation, augmented by the initial condition $P_1 = 2$, is uniquely solved by $P_n = 1 + \frac{1}{2}n(n+1)$.

- Josephus[79] problem *(Tartaglia, 1546)*

Arrange the numbers $1, 2, \ldots, n$ *consecutively (say, clockwise) about the circumference of a circle and proceed clockwise to remove every* q^{th} *number. Let* $J_q(n)$ *denote the final number which remains for a given pair* (q, n)*, i.e. the last survivor.*

For $q = 2$ *(removing number* 2 *and then every other number) the survivor's-table has the form*

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$J_2(n)$	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1

Suppose that we have $2n$ *people originally. After the first go-around, all even numbers will be eliminated and* 3 *will be the next to go. This is just like starting out with* n *people, except that each person's number has been doubled and decreased by* 1*. That is* $J_2(2n) = 2J_2(n) - 1$*, for* $n \geq 1$*. If we start with* $(2n + 1)$ *people, it turns out that person number* 1 *is wiped out just after person number* $2n$*. Again, we almost have the original situation as with* n *people, but this time their numbers are doubled and increased by* 1*. Thus* $J_2(2n + 1) = 2J_2(n) + 1$ *for* $n \geq 1$*.*

Altogether we have

$$\begin{aligned} J_2(1) &= 1 \\ J_2(2n) &= 2J_2(n) - 1, \qquad \text{for } n \geq 1, \\ J_2(2n+1) &= 2J_2(n) + 1, \qquad \text{for } n \geq 1. \end{aligned}$$

[79] The Latin writer **Hegesippus** (340–397 CE) tells us that the Jewish historian **Josephus** saved his life by knowing the solution to this problem for $q = 3$, $n = 41$. According to his account, after the Romans had captured Yodfat, Josephus and 40 other Judean freedom fighters took refuge in a cave. His companions were resolved to die rather than fall into the hands of the Romans. Josephus and one friend, not wishing to die yet not daring to dissent openly, feigned to agree. Josephus even proposed an arrangement by which the deaths might take place in an orderly manner: The men were to arrange themselves in a circle; then every third man was to be killed until but one was left, and he must commit suicide. Josephus and his friend placed themselves in places 16 and 31. This kind of 'lottery', which Josephus adopted, was similar to that used by the priests of the Second Temple (515 BCE–70 CE) in Jerusalem to win their various daily service-jobs [*Yoma* **2**, 2 (Mishna); *Yoma* **2**, 1 (Yerushalmi); *Yoma* **22**, 2 (Bavli)]. The priests stood in a circle, each one pointing one or two fingers toward the man in charge at the center. This man would then announce a number (usually 100 or 60) which was larger than the total number of participating priests, and then *count fingers* in a specified direction from a certain fiducial person, ending the count of the preassigned number at the winner.

It follows from these recursion relations that $J_2(2) = 2J_2(1) - 1 = 1$, $J_2(4) = 2J_2(2) - 1 = 1$, *etc. and in general, for all* m, $J_2(2^m) = 1$. *Hence we know that the first person will survive whenever* n *is a power of* 2. *In the general case* $n = 2^m + \ell$, *where* 2^m *is the largest power of 2 not exceeding* n, *the number of people is reduced to a power of 2 after there have been* ℓ *"executions". The eventual survivor is number* $2\ell + 1$ *in the original ordering, i.e.*

$$J_2(2^m + \ell) = 2\ell + 1, \quad \text{for } m \geq 0 \quad \text{and} \quad 0 \leq \ell < 2^m.$$

This can be proved by induction in two steps, depending on whether ℓ *is even or odd: If* $m > 0$ *and* $2^m + \ell = 2n$, *then* ℓ *is even and*

$$J_2(2^m + \ell) = 2J_2(2^{m-1} + \ell/2) - 1 = 2(2\ell/2 + 1) - 1 = 2\ell + 1,$$

by the induction hypothesis. A similar proof works in the odd case, when $2^m + \ell = 2n + 1$. *We might also note that*

$$J_2(2n + 1) - J_2(2n) = 2.$$

Either way, the induction is complete and the closed-form solution is established.

To illustrate the solution we compute $J_2(100)$. *In this case we have* $100 = 2^6 + 36$, *so* $J_2(100) = 2 \cdot 36 + 1 = 73$.

There is no closed-form solution to the Josephus problem for $q > 2$, *not even a recurrence relation. There is however a* computer recipe

$$J_q(n) = qn + 1 - D_k^{(q)},$$

where $D_0^{(q)} = 1$, $D_n^{(q)} = \frac{q}{q-1} D_{n-1}^{(q)}$ *for* $n > 0$, *and* k *is* as small as possible *such that* $D_k^{(q)} > (q-1)n$.

- Rook problems

In how many ways can n *rooks be placed on an* $n \times n$ *chessboard so that no rook can attack another?*

If the rooks are unnumbered *the answer is* $n!$ *since there is exactly 1 rook in each row and each column and thus each configuration of the* n *rooks is a different permutation of* n *objects (numbers). If the rooks are* numbered *from*

1 to n, the answer is $(n!)^2$, since there are $n!$ ways of placing the numbered rooks (the result of permuting the latter).

Now, if the rooks are restricted to avoid the main diagonal, every position of the non-attacking rooks is a derangement of n objects, and we fall back on the Bernoulli-Euler problem of misaddressed letters, with the result (for un-numbered rooks)

$$n!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots+(-1)^n\frac{1}{n!}\right].$$

Another interesting problem is to find the number of ways of arranging n un-numbered rooks on an $n \times n$ chessboard such that every square of the board is controlled by at least one of them.

The number of ways of arranging n rooks, one in each column, is n^n (the first rook can be placed on any of the n squares of the first column; no matter which square it is put on, the second rook can be put on any of the n squares of the second column, etc.).

The same argument applies to the rows, and it would seem at first glance that the number of arrangements of n rooks for which the rooks controlled all squares of the board would equal to $2n^n$. But in this enumeration we have counted twice each arrangement of the rooks for which there is one rook in each column and simultaneously one rook in each row. Since the total number of such arrangement is $n!$, the correct answer is $2n^n - n!$.

In particular, for an ordinary chessboard $(n = 8)$, we obtain $2 \cdot 8^8 - 8! = 33,514,312$ different arrangements.

- Problems of occupancy

Starting with the basic permutations and combinations of N objects, there is a large class of more complicated cases, some of which require very involved and tricky reasoning that leads to fancy mathematical formulations. We shall look at a few just to appreciate how powerful such methods can be.

Let there be a set of N objects (e.g. balls) which should be placed in a set of n compartments (e.g. urns, boxes, etc.). The number of ways in which the distribution can be effected will depend upon two factors:

(1) Whether the order of the urns, even including empty ones, is taken into account, i.e. whether the urns are distinguishable (alias different, alias labeled, distinct) or indistinguishable (alias identical).

(2) Whether the order of the balls within the urns is taken into account, i.e. whether the balls are distinguishable or not.

The simplest case is when both balls and urns are different. There are n choices for each ball as to which urn it will be placed in. The total number of independent choices is therefore n^N (empty urns are allowed).

If in the previous problem the balls are identical and they are placed in n distinct urns or fewer (i.e. empty urns are allowed) the counting proceeds as follows:

Due to the indistinguishability of the balls, any single distribution can be symbolically represented by short vertical bars (walls of urns) and circles (balls), e.g. $|00|000|0|\cdots$. We must begin and end with walls, and we must have in each distribution N balls and $n-1$ internal walls. So we merely have to count the number of ways to line up N balls and $n-1$ internal walls. There are $N+(n-1)$ positions in a line-up and N of them must be balls. Therefore the answer is $\binom{N+n-1}{N} \equiv \binom{N+n-1}{n-1}$. When no cell is empty, the result turns out to be $\binom{N-1}{n-1}$.

Note that we are actually asking here how many solutions are there, in non-negative (or alternatively positive) integers, to the equation $x_1+x_2+\cdots+x_n=N$, where $(x_1,\ldots,x_n)$ is an ordered n-tuple.

The next problem is to count the number of ways in which N different balls can be arranged in exactly n different urns (no empty urn).

The number of arrangements in which empty urns are admissible is n^N; the number of arrangements in which one assigned empty urn is admissible is $(n-1)^N$, and so on. Hence, by the principle of inclusion and exclusion, the sought number[80] is $T(N,n)=\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}(n-k)^N$.

Using this we can find the number of ways $B(N,n)$ to put N different balls into n identical urns, with no empty urns. This amounts to $B(N,n)=\sum_{j=0}^{n}\frac{1}{j!}T(N,j)$.

Finally, in how many ways can N identical balls be put in n identical urns? There is no nice, closed-form solution to this problem.

- Rolling dice

Count the number of ways of obtaining the sum N with n dice?

The number of arrangements in which the partition $x_1+x_2+\cdots+x_n=N$ is effected, with no zero values for any x_i, is $\binom{N-1}{n-1}$. Then, since each x_i is

[80] This is shown to equal the coefficient of x^N in the expansion of $N!(e^x-1)^n$.

limited by the set $\{1,2,3,4,5,6\}$, *the principle of inclusion and exclusion yields the result*

$$\binom{N-1}{n-1} - \binom{n}{1}\binom{N-7}{n-1} + \binom{n}{2}\binom{N-13}{n-1}\cdots$$

- Sum of divisors

Let $p_1,\dots,p_n$ *be distinct primes. What is the number of divisors of the number* $q=(p_1)^{\alpha_1}(p_2)^{\alpha_2},\dots,(p_n)^{\alpha_n}$ *where* $\alpha_1,\dots,\alpha_n$ *are natural numbers (including the divisors* 1 *and* q*) and what is the sum of these divisors?*

Each prime p_k *can enter a divisor of* q *with one of* (α_k+1) *exponents* $0,1,\dots,\alpha_k$. *By the rule of product, the number of divisors is*

$$(\alpha_1+1)\cdots(\alpha_n+1).$$

To compute their sum, we consider the expression

$$(1+p_1+\cdots+p_1^{\alpha_1})\cdots(1+p_n+\cdots+p_n^{\alpha_n}).$$

In performing the product, we obtain a sum in which each divisor of q *appears exactly once. Using the formula for the sum of a geometric progression, the above product — and therefore also the required sum of the divisors of* q *— is seen to have the value*

$$\frac{p_1^{\alpha_1+1}-1}{p_1-1}\cdots\frac{p_n^{\alpha_n+1}-1}{p_n-1}$$

- The birthday problem

If you know more than 23 people's birthdays, it is more likely than not that two of them occur on the same day.

Consider the probability that n *people's birthdays are all different, i.e. that in a random selection of* n *days out of 365 there shall be no day counted more than once. The total number of possible selection is* $(365)^n$, *and the number of selection in which no day is counted more than once is* $365\cdot364\cdot\cdots\cdot(365-n+1)$. *The probability is therefore*

$$p(n)=\left(1-\frac{1}{365}\right)\left(1-\frac{2}{365}\right)\cdots\left(1-\frac{n-1}{365}\right).$$

The occurrence is as likely as not if $p(n)=\frac{1}{2}$. *By taking logarithms, we obtain approximately* $\frac{1}{365}+\frac{2}{365}+\cdots+\frac{n-1}{365}=\log_e 2$ *or* $n(n-1)\cong 506$, *whence*

$n = 23$. The probability that at least two persons in a room containing n persons will have the same birthday is $q(n) = 1 - p(n)$. For $n = 64$, $q(n) = 0.997$.

- *n people are seated around a table, $n > 2$. What is the probability that two persons will be neighbors?*

There are $\frac{n!}{n} = (n-1)!$ arrangements in toto. Of these, $2(n-2)!$ are favorable. Therefore, the sought probability is $p(n) = \frac{2}{n-1}$.

- *Given a convex planar irregular (no three diagonals intersect) n-gon. Enumerate:*

(a) *the total number of diagonals;*

(b) *the number of interior intersection points generated by the diagonals;*

(c) *the number of interior regions generated by the intersecting diagonals.*

A diagonal corresponds to a 2-subset of vertices, of which there are $\binom{n}{2}$. However, not every 2-subset gives a diagonal: the n-pairs of adjacent vertices give sides. *Thus, the first requested number is $\binom{n}{2} - n = \frac{n(n-3)}{2}$.*

Likewise, there is a one-to-one correspondence between interior intersection points and combinations of vertices taken 4 at a time. Hence the second sought number is $\binom{n}{4}$.

Finally, using mathematical induction, the number of regions is found to be $\binom{n}{4} + \binom{n-1}{2} = \frac{1}{24}(n-1)(n-2)(n^2 - 3n + 12)$.

For a given n-gon, all these results are independent of the specific shape of the polygon.

- KIRKMAN'S[81] SCHOOL-GIRLS PROBLEM *(1850)*

In a boarding school there are 15 schoolgirls who always take their daily walks in 5 rows of threes. How can it be arranged so that each schoolgirl walks in the same row with every other schoolgirl exactly once a week?

[81] **Thomas Kirkman** (1806–1895, England). Mathematician. Contributed to combinatorial mathematics. Showed the existence of *Steiner systems* seven years before Steiner's article on the subject.

Let the girls be labeled $X_1, \ldots, X_7, Y_1, \ldots, Y_7, Z$. *The solution is*

Day 1	X_1Y_1Z	$X_2X_6Y_4$	$X_3X_4Y_7$	$X_5X_7Y_6$	$Y_2Y_3Y_5$
Day 2	X_2Y_2Z	$X_3X_7Y_5$	$X_4X_5Y_1$	$X_6X_1Y_7$	$Y_3Y_4Y_6$
Day 3					
Day 4					
Day 5					
Day 6					
Day 7	X_7Y_7Z	$X_1X_5Y_3$	$X_2X_3Y_6$	$X_4X_6Y_5$	$Y_1Y_2Y_4$

The solution has the nice property that the triplets for each day can be obtained from those of the previous day by replacing X_i *by* X_{i+1}, Y_i *by* Y_{i+1} $(i \leq 6)$, X_7 *by* X_1, Y_7 *by* Y_1.

- CHANGING A DOLLAR

In how many ways can a dollar be changed into pennies, nickels, dimes, quarters and half-dollars? A painstaking naive counting yields at length the number 292. Sylvester (1855) developed a general theory *which enables one to derive results like these through a systematic fast algorithm. But even with this tool, the following consideration is useful: In general, if the change adds up to N cents and consists of coins of denomination* $n_1, n_2, \ldots, n_k$ *(no restriction on the number of coins of different denominations), one easily derives the recursion relation*

$$D(N; n_1, n_2, \ldots, n_k) = D(N; n_1, n_2, \ldots, n_{k-1}) + D(N - n_k; n_1, n_2, \ldots, n_k)$$

where the left hand side is the sought number of ways to change N *cents with denominations* $\{n_1, n_2, \ldots, n_k\}$. *The above relation shows that if no* n_k*-cent coin is included in the change, then the full sum* N *is made up of coins of lesser denominations* $\{n_1, n_2, \ldots, n_{k-1}\}$, *and if* at least *one* n_k*-cent coin is used, then the remainder* $(N - n_k)$ *may include coins of denomination* $\{n_1, n_2, \ldots, n_k\}$. *Applying this to our problem we get*

$$D(100; 1, 5, 10, 25, 50) = D(100; 1, 5, 10, 25) + D(50; 1, 5, 10, 25, 50).$$

A repeated application of this relation, leads to the above result after a few steps.

It is clear from the above examples that combinatorial mathematics *is first of all concerned with counting the number of ways of arranging given objects*

in a prescribed way (i.e. satisfying certain conditions). Generally speaking, both the theoretical analysis and the actual construction of discrete sets *are much more difficult than those problems in analysis concerning infinite sets.*

The main emphasis and the name of this field have changed from time to time and from person to person. Other names such as combinatorial analysis, combinatorial theory *and recently* discrete mathematics *have also been used to describe the same field*[82].

- COMBINATORIAL ANALYSIS *(1846–1898)*

Up to the middle of the 19^{th} *century, problems of combination were generally undertaken as they became necessary for the advancement of some particular part of mathematical science; it was not recognized that the theory of combinations is in reality a science in and of itself, well worth studying for its own sake irrespective of applications to other parts of analysis. There was a total absence of orderly development, and until* 1846*, Euler's classical paper remained the only method of combinatorial analysis*[83]*. [Other writers who have contributed to the solution of special problems are* ***James Bernoulli****,* ***Ruggiero Boscovich****,* ***Karl Friedrich Hindenburg*** *(1741–1808),* ***William Emerson*** *(1701–1782),* ***Robert Woodhouse*** *(1733–1827),* ***Thomas Simpson*** and ***Peter Barlow****.]*

In 1846 ***Carl G.J. Jacobi*** *studied the partitions of numbers by means of certain identities involving infinite series that are met in the theory of elliptic functions. Further advance was made by* ***Arthur Cayley*** and ***Joseph Sylvester*** *(1855) and during 1888–1898 by* ***Pery Alexander MacMahon*** *(1854–1929).*

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82 In the 20^{th} century, the subject has come a long way since Kirkman's time and the days are past when the calculus was thought to be the undisputed queen of applied mathematics.

83 *De Partitione Numerorum* (1748), in which the consideration of the reciprocal of the product $(1-ax)(1-ax^2)(1-ax^3)\cdots$ establishes a fundamental connection between arithmetic and algebra through the identity

$$\frac{1}{(1-ax)(1-ax^2)\cdots(1-ax^n)} = 1+ax\frac{1-x^n}{1-x}+a^2x^2\frac{(1-x^n)(1-x^{n+1})}{(1-x)(1-x^2)}+\cdots .$$

Here **Euler** showed that he could convert arithmetical addition into algebraic multiplication and by that he gave the complete formal solution of the main problem of the *partition of numbers*.

On Chance

* * *

"*I returned and saw under the sun, that the race is not to the swift, nor the battle to the strong, neither yet bread to the wise, nor yet riches to men of understanding, nor yet favour to men of skill: but time and chance happeneth them all*".

Ecclesiastes **9** 11

* * *

"*Everything existing in the Universe is the fruit of chance and necessity*".

Democritos of Abdera (ca 460–370 BCE)

* * *

"*The probable is what usually happens*".

Aristotle (384–322 BCE)

* * *

"*Probability is the very guide of life*".

Marcus Tullius Cicero (ca 50 BCE)

* * *

"*The only certainty is that there is nothing certain*".

Pliny the Elder (23–79 *CE*)

* * *

"*It is a truth very certain that when it is not in our power to determine what is true we ought to follow what is most probable*".

René du Perron Descartes (1596–1650)

* * *

"*It is remarkable that a science which began with the consideration of games of chance should have become the most important object of human knowledge*".

Pierre Simon de Laplace (1749–1827)

* * *

"*The most important questions of life are, for most part, really only problems of probability*".

Pierre Simon de Laplace

* * *

"*Fate, time, occasion, chance, and change — to these all things are subject*".

Percy Bysshe Shelley (1792–1822)

* * *

"*In the field of observation, chance favours the prepared mind*".

Louis Pasteur (1822–1895)

* * *

"*No victor believes in chance*" (Kein sieger glaubt an den zufall).

Friedrich Wilhelm Nietzsche (1844–1900)

* * *

"*Chance is the pseudonym of God when he did not want to sign*".

Anatole France (1844–1924)

* * *

"*I can believe anything, provided it is incredible*".

Oscar Wilde (1854–1900)

* * *

"*The record of a month's roulette playing at Monte Carlo can afford us material for discussing the foundations of knowledge*".

Karl Pearson (1857–1936)

* * *

"*The conception of chance enters into the very first steps of scientific activity in virtue of the fact that no observation is absolutely correct. I think chance is more fundamental concept than causality; for whether in a concrete case, a cause-effect relation holds or not can only be judged by applying the laws of chance to the observation*".

Max Born (1882–1970)

* * *

"*The luck of having talent is not enough; one must also have a talent for luck*".

Hector Berlioz

* * *

"*The harder you work, the luckier you get*".

Anon

* * *

"*What is good luck for the early bird is bad luck for the early worm*".

Anon

* * *

"*Depend on the rabbit foot if you will, but remember — it didn't work for the rabbit*!"

Anon

* * *

1641–1672 CE **Franciscus (Franz) Sylvius de la Boë** (1614–1672, The Netherlands). German born physician, anatomist and chemist. Taught that the functions of the living organism were mainly determined by chemical activities ("effervescences"), particularly by acidic or alkaline characters of body fluids (a precursor of the modern cult of pH). He represents the culmination of chemical medicine (*Iatrochemistry*).

Professor of medicine at Leyden (1658–1672), where he built the first university chemical laboratory. Discovered (1614) *Sylvian fissure* in brain.

1642–1644 CE **Abel Janszoon Tasman** (1603–1659, The Netherlands). The greatest of Dutch navigators and explorers. Discovered *Tasmania* and *New Zealand* (1642), and *Tonga* and the *Fiji* Islands (1643); On his second voyage (1644) he discovered the *Gulf of Carpentaria*.

Tasman entered the service of Dutch East India Co. (1633); sent by Anthony van Diemen, governor general of Dutch East Indies, on expeditions to Indian and Australian waters (Aug. 1642) in quest of "islands of gold and silver".

Tasman was born at Lutjegast at Groningen. Although Tasman contributed to the extension of the Dutch colonial empire, his achievements were coldly received by the Dutch colonial authorities.

The first people to live in New Zealand were the *Maoris*. They arrived around 750 *CE* from the Cook, Marquesas, or Society Islands (NE of New Zealand) by canoes.

Tasman tried (1642) to send a group of men ashore, but the Maori attacked their two small landing crafts and killed several of the men. Tasman made no further attempt to land. No other European came to New Zealand until 1769, when Captain James Cook landed on the North Island, made friends with the Maoris, and explored and charted both the North Island and the South Island. In 1840 the Maoris signed the *Treaty of Waitangi*, which gave Great Britain the sovereignty over New Zealand.

1642–1655 CE **Cyrano de Bergerac** (**Savinien de**) (1619–1655, France). Playwright, soldier and writer of science-fiction and philosophical fiction. He was acquainted with all the philosophical trends of his period (Scholasticism, which he attacked; Skepticism, Epicureanism as revived by Gassendi, Cartesianism, and the Italian philosophers of the Renaissance), and was aware of all the recent discoveries in astronomy and physics since Copernicus, Kepler and Galileo, and in medicine since Harvey. His novels show him as a keen and talented popularizer, and contain amazing forecasts of many

later developments in science and technology such as: *the unity of matter, its atomic structure, phagocytes, animal intelligence, aviation, the gramophone, and X-rays*. Known for his sword-fighting and for his long nose [Edmond's Rostand's famous play *Cyrano de Bergerac* (1897) contains a somewhat fanciful account of Cyrano's colorful life].

Cyrano was born in Paris. He received his first education from a country priest. At the age of 19 he entered the corps of the guards, serving in the campaigns of 1639 and 1640, and began the series of exploits that were to make him a veritable hero of romance. After 2 years of this life Cyrano left the service and returned to Paris to pursue literature and science studies, becoming a pupil of **Gassendi**.

Among his writings are two fantastic voyages: *L'autre monde ou les états et empires de la lune* and *Des états et empires du soleil* (1654) (published posthumously 1657 and 1662, after being purged of many religious and philosophical audacities; tr. *Voyages to the Moon and the Sun*, 1923).

Only after 20^{th} century scholarship made the complete text of his novels available, did his talent and originality receive full recognition.

Cyrano's ingenious mixture of science and romance has furnished a model for many subsequent writers, among them **Jonathan Swift** (1667–1745, England) and **Edgar Allan Poe** (1809–1849, U.S.A.). He adopted his fanciful style both for safely conveying ideas that might be regarded as unorthodox, and to relax from the serious study of physics.

Cyrano spent a stormy existence in Paris and was involved in many duels. He entered the household of duke d'Arpajon as secretary in 1653, and died two years later as a result of injuries following an accident.

1642–1656 CE **Thomas Hobbes** (1588–1679, England). Philosopher. Best known for his political philosophy, based on the idea of social contract, for purpose of security of each individual, and absolute authority of a sovereign[84] (*Leviathan*, 1651).

In his travels on the Continent he met **Galileo**, **Gassendi** and **Mersenne**. In England he was friendly with **Bacon** and **Harvey**.

Hobbes was influenced by two developments of his time: the new system of *physics* that Galileo and others were working on and the English Civil War. Men, he concluded, are selfish. They are moved chiefly by desire for power

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[84] According to Hobbes, man creates social laws autonomically, with no dependence on God. This is clearly an antithesis to the teaching of Luther, who claimed that the death of Christ relieves us from all moral obligations to each other.

and by fear of others. Therefore, without an all powerful sovereign to rule them, men's lives would be 'poor, nasty, brutish and short'.

Though modern physics is not so materialistic as it seemed to be in the days of Hobbes and though men's motives are more complex than he supposed, Hobbes influence continues. He raised fundamental and challenging questions about the relationship between science and religion, between thought and physiological processes on which it is based, and the nature and limitations of political power. These are questions that men still struggle to answer. He is probably more important for the questions he asked than for the answers he gave.

Hobbes was born in Westport, England. He was educated at Oxford University, and served as secretary to **Francis Bacon**. During the Civil War in England he fled to the European continent, returning to England while Cromwell's protectorate was still in power.

1642–1680 CE **Johannes Hevelius** (Hevel, Hovels or Höwelcke, 1611–1680, Danzig). German astronomer. Founder of lunar topography. Discoverer of comets and the moon's libration in longitude.

He was born in Danzig (now Gdansk, Poland). Studied law at Leyden in 1630; traveled in England and France, and in 1634 settled in his native town as a brewer and town councilor. From 1639 his chief interest became centered on astronomy, though he took, throughout his life, a leading part in municipal affairs. In 1641 he built an observatory in his house, provided with a telescope (46 m focal length) which he constructed by himself.

Hevelius made observations of *sunspots* (1642–1645), devoted four years to charting the lunar surface, and published his results in *Selenographia* (1647). It is the first map of the side of the moon observable from earth. He discovered 4 comets (1652, 1661, 1661, 1672), and suggested the motion of such bodies in parabolic orbits round the sun.

On 26 September 1679, his observatory, instruments and books were destroyed by arson. He promptly repaired the damage, so far as to enable him to observe the great comet of December 1680, but his health suffered from the shock.

His catalogue of 1564 stars appeared posthumously (1690).

1643 CE Typhoid fever first identified or described with accuracy.

1643 CE **Evangelista Torricelli** (1608–1647, Italy). Geometer and physicist. A pupil of Galileo. Engaged in pre-calculus calculations of areas, arc-lengths and extremum. Applied Galileo's laws of motion to fluids, and invented the first barometer to measure air pressure, using mercury as fluid

in a 185 cm glass column sealed at the top. When the tube is upended in a dish, the mercury sinks to about 76 cm, leaving a partial vacuum at the top. Torricelli was motivated by his desire to understand why lift pumps were unable to raise water more than 10.37 m. He then correctly concluded that the 760 mm of mercury balanced the air pressure in the dish.

Mining engineers were long aware of the fact that a suction-pump could not draw water from depths greater than some 10 meters and there was no explanation of why such limit should exist. When the engineers of Cosimo de'Medici II failed in an attempt to build a suction-pump capable of lifting water from a depth of 17 meters, the problem was referred to Galileo, and finally solved by his brilliant pupil Torricelli (1644), who then announced that the pressure of the atmosphere was equivalent to a column of water over 10 meter in height. He predicted that the pressure of the atmosphere would fall with increasing altitude, a truth which he confirmed experimentally in 1647, when a barometer was carried to the top of a 1450 m high mountain in the Auvergne; the height in the mercury in it fell by 7.5 cm during the ascent.

Torricelli created the first man-made vacuum known to science, thus refuting the 2000 year old Aristotelian view that vacuum was impossible.

Aristotle was Exactly Wrong

The ancient Greeks believed that the air through which birds fly extended to the moon (Daedalus failed because human arms were insufficiently strong, but that was their only objection). As ***Aristotle*** *said: "Nature abhors a vacuum". Even* ***Kepler****, in his Somnium (1634) never considered air as a problem.*

In the mid 17^{th} century, the Catholic Church had its own "theory" of why there was no vacuum: "vacuum is nothing; since God is everywhere and in everything he could not be nowhere and in nothing". So the pope decreed that the vacuum did not exist and to talk about vacuum was considered heresy. And that was exactly the reason why the air pressure became a Protestant *program.*

Torricelli *not only created the first artificial vacuum and demonstrated that air had weight — he essentially* discovered outer space! *for if air did have*

weight, and if one assumes its density was uniform throughout the atmosphere, its weight implied that the atmosphere was only 8 *km high. Even if one assumed it thinned as it rose, the atmosphere certainly could not be more than some* 150 *km high. From that instant onward, man realized that he inhabited not a unique land in the universe filled with possible places of habitation, but a speck-sized island of life in a vast cosmos of life-inimical emptiness.*

Science has suddenly isolated man, and showed how precarious was his grip on nature. Nature, in fact, prefers a vacuum, and Aristotle was shown to be exactly wrong. How puny man suddenly became, and how horrific his universe. This was one of the first dreadful fears created by science, fears confirmed again and again when the Scottish astronomer ***Thomas Henderson*** *(1798–1844) was able to show (1831) that the Sun's nearest stellar neighbor was an ungodly* 40 *trillion km away; when* ***Charles Darwin*** *showed (1859) that man was just another animal; and when* ***Einstein*** *(1905) turned common sense regarding the most fundamental categories upside-down; and when (1945) science coupled with technology showed that they could destroy men's sense of the world together with the world itself.*

1644 CE **Marin Mersenne** (1588–1648, France). Mathematician and natural philosopher. A Franciscan friar who lived in one of the critical periods of scientific history, overlapping the lives of **Galileo** (1564–1642), **Fermat** (1601–1665) and **Descartes** (1596–1650). Contributing little himself, Mersenne's unique historical importance was his gift for stirring up profitable controversies among his friends. His main accomplishment was his correspondence with many of the intellectuals of his time and the meetings held in his quarters in the Minim convent in Paris. At one such gathering (1647), Pascal first met Descartes. Some 18 years after his death, this group of acquaintances had become the French Academy of Sciences. Thus we can see Mersenne as a catalyst, speeding up the exchange of ideas between others.

Mersenne was born at Oise, France. In 1604 he entered the Jesuit school of La Flèche, where he met Descartes. In 1609 he went to the Sorbonne in Paris. In 1611 he joined the order the Minims and moved (1619) to a cloister at the Place Royale, where he remained most of his life. In 1647 he traveled to meet with Fermat. The hot journey over bad roads wore him out and he died soon afterwards. All his mathematical works, except his bad guess about perfect numbers, quickly became obsolete.

Since the days of Greek science, philosophers were concerned with the mystical significance of natural numbers. The 6 days of creation and the 28 days of the lunar month drew attention to the so-called perfect numbers. Euclid had already proved in his '*Elements*' that if $2^k - 1$ is prime, then $2^{k-1}(2^k - 1)$ is a perfect number. Thus, interest aroused early in primes of the form $2^k - 1$. Clearly, if this number is prime, k itself must be a prime (the converse is however not true). In 1644, Mersenne made the incorrect conjecture that for $p \leq 257$, $2^p - 1$ is prime when and only when $p = 2$, 3, 5, 7, 13, 17, 19, 31, 67, 127 and 257. He made 2 sins of commission (67, 257) and 3 sins of omission (61, 89, 107).

Mersenne, believing in inquiry by experiments, measured in 1636 the speed of sound in air. He also observed during 1623–1647 that the intensity of sound is inversely proportional to the square of the source-observer distance. He made a distinction between sound frequency and sound intensity and found that sound velocity is independent of pitch and loudness. He discovered, before Galileo, that the frequency of swing of a pendulum is inversely proportional to the square root of its length. Although relationships between the frequencies of vibrating strings had been known since the time of Pythagoras (ca 540 BCE), the *absolute* frequency of a musical note was first measured by Mersenne, who published his results in "*Harmonie Universelle*" in 1636.

The Mersenne Primes[85] (1644–2003)

It took ***Frank Nelson Cole*** *(1861–1926) about 1000 hours to calculate, by 1903, that*

$$2^{67} - 1 = 193,707,721 \times 761,838,257,287.$$

(Nowadays, a fairly standard computer will render the result in 0.1 sec.)

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85 To dig deeper, see:

- Hardy, G.H. and E.M. Wright, *An Introduction to the Theory of Numbers*, Oxford University Press: Oxford, 1989, 426 pp.
- Ore, O., *Number Theory and its History*, McGraw-Hill Book Co., 1948, 370 pp.

In 1911, **R.E. Powers** *showed that Mersenne had also missed the primes* $2^{89}-1$ *and* $2^{107}-1$ *and in 1947 it was found that* $2^{257}-1$ *(having 78 digits) is composite, having three prime divisors*

$$\begin{aligned}&231,584,178,474,632,390,847,141,970,017,375,815,706,539,969,\\&331,281,128,078,915,168,015,826,259,279,871\\&=[535,006,138,814,359]\\&\quad\times[1,155,685,395,246,619,182,673,033]\\&\qquad\times[374,550,598,501,810,936,581,776,630,096,313,181,393]\end{aligned}$$

Table 3.3: Known Mersenne Primes

	p	Digits	Year	Discoverer	$M_p = 2^p - 1$
1	*2*	*1*	—	—	*3*
2	*3*	*1*	—	—	*7*
3	*5*	*2*	—	—	*31*
4	*7*	*3*	—	—	*127*
5	*13*	*4*	*1456*	*anonymous*	*8,192*
6	*17*	*6*	*1588*	*Cataldi*	*131,071*
7	*19*	*6*	*1588*	*Cataldi*	*524,287*
8	*31*	*10*	*1772*	*Euler*	*2,147,483,647*
9	*61*	*19*	*1883*	*Pervushin*	*2,305,843,009,213,693,951*
10	*89*	*27*	*1911*	*Powers*	
11	*107*	*33*	*1914*	*Powers*	
12	*127*	*39*	*1876*	*Lucas*	
13	*521*	*157*	*1952*	*computers*	
14	*607*	*183*	*1952*	*computers*	
15	*1279*	*386*	*1952*	*computers*	
16	*2203*	*664*	*1952*	*computers*	
17	*2281*	*687*	*1952*	*computers*	
18	*3217*	*969*	*1957*	*computers*	
19	*4253*	*1281*	*1961*	*computers*	
20	*4423*	*1332*	*1961*	*computers*	
21	*9689*	*2917*	*1963*	*computers*	
22	*9941*	*2993*	*1963*	*computers*	
23	*11213*	*3376*	*1963*	*computers*	
24	*19937*	*6002*	*1971*	*computers*	
25	*21701*	*6533*	*1978*	*computers*	
26	*23209*	*6987*	*1979*	*computers*	

Table 3.3: (Cont.)

	p	Digits	Year	Discoverer	$M_p = 2^p - 1$
27	44497	13395	1979	computers	
28	86243	25962	1982	computers	
29	110503	33265	1988	computers	
30	132049	39751	1983	computers	
31	216091	65050	1985	computers	
32	756839	227832	1992	computers	
33	859433	258716	1994	computers	
34	1257787	378632	1996	computers	
35	1398269	420921	1996	computers	
36	2976221	895932	1997	computers	
37	3021377	909526	1998	computers	
38	6972593	2098960	1999	computers	
39	13466917	4053946	2001	computers	

Table 3.4: Prime factors of $2^n - 1, \quad n \leq 128$

n		n	
2	**3**	30	$3 \cdot 3 \cdot 7 \cdot 11 \cdot 31 \cdot 151 \cdot 331$
3	**7**	**31**	**2147483647**
4	$3 \cdot 5$	32	$3 \cdot 5 \cdot 17 \cdot 257 \cdot 65537$
5	**31**	33	$7 \cdot 23 \cdot 89 \cdot 599479$
6	$3 \cdot 3 \cdot 7$	34	$3 \cdot 43691 \cdot 131071$
7	**127**	35	$31 \cdot 71 \cdot 127 \cdot 122921$
8	$3 \cdot 5 \cdot 17$	36	$3 \cdot 3 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 19 \cdot 37 \cdot 73 \cdot 109$
9	$7 \cdot 73$	37	$223 \cdot 616318177$
10	$3 \cdot 11 \cdot 31$	38	$3 \cdot 174763 \cdot 524287$
11	$23 \cdot 89$	39	$7 \cdot 79 \cdot 8191 \cdot 121369$
12	$3 \cdot 3 \cdot 5 \cdot 7 \cdot 13$	40	$3 \cdot 5 \cdot 5 \cdot 11 \cdot 17 \cdot 31 \cdot 41 \cdot 61681$
13	**8191**	41	$13367 \cdot 164511353$
14	$3 \cdot 43 \cdot 127$	42	$3 \cdot 3 \cdot 7 \cdot 7 \cdot 43 \cdot 127 \cdot 337 \cdot 5419$
15	$7 \cdot 31 \cdot 151$	43	$431 \cdot 9719 \cdot 2099863$
16	$3 \cdot 5 \cdot 17 \cdot 257$	44	$3 \cdot 5 \cdot 23 \cdot 89 \cdot 397 \cdot 683 \cdot 2113$
17	**131071**	45	$7 \cdot 31 \cdot 73 \cdot 151 \cdot 631 \cdot 23311$
18	$3 \cdot 3 \cdot 3 \cdot 7 \cdot 19 \cdot 73$	46	$3 \cdot 47 \cdot 178481 \cdot 2796203$
19	**524287**	47	$2351 \cdot 4513 \cdot 13264529$
20	$3 \cdot 5 \cdot 5 \cdot 11 \cdot 31 \cdot 41$	48	$3 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 97 \cdot 241 \cdot 257 \cdot 673$
21	$7 \cdot 7 \cdot 127 \cdot 337$	49	$127 \cdot 4432676798593$
22	$3 \cdot 23 \cdot 89 \cdot 683$	50	$3 \cdot 11 \cdot 31 \cdot 251 \cdot 601 \cdot 1801 \cdot 4051$
23	$47 \cdot 178481$	51	$7 \cdot 103 \cdot 2143 \cdot 11119 \cdot 131071$
24	$3 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 241$	52	$3 \cdot 5 \cdot 53 \cdot 157 \cdot 1613 \cdot 2731 \cdot 8191$
25	$31 \cdot 601 \cdot 1801$	53	$6361 \cdot 69431 \cdot 20394401$
26	$3 \cdot 2731 \cdot 8191$	54	$3 \cdot 3 \cdot 3 \cdot 3 \cdot 7 \cdot 19 \cdot 73 \cdot 87211 \cdot 262657$
27	$7 \cdot 73 \cdot 262657$	55	$23 \cdot 31 \cdot 89 \cdot 881 \cdot 3191 \cdot 201961$
28	$3 \cdot 5 \cdot 29 \cdot 43 \cdot 113 \cdot 127$	56	$3 \cdot 5 \cdot 17 \cdot 29 \cdot 43 \cdot 113 \cdot 127 \cdot 15790321$
29	$233 \cdot 1103 \cdot 2089$	57	$7 \cdot 32377 \cdot 524287 \cdot 1212847$
		58	$3 \cdot 59 \cdot 233 \cdot 1103 \cdot 2089 \cdot 3033169$
		59	$179951 \cdot 3203431780337$

Table 3.4: (Cont.)

60	3 · 3 · 5 · 5 · 7 · 11 · 13 · 31 · 41 · 61 · 151 · 331 · 1321
61	**2305843009213693951**
62	3 · 715827883 · 2147483647
63	7 · 7 · 73 · 127 · 337 · 92737 · 649657
64	3 · 5 · 17 · 257 · 641 · 65537 · 6700417
65	31 · 8191 · 145295143558111
66	3 · 3 · 7 · 23 · 67 · 89 · 683 · 20857 · 599 · 79
67	193707721 · 761838257287
68	3 · 5 · 137 · 953 · 26317 · 43691 · 131071
69	7 · 47 · 178481 · 10052678938039
70	3 · 11 · 31 · 43 · 71 · 127 · 281 · 86171 · 122921
71	228479 · 48544121 · 212885833
72	3 · 3 · 3 · 5 · 7 · 13 · 17 · 19 · 37 · 73 · 109 · 241 · 433 · 38737
73	439 · 2298041 · 9361973132609
74	3 · 223 · 1777 · 25781083 · 616318177
75	7 · 31 · 151 · 601 · 1801 · 100801 · 10567201
76	3 · 5 · 229 · 457 · 174763 · 524287 · 525313
77	23 · 89 · 127 · 581283643249112959
78	3 · 3 · 7 · 79 · 2731 · 8191 · 121369 · 22366891
79	2687 · 202029703 · 1113491139767
80	3 · 5 · 5 · 11 · 17 · 31 · 41 · 257 · 61681 · 4278255361
81	7 · 73 · 2593 · 71119 · 262657 · 97685839
82	3 · 83 · 13367 · 164511353 · 8831418697
83	167 · 57912614113275649087721
84	3 · 3 · 5 · 7 · 7 · 13 · 29 · 43 · 113 · 127 · 337 · 1429 · 5419 · 14449
85	31 · 131071 · 9520972806333758431
86	3 · 431 · 9719 · 2099863 · 2932031007403
87	7 · 233 · 1103 · 2089 · 4177 · 9857737155463
88	3 · 5 · 17 · 23 · 89 · 353 · 397 · 683 · 2113 · 2931542417
89	**618970019642690137449562111**
90	3 · 3 · 3 · 7 · 11 · 19 · 31 · 73 · 151 · 331 · 631 · 23311 · 18837001 · 1891 · 1127 · 911 · 8191 · 112901153 · 23140471537
92	3 · 5 · 47 · 277 · 1013 · 1657 · 30269 · 178481 · 2796203
93	7 · 2147483647 · 658812288653553079
94	3 · 283 · 2351 · 4513 · 13264529 · 165768537521

Table 3.4: (Cont.)

95	31 · 191 · 524287 · 420778751 · 30327152671
96	3 · 3 · 5 · 7 · 13 · 17 · 97 · 193 · 241 · 257 · 673 · 65537 · 22253377
97	11447 · 13842607235828485645766393
98	3 · 43 · 127 · 4363953127297 · 4432676798593
99	7 · 23 · 73 · 89 · 199 · 153649 · 599479 · 33057806959
100	3 · 5 · 5 · 5 · 11 · 31 · 41 · 101 · 251 · 601 · 1801 · 4051 · 8101 · 268501
101	7432339208719 · 341117531003194129
102	3 · 3 · 7 · 103 · 307 · 2143 · 2857 · 6529 · 11119 · 43691 · 131071
103	2550183799 · 3976656429941438590393
104	3 · 5 · 17 · 53 · 157 · 1613 · 2731 · 8191 · 858001 · 308761441
105	7 · 7 · 31 · 71 · 127 · 151 · 337 · 29191 · 106681 · 122921 · 152041
106	3 · 107 · 6361 · 69431 · 20394401 · 28059810762433
107	**162259276829213363391578010288127**
108	3 · 3 · 3 · 3 · 5 · 7 · 13 · 19 · 37 · 73 · 109 · 87211 · 246241 · 262657 · 279073
109	745988807 · 870035986098720987332873
110	3 · 11 · 11 · 23 · 31 · 89 · 683 · 881 · 2971 · 3191 · 201961 · 48912491
111	7 · 223 · 321679 · 26295457 · 319020217 · 616318177
112	3 · 5 · 17 · 29 · 43 · 113 · 127 · 257 · 5153 · 15790321 · 54410972897
113	3391 · 23279 · 65993 · 1868569 · 1066818132868207
114	3 · 3 · 7 · 571 · 32377 · 174763 · 524287 · 1212847 · 160465489
115	31 · 47 · 14951 · 178481 · 4036961 · 2646507710984041
116	3 · 5 · 59 · 233 · 1103 · 2089 · 3033169 · 107367629 · 536903681
117	7 · 73 · 79 · 937 · 6553 · 8191 · 86113 · 121369 · 7830118297
118	3 · 2833 · 37171 · 179951 · 1824726041 · 3203431780337
119	127 · 239 · 20231 · 131071 · 62983048367 · 131105292137
120	3 · 3 · 5 · 5 · 7 · 11 · 13 · 17 · 31 · 41 · 61 · 151 · 241 · 331 · 1321 · 61681 · 45622845
121	23 · 89 · 727 · 1786393878363164227858270210279
122	3 · 768614336404564651 · 2305843009213693951
123	7 · 13367 · 3887047 · 164511353 · 177722253954175633
124	3 · 5 · 5581 · 8681 · 49477 · 384773 · 715827883 · 2147483647
125	31 · 601 · 1801 · 269089806001 · 4710883168879506001
126	3 · 3 · 3 · 7 · 7 · 19 · 43 · 73 · 127 · 337 · 5419 · 92737 · 649657 · 77158673929
127	**170141183460469231731687303715884105727**
128	3 · 5 · 17 · 257 · 641 · 65537 · 274177 · 6700417 · 67280421310721

Table 3.4: (Cont.)

$$
\begin{aligned}
M_{131} &= 263 \cdot ... \\
M_{137} &= 32,032,215,596,496,435,569 \cdot \\
&\quad 5,439,042,183,600,204,290,159 \\
M_{139} &= 5,625,767,248,687 \cdot ... \\
M_{149} &= 86,656,268,566,282,183,151 \cdot ... \\
M_{151} &= 18,121 \cdot 55,871 \cdot 165,799 \cdot 2,332,951 \cdot ... \\
M_{157} &= 852,133,201 \cdot 60,726,444,167 \cdot 1,654,058,017,289 \cdot ... \\
M_{163} &= 150,287 \cdot 704,161 \cdot 110,211,473 \cdot 27,669,118,297 \cdot ... \\
M_{167} &= 2,349,023 \cdot ... \\
M_{173} &= 730,753 \cdot 1,505,447 \cdot 70,084,436,712,553,223 \cdot ... \\
M_{179} &= 359 \cdot 1,433 \cdot ... \\
M_{181} &= 43,441 \cdot 1,164,193 \cdot 7,648,337 \cdot ... \\
M_{191} &= 383 \cdot 7,068,569,257 \cdot 39,940,132,241 \cdot \\
&\quad 332,584,516,519,201 \cdot \\
&\quad 147322265321145317331353282383 \\
M_{193} &= 13,821,503 \cdot 61,654,440,233,248,340,616,559 \cdot ... \\
M_{197} &= 7,487 \cdot ... \\
M_{199} &= 164,504,919,713 \cdot ... \\
M_{211} &= 15,193 \cdot 60,272,956,433,838,849,161 \cdot ... \\
M_{223} &= 18,287 \cdot 196,687 \cdot 1,466,449 \cdot 2,916,841 \cdot \\
&\quad 1,469,495,262,398,780,123,809 \cdot ... \\
M_{227} &= 26,986,333,437,777,017 \cdot ... \\
M_{229} &= 1,504,073 \cdot 20,492,753 \cdot 59,833,457,464,970,183 \cdot ... \\
M_{233} &= 1,399 \cdot 135,607 \cdot 622,577 \cdot ... \\
M_{239} &= 479 \cdot 1,913 \cdot 5,737 \cdot 176,383 \cdot 134,000,609 \cdot \\
&\quad 7,110,008,717,824,458,123,105,014,279,253,754,096, \\
&\quad 863,768,062,879 \\
M_{241} &= 22,000,409 \cdot ... \\
M_{251} &= 503 \cdot 54217 \cdot 178,230,287,214,063,289,511 \cdot \\
&\quad 61,676,882,198,695,257,501,367 \cdot ... \\
M_{257} &= 535,006,138,814,359 \cdot \\
&\quad 1,155,685,395,246,619,182,673,033 \cdot ... \\
M_{263} &= 23,671 \cdot 13,572,264,529,177 \cdot \\
&\quad 120,226,360,536,848,498,024,035,943 \cdot ...
\end{aligned}
$$

The number M_{59} was factored by ***Landry*** *(1869), M_{67} by* ***Cole*** *(1903), M_{73} by* ***Poulet*** *(1923) and M_{113} by* ***Lehmer*** *(1947).*

It is obvious that Mersenne could not have tested the correct *results for* $p = 19, 31, 127$. *Some have believed that Fermat had communicated to him an as yet undiscovered theorem, since empirical methods could hardly have been used in Mersenne's time.*

So by 1947, Mersenne's range $n \leq 258$, *had been completely checked and it was determined that the correct list is:*

$$n = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107 \text{ and } 127.$$

It thus, took 304 years to set Mersenne right!

Apart from the challenge of calculating *Mersenne numbers for higher values of* n, *number theorists since Fermat endeavored ceaselessly to produce aids for recognizing if a Mersenne number is a prime, and if not, to determine its factors (a daunting task in light of the limited calculating devices at their disposal up to the middle of the* 20^{th} *century). Most theoretical results were found by Fermat, Euler and Lucas inside of a period of 250 years.*

The Mersenne primes $M_n = 2^n - 1$ *have the following remarkable properties:*

- *If* n *is composite, so is* $2^n - 1$. *For let* $n = rs, \quad r > 1, \quad s > 1$. *Then*

  $$2^n - 1 = (2^r)^s - 1 = (a-1)(a^{s-1} + a^{s-2} + \cdots + a + 1); \quad a = 2^r,$$

  and so $2^n - 1$ *is divisible by* $2^r - 1 > 1$ *and (since* $r > 1$*) cannot be prime.*

  This means that when $2^n - 1$ *is prime,* n *cannot be composite and must be prime. However,* $2^n - 1$ *is frequently composite when* n *is prime (e.g.* $2^{11} - 1 = 23 \cdot 89, \qquad 2^{23} - 1 = 47 \cdot 178,481$*).*

  Thus

  $$\begin{array}{llcll} n & = \text{composite} & \longrightarrow & 2^n - 1 & = \text{composite} \\ n & = \text{prime} & \longrightarrow & 2^n - 1 & = \text{composite or prime} \\ 2^n - 1 & = \text{prime} & \longrightarrow & n & = \text{prime} \end{array}$$

  In general, if $n > 1$ *and* $a^n - 1$ *is prime, then* $a = 2$ *and* n *is prime, for if* $a > 2$, *then* $a^n - 1$ *is divisible by* $a - 1$, *so* $a^n - 1$ *cannot be prime. Note that on account of the above result, the problem of the primality of* $2^n - 1$ *is reduced to that of* $2^p - 1$ *where* p *is prime.*

- *If $r > 2$ is a prime, each prime factor p of $M_r = 2^r - 1$ must be of the form $p = 1 + 2kr$. For example*

$$\begin{aligned} M_{43} &= 431 \cdot 9719 \cdot 2,099,863 = 2^{43} - 1 \\ p_1 &= 431 = 1 + 2 \cdot 5 \cdot 43 \\ p_2 &= 9719 = 1 + 2 \cdot 113 \cdot 43 \\ p_3 &= 2,099,863 = 1 + 2 \cdot 3 \cdot 3 \cdot 2713 \cdot 43 \end{aligned}$$

Fermat (1640) proved this statement in the following way: Let $r > 2$ be a prime and p a prime divisor of $M_r = 2^r - 1$. So

$$2^r \equiv 1 (mod\ p); \quad 2^{p-1} \equiv 1 (mod\ p) \quad \text{by FLT}$$

Let $d =$ highest common factor of r and $p-1$. Then by Euclid's algorithm $d = \alpha r + \beta(p-1)$ for suitable integers α, β. It follows that

$$2^d = (2^r)^\alpha \left(2^{p-1}\right)^\beta \equiv 1^\alpha \cdot 1^\beta \equiv 1 (mod\ p).$$

Since M_r is odd we see that $p > 2$; since p divides $(2^d - 1)$ we infer that $d > 1$. Because r is a prime and $d > 1$, $d = r$ and $(p-1)$ is divisible by r. Consequently $p - 1 = sr$ for some s. Finally, $p - 1$ is even and r is odd. Hence s is even, $s = 2k$, say, as claimed.

- *Euler (1750) found a simple criterion for the factorizability of M_p: If both $p = 4k + 3 > 3$ and $(2p + 1)$ are prime then $(2p + 1)$ divides M_p. Thus if $p =$ 11, 23, 83, 131, 179, 191, 239, 251, then M_p has the factors 23, 47, 167, 359, 383, 479, 503 respectively. The theorem was proved by* ***Lagrange*** *(1775) and again by* ***Lucas*** *(1878).*

- *Every $M_p = 2^p - 1$ is prime to every other Mersenne number.*

- *If M_p is a Mersenne prime, then $M_{M_p} = 2^{M_p} - 1$ is not necessarily a prime number. For example*

$M_{13} = 2^{13} - 1 = 8151$ *is a prime*

$M_{8151} = 2^{8151} - 1$ *is composite and has the prime factor*

$$2 \cdot (20,644,229) M_{13} + 1 = 338,193,759,479$$

In this connection one may consider the sequence of numbers

$$\begin{aligned}
&C_1 = 2^2 - 1 = 3 = M_2,\\
&C_2 = 2^{C_1} - 1 = 7 = M_3,\\
&C_3 = 2^{C_2} - 1 = 2^7 - 1 = 127 = M_7,\\
&C_4 = 2^{C_3} - 1 = 2^{127} - 1 = M_{127}, \dots, C_{n+1} = 2^{C_n} - 1, \dots.
\end{aligned}$$

It is not known whether all number C_n are primes and even if there exist infinitely many which are primes. It is impossible yet (2004 CE) to test C_5, which has more than 10^{38} digits!

1645–1667 CE **Ismael Boulliau** (1605–1694, France). Astronomer and classical scholar. An early Copernican, Keplerian and defender of Galileo. First to suggest (without proof) in *Astronomia Philolaïca* (1645) that the central force keeping the planets in their Keplerian elliptical orbits, must be proportional to their inverse-square distance from the sun[86]. This work is arguably the most important book in astronomy between Kepler and Newton.

Newton (1684) *proved* that planets moving under such law will obey the three laws of Kepler. Among other astronomers who preceded Newton in astronomical inquiries and contributed some ideas to the establishment of the true laws that govern motion of planets in their courses, are: **Giovanni Borelli** (1664, Pisa), **Huygens** (1673), **Hooke** (1674) and **Halley** (1684).

Boulliau established (1667) the brightness periodicity of the first known long period variable star, *Mira Ceti*.

Born to Calvinist parents in London, Boulliau converted to Catholicism and moved to Paris in the early 1630s. During the next thirty years he enjoyed the patronage of the family de Thou and assisted the Brothers Dupuy at the Bibliotheque du roi. Boulliau was a friend of **Pascal** and **Gassendi** and a close associate of Huygens.

Newton, in his *Principia*, praised *Astronomia Philolaïca*, particularly for the inverse-square hypothesis and its accurate tables.

1645–1675 CE **Jean Picard** (1620–1682, France). Astronomer and a founding member of the *French Academie Royale des Sciences* (1666). First to

[86] Kepler had claimed proportionality to the inverse distance.

apply the telescope to measurements of angles. Known especially for accurate measurements of a degree of a meridian, from which he computed the size of the earth (1668–1670). Credited with first use of telescopic sights and of pendulum clocks in astronomical observations. Determined latitude and longitude of Tycho Brahe's observatory in Uraniborg so Tycho's observations could be directly compared with others. His measurements of the earth's size were used by Newton in his gravitational theory.

Picard became professor of astronomy (1655) at the College de France in Paris. In 1673 he moved to the Paris Observatory where he collaborated with **Cassini**, **Römer** and **La Hire**.

1646 CE **Athanasius Kircher** (1601–1680, Germany). Jesuit and scholar. Credited with the invention of the *magic lantern*[87] (Laterna Magica), the first early projection device and a forerunner of the modern slide and motion picture projectors.

The device, in its simpler forms, consisted of: (1) the lantern body, (2) a source of light, (3) an optical system for projecting the images. It projected on a white wall or screen largely magnified images of transparent pictures painted (or later, photographed) on glass, or of objects (crystals, animals, etc.) carried on glass slides. The projection was made by means of a concave mirror (acting as condenser) and a projection lens, using sunlight, oil or candle light.

Laterna Magica was used during 1726–7 at the Opera in Hamburg.

Kircher was a professor of mathematics in Rome (1650). He was one of the first to experiment with *moving images*.

1646–1658 CE **Johann Rudolph Glauber** (1604–1668, Germany). Chemist. First to distill coal and obtain benzene and phenol; investigated decomposition of common salt through action of acids and bases. *Glauber's salt* [$Na_2SO_4 \cdot 10H_2O$ (1658)] is named after him. Glauber had a fairly clear

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[87] Although Kircher was first to set up various types of the apparatus, is difficult to identify the original inventor. In about 1653 the mathematician **Andreas Tacquet** displayed at Löwen a journey from China to the Netherlands by means of Kircher's lantern — the first lantern lecture ever delivered. **Thomas Walgenstein** (a Danish student at Leyden) got the idea from Tacquet, and demonstrated (1658) in a number of cities a magic lantern with interchangeable painted glass slides and a two-lens objective, creating a great sensation. **Huygens**, who had dealings with Walgenstein, also possessed a proper magic lantern at about that time. He even used a lens as a condenser, whereas Walgenstein still retained the use of a concave mirror.

idea that salts consist of acid and base, and a correct notion on *affinity*. Resided in Amsterdam from 1655.

1648–1656 CE *Fire and sword over Eastern Europe*; the uprising of the serfs and the Greek orthodox Ukrainian Cossacks against the weak regime of the Roman Catholic Polish gentry. The chieftain of the Cossacks, Bogdan Chmialnicki, placed himself under the protection of Russia, thus precipitating a prolonged conflict. Over 200,000 Jews, about one half of the total Jewish population of the Ukraine and Galicia, perished in the decade of this revolution and over 700 of their communities were destroyed. Many more fled to Holland, Germany, Bohemia and the Balkans.

This tragic event brought forth a new spiritual movement generated by **Israel Ba'al Shem Tov** (1700–1760) and known as *Chassidism*. It was a new interpretation of Judaism based not upon reason but faith, not upon intellect but emotion; man could literally escape his unbearable miseries by immersing himself in a mystical-esoteric kindling of the soul with God. To the masses who hungered for a direct, simple, stimulating religion which they could follow without any philosophical sophistications, the doctrine of salvation through prayer and humility rather than study was appealing. The unsuppressed emotions and optimistic Chassidic spirit served as a buffer against the depressing environment of dissolution and terror.

1650 CE **Bernhard Varen** (**Bernhardus Varenius**, 1622–1650, Germany). Geographer and physician. In *Geographia generalis* (1650) he endeavored to lay down the general principles of environmental science on a wide scientific basis, according to the knowledge of his day.

His work long held its position as the best treatise in existence on scientific and comparative geography. The work is divided into:

- *Absolute geography*: investigates mathematical facts relating to the earth as a whole, its figure, dimensions, motions, etc.
- *Relative geography*: considers the earth as affected by the sun and the stars, climates, seasons, the difference of apparent time at different places, variations in the length of the day, etc.
- *Comparative geography*: treats the actual divisions of the surface of the earth, their relative positions, globe and map construction, longitude, navigation, etc.

Geography is viewed as encompassing all aspects of the surface of the earth, including its geologic and oceanographic features, climate, plant and animal life. Varen explains the global wind system by a physical process through which air in the equatorial regions is thinned by the sun's heat and in response the cold, heavier air of the polar regions flows equatorward.

Isaac Newton thought so highly of this book[88] that he prepared an annotated edition which was published in Cambridge (1672), with the addition of the plates which had been planned by Varen, but not produced by the original publishers.

Varen was born at Hitzacker on the Elbe, in the Lüneburg district of Hanover. His early years (from 1627) were spent at Uelzen, where his father was court preacher to the Duke of Brunswick. He studied at the Universities of Königsberg (1643–1645) and Leiden (1645–1649), where he devoted himself to mathematics and medicine, taking his medical degree at Leiden (1649). He then settled at Amsterdam, intending to practice medicine. But the recent discoveries of **Tasman** (1642–1644), and **Schouten** (1615–1616), attracted him to geography. He died only 28 years of age, a victim to the privations and miseries of a poor scholar's life.

ca 1650 CE The intensity of the earth's magnetic field began to decline; it diminished 15 percent during the next 350 years[89]. If the trend continues at the same rate, a reversal of the earth's magnetic field may occur around 4000 CE.

1650–1654 CE **James Ussher** (1581–1656, Ireland). Prelate and scholar. Calculated that God created the world at 9:00 a.m., Sunday, 23 October, 4004 BCE. The calculations of this archbishop were somewhat less precise than the result would seem to indicate. The year 4004 BCE was arrived at by taking **Luther**'s estimate[90] of 4000 BCE [obtained by rounding off various arithmetical calculations of *Biblical chronology*], and then correcting it by four years to allow for Kepler's dating of the birth of Christ in 4 BCE

[88] The reason why Newton took so much care in correcting and publishing Varen was, because he thought him necessary to be read by his audience while he was delivering lectures on the same subject from the Lucasian Chair.
The book was still recommended for students at Cambridge in 1910!

[89] The fact that the magnetic compass was a key factor in navigation led governments to subsidize the science of geomagnetism. As a result magnetic observations have been made since the 16^{th} century.

[90] The literal interpretation of the biblical book of *Genesis* gained increasing devotion by Churchmen, not in the early Middle Ages, when the teaching of the Greeks were still largely accepted in the secular world and the New Testament was still new, but in the later Middle Ages and the beginning of modern times, in a reaction to the scientific explorations of the Renaissance. Early Christian scholars, such as St. Augustine, continued in all essentials the tradition of the Greek philosophers, but the thread of that kind of thinking was lost in late medieval outgrowths of Christian scholasticism and theological idealism.

[based on discrepancy between solar eclipses and New Testament dating of the crucification].

1650–1671 CE **Nicolaus Mercator-Kaufmann** (1619–1687, England). Independently of James Gregory introduced and summed infinite series[91], in connection with the calculations of areas under plane curves.

Mercator [*not* to be confused with **Gerhardus Mercator** (1512–1594), who is known for the *Mercator projection*] was born in Holstein (then a part of Denmark) but spent most of his life in England and was one of the first members of the Royal Society of London. He died in Paris.

1652 CE **Thomas Bartholinus**[92] (1616–1680, Denmark). Physiologist, physician and mathematician. Discovered the *lymphatic system* (1652) and determined its relationship to the circulatory system. Professor of mathematics (1646–1648) and of anatomy (1648–1680) at Copenhagen University; physician to King Christian V (1670–1680). Defended Harvey's doctrine of the circulation of the blood.

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91 The series

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}\cdots$$

is sometimes referred to as the *Mercator series*. It was independently discovered by **G. Saint-Vincent** (1584–1667). In the early days of the calculus, this series was probably derived through a term by term integration of the geometric series expansion

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots, \qquad -1 < x \le 1.$$

Since

$$\log_e\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right),$$

the substitution $x = \frac{1}{2N+1}$ yields the formula

$$\log_e(N+1) = \log_e N + 2\left[\frac{1}{2N+1} + \frac{1}{3(2N+1)^3} + \frac{1}{5(2N+1)^5} + \cdots\right].$$

This series converges rather rapidly for all positive N and was used in the calculations of logarithms.

92 His father **Caspar Berthelsen** (Lat. Bartholinus) (1585–1629), physician, was first to describe the olfactory nerve as first cranial nerve. His brother **Erasmus Bartholinus** (1625–1698), was a physician, mathematician and physicist; discovered the phenomenon of *double refraction of light* in Icelandic feldspar. Both were professors of medicine at the University of Copenhagen.

1653 CE The construction of *Taj Mahal* in Agra, India was completed. About 20,000 workmen built it over 21 years. The Indian ruler Shah Jahan ordered it built in memory of his wife Mumtaz-i-Mahal.

The Taj Mahal is made of white marble. It rests on an eight-sided platform of red sandstone. Each side is about 40 m long. At each corner of the platform stands a slender white minaret. The central structure has four smaller domes surrounding the huge, bulbous central dome. The tombs of the Shah and his wife are in a basement room. Above them in the main chamber are false tombs. Light is admitted into the central chamber by finely cut marble screens. The Taj is amazingly graceful from almost any angle of view, and the precision and care which went into its design and construction are impressive. It is one of the most beautiful and costly tombs in the world.

Scientists fear that after centuries of undiminished glory, industrial pollution could cause irreparable damage to the marble.

1654–1672 CE **Otto von Guericke** (1602–1686, Germany). Soldier, engineer and natural philosopher. Believed in a finite starry cosmos surrounded by an infinite void, as in the Stoic system. Performed spectacular public demonstrations in Magdeburg in which two teams of horses were unable to break apart two large evacuated brass hemispheres, held together by external atmospheric pressure. The vacuum was achieved by means of an air-pump which he developed in 1650. He was also able to show that sound could not travel, flames could not burn and animals could not live in vacuum.

Von Guericke was born at Magdeburg, in Prussian Saxony. He studied law at Leipzig, Helmstadt and Jena, and mathematics and mechanics at Leyden. He then visited France and England and in 1636 became engineer-in-chief at Erfurt. Toward the end of the Thirty Years War he returned to Magdeburg and helped rebuild it. He became mayor (1646–1676) and a magistrate at Brandenburg. His leisure was devoted to scientific pursuits, especially in pneumatics. Enticed by the discoveries of **Galileo**, **Pascal** and **Torricelli**, he attempted to create a vacuum. He also experimented with static electricity and made successful researches in astronomy, predicting the periodicity of the return of comets. In 1672, at the age of 70, he published his ideas and experimental results in *The New Magdeburg Experiments on Void Space*.

Only God and space can be infinite, Guericke said, and though the starry cosmos may be immense, it is nonetheless finite in size. Guericke believed in a *finite* Stoic cosmos, and thought that the gaps between stars reveal to us the emptiness and darkness of an extracosmic void, i.e. the sky is dark at night because we look between the stars and see the starless void beyond.

1655–1663 CE **Francesco Maria Grimaldi** (1618–1663, Italy). Physicist. First to suggest the *wave theory of light* in a book entitled "*Physico-Mathesis de lummine coloribus et iride*", published after his death in 1665. Grimaldi's major contribution to the optics was the discovery of *diffraction* (1660).

Grimaldi found that light did not travel exactly in straight lines, for he discovered that shadows were a little larger than they should be on the supposition that the propagation of light was rectilinear. Moreover, he found that the edges of shadows were often colored, and so he suggested that light was a fluid capable of wave-like motions, *different frequencies being different colors*(!) If the motions of the light-fluid were wave-like, then the edges of shadows should be blurred and colored, he said, for water waves can easily go round an obstacle they encounter. He supposed further that his light-fluid moved with great speeds, undulating all the time.

Grimaldi developed experiments to study phenomena associated with diffraction, interference, reflection and the color of light. His experiments were wide in scope and subtle in arrangement. He succeeded in detecting interference fringes even with such a quasi-coherent source as the pinhole source. He illuminated two closely spaced pinholes with a pinhole source and, in the pattern projected onto a screen, he discovered that some areas of the projected patterns were even darker than when one of the holes were plugged. His observations are basically the same as those of Thomas Young (1773–1829) in an experiment performed 150 years later.

His book planted many seeds which were later cultivated to full bloom by **Huygens**, **Newton**, **Young** and **Fresnel**.

Grimaldi was born in Bologna, son of a wealthy silk merchant. At the age of 14 he joined the *Society of Jesus*, and was educated at his Order's houses at Parma, Ferrar and Bologna, where he became Professor of Mathematics at the Jesuit College (1648).

1655–1678 CE **Christiaan Huygens**[93] (1629–1695, The Netherlands). An eminent Dutch mathematician, mechanician, physicist and astronomer.

He was born at the Hague, the second son of Sir Constantin Huygens, poet and diplomat (1596–1687). From his father he received the rudiments of his

[93] For further reading, see:

- Wolf, E., The life and work of Christiaan Huygens, in Blok, H., H.A. Ferwerda and H.K. Kuiken (Editors), *Huygens Principle: 1690–1990: Theory and Applications*, Elsevier Science Publishers, 1992.

education, which he continued at Leyden (1645 to 1647) and Breda, where he studied law and mathematics.

In 1655 he discovered Titan, satellite of Saturn, with a telescope that he built himself, and suggested that the appendages of Saturn seen earlier by Galileo (1610) are edges of a flat disk surrounding the planet. In 1656, Huygens was the first effective observer of the Orion nebula.

In November 1659 Huygens made the first reliable record of surface features of *Mars*, using a refracting telescope of his own design. After observing a prominent, dark, triangular feature (now called *Syrtis Major*) for several weeks, Huygens concluded that the rotation period of Mars is approximately 24 hours (modern value = $24^h\,37^m\,23^s$). This was the first in a series of observations that would soon lead to speculations about *life on Mars*.

In 1663, on his visit to England, he was elected a fellow of the Royal Society. During the period 1666–1681 he resided in Paris as a guest of King Louis XIV. He returned to Holland to conclude his studies on physical optics.

In 1656 he built the first reliable isochronous pendulum clock with an accuracy of 10 sec/day. Isochronism was achieved by forcing the bob of the pendulum to mark a *cycloidal arc*[94]. Huygens' clock incorporated the verge

[94] *The cycloidal pendulum*: A vertical pendulum having a bob of mass m suspended from the fixed point O.

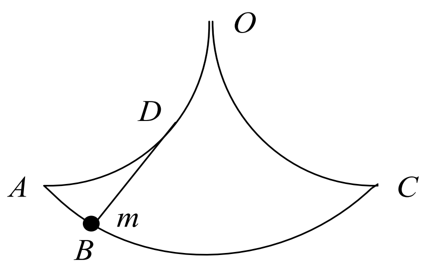

Fig. 3.1: The cycloidal pendulum

As it oscillates, the string winds up on the constant curve ODA (or OC) as indicated in the Fig. 3.1. If the curve ABC is a *cycloid*, the curves ODA and OC are the *evolutes* of the cycloid, and in fact are themselves two halves of an equal cycloid.
Consider the motion of a *free particle* of mass m under gravity on a smooth cycloid ABC whose axis is vertical and vertex lowest, as in the figure.
Let the cycloid be generated by a circle of radius a and let s be the arclength from its lowest point to a general point B on it. Then, it follows from the geometry of the curve that $s = 4a\sin\vartheta$, where ϑ is the angle made by the tangent at B with the horizontal tangent at the vertex (x − axis).

type escapement [In 1676 a much improved *anchor* type was invented, that interfered less with the pendulum's free motion; this device allowed, for the first time, the uniform division of a given time interval] and a spiral balanced spring of his invention. Huygens used continued fractions for the purpose of approximating the correct design for the toothed wheels of a planetarium [rational approximation for irrational gear ratio]. In 1657 Huygens came to Paris and became interested in the new theory of probability. He introduced the concept of *mathematical expectation*.

As an outgrowth of his experimentation with the pendulum and with circular motion, Huygens was able to derive in 1673 the law of *centripetal acceleration* of a mass which moves uniformly in a circle of radius r with velocity v; its acceleration $a = v^2/r$ is directed toward the center of the circle. In his studies of mechanics Huygens gave a clear and concise account of the laws governing the collision of *elastic bodies*[95] (1669) and introduced the important concept of the *moment of inertia*. His studies of pendulum motion made it possible for him to make the first accurate determination of the value of the acceleration of gravity and to show that it varied with latitude[96].

Although his contribution to the calculus was somewhat indirect, he was the first person since Archimedes to calculate areas of portions of surfaces of revolution, such as the paraboloid and hyperboloid.

Huygens used his self-made telescope to make important astronomical discoveries throughout the solar system: the existence of Titan, explanation of

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The force along the tangent at B is given by

$$m\frac{d^2s}{dt^2} = -mg\sin\vartheta \equiv -\frac{mg}{4a}s,$$

so that the motion is *simple harmonic* with the time to the lowest point being $\pi\sqrt{\frac{a}{g}}$, independent of the initial position of the particle. This property will still be true if, instead of the material curve, we substitute a string tied to the particle in such a way that the particle describes a cycloid and the string is always normal to the curve. This will be the case if the string *unwraps and wraps itself on the evolute of the cycloid*. Hence, if a string of length $4a$ is allowed to wind and unwind itself upon fixed metal cheeks in the form of half of the original cycloid each, a particle of mass m attached to its end will have its time of oscillation always *isochronous*, whatever the angle through which the string oscillates. In order words: the period of oscillation will be the same regardless of the amplitude of the oscillations.

95 These conclusions were arrived at independently by **Christopher Wren** in 1668.

96 Emil Wolf: The life and work of Christian Huygens, in *Huygens' Principle 1690–1990*, Elsevier Science Publications, pp. 3–17, 1992.

Saturn's rings, Martian rotation period, cloud cover of Venus. In addition he contributed to the design of the first microscopes.

In 1678, Huygens made an explicit development of the wave theory of light, and stated his celebrated principle[97] (in its extended formulation): each surface element of a wavefront at time t_0 is regarded as a source of secondary spherical waves. The wavefront at later time t, is the envelope of all the interfering secondary spherical waves with radius $c(t - t_0)$.

In the same year, Huygens made the fundamental discovery of *polarization*: each of the two rays arising from refraction by *Iceland spar* may be extinguished by passing it through a second crystal of the same material if the latter crystal be rotated by 90° about the direction of the ray[98].

Huygens never married. He died at the Hague, bequeathing his manuscripts to the University of Leyden and his considerable property to the sons of his brother.

1655–1660 CE **William Brouncker** (1620–1684, England). Mathematician. Among the founders of the Royal Society of London and its first president. Worked on continued fractions and calculating logarithms by infinite series. Discovered the expansion (1655)

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cdots}}}.$$

Gave a method of solution to the Diophantine equation $x^2 - ay^2 = 1$ (1657), which first appeared in Fermat's work (1640), to be known later as the *Pell equation*.

[97] Huygens' Principle continued to play an important role in the development of physics: In 1926, **Schrödinger** invoked the Principle to elucidate the transition from classical to quantum mechanics. In 1938, **Zernike** used Huygens' Principle to show how certain statistical features of light, known as its *coherence properties*, are transmitted on propagation.

Feynman (1948) made use of Huygens' Principle, in the so-called path integral formulation of quantum mechanics.

[98] It was however left to **Newton** (1717) to interpret these phenomena. He assumed that rays have "sides"; and indeed this "transversality" seemed to him an insuperable objection to the acceptance of the wave theory, since at that time scientists were familiar only with longitudinal waves. Later (1808), the *polarization of light by reflection* was discovered by **Etrienne-Louis Malus** (1775–1812, France). But Malus did not attempt the interpretation of this phenomenon. Only in 1818 did **Fresnel** establish the transversality of light waves.

Continued Fractions

The origin of continued fractions is hard to pinpoint: we can find examples of these fractions throughout mathematics in the last 2000 years, but its true foundations were not laid until the late 1600's, early 1700's.

The origin of continued fractions is traditionally placed at the time of the creation of Euclid's Algorithm, used to find the greatest common divisor (`gcd`) of two numbers. However, by algebraically manipulating the algorithm, one can derive the simple continued fraction of the rational p/q as opposed to the GCD of p and q. It is doubtful whether Euclid or his predecessors actually used this algorithm in such a manner. But due to their close relationship, the creation of Euclid's Algorithm signifies the initial development of continued fractions.

For more than a thousand years, any work that used continued fractions was restricted to specific examples. The Indian mathematician ***Aryabhata*** *(d. 550 CE) used a continued fraction to solve a linear indeterminate equation. Rather than generalizing this method, he used continued fractions solely in specific examples.*

Throughout Greek and Arab mathematical writing, we can find examples and traces of continued fractions. But again, its use is limited to specifics. More examples were provided during the Late Renaissance by ***Bombelli*** *(ca 1570 CE) and* ***Cataldi*** *(ca 1600 CE), who expressed the square roots of* 13 *and* 18*, respectively, as repeated continued fractions. However, neither of them investigated the properties of the continued fractions.*

Christiaan Huygens *was first to demonstrate practical applications of continued fractions (1695). He used it for the purpose of approximating the correct design for the toothed wheels of a planetarium.*

William Brouncker *discovered in 1655 a continual fraction expansion for $\frac{4}{\pi}$. He then discovered a method to solve the Diophantine equation*

$$x^2 - Ny^2 = 1$$

by a continued fraction. The theory shows that a particular solution is

$$x = p_n, \qquad y = q_n$$

where $\frac{p_n}{q_n}$ is a certain convergent of $\sqrt{N}$. Moreover, from one solution, an infinite number of solutions may be found. Thus from the least-values solution

$$x = 3, \qquad y = 2 \qquad \text{of} \qquad x^2 - 2y^2 = 1,$$

one derives

$$x_n = \frac{1}{2}\left[(3+2\sqrt{2})^n + (3-2\sqrt{2})^n\right]$$

$$y_n = \frac{1}{2\sqrt{2}}\left[(3+2\sqrt{2})^n - (3-2\sqrt{2})^n\right].$$

The field began to flourish when **Leonhard Euler** *(1707–1783),* **Johann Heinrich Lambert** *(1728–1777), and* **Joseph Louis Lagrange** *(1736–1813) embraced the topic. Euler laid down much of the modern theory in his work De Fractionibus Continius (1737). He showed that every rational can be expressed as a terminating simple continued fraction. He also provided an expression for e in a continued fraction form*

$$e - 1 = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}}}}}$$

He used this expression to show that e and e^2 are irrational. He also demonstrated how to pass from a series to a continued fraction representation of the series, and conversely.

Lambert generalized Euler's work on e to show that both e^x and $\tan x$ are irrational if x is rational. Lagrange used continued fractions to find the value of irrational roots. He also proved that a real root of a quadratic irrational[99] *is a periodic continued fraction.*

The 19^{th} century can probably be described as the golden age of continued fractions. The subject was known to every mathematician and, as a result, there was an explosion of growth within this field. Some of the more prominent mathematicians to make contributions to this field include **Gauss**, **Cauchy**, **Jacobi** *and* **Hermite**.

[99] Any number of the form $\frac{P \pm \sqrt{D}}{Q}$, where $P,\ D,\ Q$ are integers and D is a positive integer which is *not* a square.

1655–1695 CE **John Wallis** (1616–1703, England). Mathematician. Contributed substantially to the origins of calculus. Wrote a book, '*Arithmetica infinitorum*' (1655), in which he introduced the concept of *limit*, negative and fractional exponents, and the symbol ∞ for infinity. The whole thrust of his work was to replace geometrical with algebraic concepts and procedures wherever possible. Newton's study of this book was a major influence in his discovery of the general binomial theorem.

Wallis prime objective, however, was to 'square the circle', i.e., to effect the quadrature of the curve $y = (1-x^2)^{1/2}$, by expanding y in power series of x^2 of the form $a_0 + a_1x^2 + a_2x^4 + \cdots$. In this he was unsuccessful; it was left to the young **Newton** to achieve success here. In 1685 Wallis presented a graphical representation of complex numbers in his book: "*Treatise of Algebra*". He also made pioneering contributions to mechanics: In 1668 he suggested *the law of conservation of momentum*. In 1655 he found the infinite rational product $\pi = 2\frac{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdots}{1\cdot3\cdot3\cdot5\cdot5\cdot7\cdots}$.

In his book *Opera Mathematica* (1695) Wallis laid some of the basic groundwork for *continued fractions*. He explained how to compute the n^{th} *convergent* and discovered some of the now familiar properties of convergents. It was also in this work that the term "*continued fraction*" was first used.

Wallis has been described thus: "*One of those youthful prodigies who never quite pay off. But for the mishap of living in the age of Newton, Leibniz, Descartes and Fermat, he might have been a leading mathematician of the* 17*th century*..."

We also find in history books that he testified against Laud, the Archbishop of Canterbury, and was partly responsible for his execution. For his distinguished services to the crown, he was appointed Savillian professor of geometry at the University of Oxford.

1655–1698 CE **Georg Eberhard Rumpf van Hanau** (**Rumphius**, 1627–1702, Amboina, the Moluccas, East Indies). One of the great naturalists of the 17^{th} century. In his *Herbarium Amboinense* he described the first large herbal of the flora of the Eastern and tropical world (published posthumously 1741–1750, Amsterdam). Earlier (1705), his manuscript *Amboinsche Rariteitkamer*, describing Moluccan shells, was published in Delft.

Rumphius was born near Hanau (on the river Main), the son of a prosperous architect. He was educated in the local gymnasium and soon developed a very adventurous spirit; he enlisted in the Venetian army, but the ship on which he sailed to Brazil was captured by the Portuguese, and the boy was drafted into the Portuguese army. In 1649 he was allowed to return to Hanau. In 1653 he took service as a warrant officer in the Dutch East India Company and sailed to Java and from there to Amboina. This was to become the arena

of his scientific activities for the next half-century. In spite of many catastrophes which befell him [loss of eyesight, 1670; loss of his wife and youngest daughter in the earthquake of Feb. 17, 1674; loss of all his manuscript drawings in the Amboina fire of 1687; loss of the text and illustrations of half of the *Herbarium* at sea, when the carrying ship was sunk by the French in 1692], he continued his work. Four more years of labor amidst all the discomforts of tropical life — far away from every library and university, the new manuscript of the whole work (12 books) was ready. In 1696, the Amsterdam officials of the Dutch East India Co. had it in their hands — one of the masterpieces of botanical literature — but they did not considered it worth publishing. It was to remain 40 years hidden in their archives.

After a long period of oblivion, Rumphius' *magnum opus*, the *Herbarium amboinense* was rescued from the Zeventienen archives by the Amsterdam professor, **Johannes Burman** (1707–1779), who decided to edit it, to translate it into Latin, to add various notes, and to publish it with the original illustrations. The work was so enormous and so expensive to produce that no single Dutch firm would assume the whole risk. It was finally issued by a consortium of eight Dutch publishers in six folio parts appearing in Amsterdam from 1741 to 1750.

1656 CE Plague spread from Sardinia to Naples. Ca 400,000 perished. The disease returned by 1672, killing ca 400,000 more.

1656 CE **Thomas Wharton** (1614–1673, England). Physician and anatomist. Gave first thorough account of the *glands* of the human body, which he classified as excretory, reductive and nutrient. Wharton differentiated the viscera from the glands and explained their relationships, describing the *spleen* and the *pancreas*.

Wharton was born in Winston-on-Tees, Durham county. He studied at Cambridge and Oxford, obtaining his M.D at Oxford (1647). He had medical practice in London and was elected a fellow of the Royal College of Physicians (1650).

In 1456 he published his Latin treatise "*Adenographia – a description of the glands of the entire body*".

Wharton discovered the duct of the *submaxillary salivary gland* and the jelly of the umbilical cord, both of which are named for him; he also provided the first adequate account of the *thyroid*, and gave it that name. He explained the role of saliva in mastication and digestion.

1657 CE *Accademia del Cimento*, the first scientific research institute, was founded in Florence with the encouragement and support of Ferdinand II, grand duke of Tuscany (1610–1670).

1657 CE **Johann van Waveren Hudde** (1628–1704, The Netherlands). Mathematician. Introduced letter coefficients which stand for negative as well as positive number. Until this date, negative values *were not allowed*.

1657–1685 CE **Robert Hooke** (1635–1703, England). Inventor and experimental scientist. Because of his varied interests he abandoned many successful but slow-moving experiments without finishing them, originating much but perfecting little. Others profited from his findings.

In 1660, Hooke laid the foundations to the currently accepted theory of elasticity in his motto "*Ut tensio sic vis*". It states the one-dimensional stress-strain relation of linear elasticity. This he discovered while applying spiral springs to the balances of watches.

His other scientific activities were:

- Improved the air pump and used the improved version to confirm Galilei's hypothesis (with a feather and a coin) that in a vacuum all objects fall at the same rate (1657).
- Constructed the first reflecting telescope (1664).
- Invented the *anchor escapement* for clocks and was first to use a *spiral spring* to regulate watches (1658).
- Discovered plant cells (1665).
- Stated the *inverse-square-law* of gravitation prior to Newton's publication (he insisted that Newton mention this fact in his *Principia*!) and approached to a remarkable degree the discovery of universal gravitation (1679).
- His optical investigations led him to adopt, in an imperfect form, the wave theory of light, to anticipate the concept of interference and to observe, independently of **F.M. Grimaldi** (1618–1663), the phenomenon of diffraction.

In personal appearance Hooke made a sorry show: his figure was crooked, his limbs shrunken, his hair hung in disheveled locks over his haggard countenance. His temper was irritable, his habits penurious and solitary.

Many circumstances concurred to embitter the latter years of his life, and the repeated anticipation of his discoveries by others filled him with morbid jealousy. In 1691, the Royal Society made him a grant to enable him to complete his inventions. While engaged in this task he died, worn out with disease, in London.

1658–1671 CE **Johann de Witt** (1625–1672, Netherlands). Dutch statesman and amateur mathematician. He conceived a new and ingenious

way of generating conics, essentially the same as that by projective pencils of rays in modern synthetic geometry, but which he treated by means of Cartesian analytic geometry. Using the known theory of probability of his day [mainly via the works of **Fermat**, **Pascal** and **Huygens**], he gave a careful and adequate discussion of the theory of *life-annuities*. This represents his most important contribution to mathematics, and is a remarkable performance for a man deeply involved in the affairs of state.

Witt was born at Dort. He was educated at Leyden and displayed early on remarkable talents in mathematics and jurisprudence. As a student he lived in the house of **Franciscus van Schooten** (1615–1660) [a professor of mathematics at Leyden, remembered for his recommendation of the use of Cartesian coordinates in 3-dimensional geometric problems].

He led a hectic life while leading of the United Provinces through periods of war, in which he opposed the designs of Louis XIV. When in 1672 the French invaded The Netherlands, de Witt was dismissed from office by the Orange party and lynched by an infuriated mob.

1659 CE **Johann Heinrich Rahn** (1622–1676, Switzerland). Mathematician. In his book *Teutsche Algebra* (1659) he introduced the logical symbol $\therefore$ (therefore) and the operational symbol $\div$ for division.

1658–1673 CE **Jan Swammerdam** (1637–1680, The Netherlands). Naturalist. Founder of both comparative anatomy and entomology.

Conducted microscopic examination of aspects of human anatomy. First to observe and record *red blood cells* (1658). Discovered the *valves of the lymph vessels* (1664), which now bear his name.

Performed (1667) series of experiments on *animal respiration*: by compressing and expanding an air bellows attached to the windpipe and lungs of various animals, he was able to examine the effects of inflating and deflating the lungs.

Swammerdam described ovarian follicles of mammals independently of **de Graaf** (1672). Through his investigation of the human reproductive system he was first to show that female mammals produce *eggs*, analogous to birds' eggs.

Studied the anatomy of *insects*, which he classified on the basis of development. His chief works are *Historia Insectorium Generalis* (1669) and *Biblia Naturae* in which he used a simple microscope to make observations of a great range of biological phenomena.

Swammerdam was born in Amsterdam. He studied medicine at Leiden but never practiced.

1659–1661 CE *Cheques and Banknotes*: Messrs Clayton and Morris, bankers in London, handled the first known *cheque* (1659). The first European *banknotes* were issued in Stockholm, Sweden. They were originally receipts issued by bankers for gold deposited with them, promising to repay the deposition (1661).

1660 CE The Royal Society founded in London by **Jonh Wilkins** (1614–1672) and **William Brouncker** (1620–1684).

1660 CE **Robert Boyle** (1627–1691, England). Irish chemist and physicist. Laid the foundations of modern chemistry[100]. In his book *The Sceptical Chymist* (1661) he disputed and refuted the ideas of Aristotle (the 4 Greek "elements": air, earth, fire, water) and Paracelsus (the fundamental nature of sulfur, salt, and mercury; 1530) on the composition of matter. Introduced the modern concepts of *elements*, *alkali*, *acid* and defined *chemical reaction.* Although he argued against the existence of elements, he was first to make attempts to classify all substances into elements, compounds and mixtures. Experimented with his improved air-pump ('Boyle's engine') and showed that sound cannot diffuse in vacuum, whereas light can pass through it unattenuated.

Derived 'Boyle's law' for ideal gases, stating that at fixed temperature the gas volume is inversely proportional to its pressure. Theorized that gas is made of small indivisible spherical particles, in random motion.

Robert Boyle was born at Lismore Castle, in the province of Munster, Ireland, the 14^{th} child of Richard Boyle, the great earl of Cork. While still a child he learned to speak Latin and French, and was only 8 years old when he was sent to Eaton. During 1638–1642 he traveled with a tutor in Europe and being a heir to a great fortune, he decided to dedicate his life to study and scientific research. He settled in Oxford (1654) and set himself, with the

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[100] The Late Latin *alchimista* stemmed from *al kimiya*, the prefix *al* being the Arabic article. The remainder of the word may be from the Greek *chimia* ($\chi\upsilon\mu\epsilon\iota\alpha$) [meaning: pouring, infusion and used in connection with the study of juices of plants. Also *cheimeia* = transmutation of metals]. This derivation accounts for the old-fashioned spellings: *chymist* and *chymistry*.

Another view traces it to the Egyptian *kym*, god of the Nile or *khem*, which denotes black earth and occurs in the *Bible* (*Gen* **9**, 24; *Ps* **78**, 51; *Ps* **105**, 23, 37; *Ps* **106**, 22) and **Plutarch**. On this derivation alchemy is explained as meaning the *Egyptian art*. The first occurrence of the word is said to be in a treatise of **Julius Firmicus** (ca 346 CE), an astrological writer.

The prefix *al* was added by a later copyist, and dropped about the middle of the 16^{th} century.

assistance of Robert Hooke, to improve on the air-pump of Otto von Guericke (1657–1660). It was in relation to this work that he discovered in 1661 his famous law. In 1680 he was elected president of the *Royal Society of London for improving natural knowledge* (established 1660). In 1668 he left Oxford for London, where his failing health caused him to withdraw from all his public engagements. He was buried in the churchyard of St. Martin's in the Fields.

1660–1677 CE **Baruch Spinoza** (1632–1677, Netherlands). One of the greatest philosophers of modern times. His philosophical system has come to impregnate the prevailing modern scientific, social and moralistic theories. Spinoza merged in his doctrine the best gems of reason he could extract from Greek philosophy, the Talmud and the Kabbalah, Maimonides and the Christian scholars Hobbes and Descartes.

His philosophy is composed of four elements, subsequently personified by four great Jewish thinkers:

- The need for *piety* (**Leopold Zunz**, 1832)
- The passion for *freedom* and *justice* (**Karl Marx**, 1848)
- The *rational* ordering of all thought (**Sigmund Freud**, 1904)
- The conception of all-embracing *science* of the *universe* (**Albert Einstein**, 1915)

It was mainly the spread and influence of science in its more dogmatic aspects that, toward the end of the 19^{th} century, caused especial interest to be taken in Spinoza's thought. By a sort of instinct Spinoza seems to have anticipated, by deductions from first principles, many of the most fundamental principles of modern science; e.g., the conservation of energy (in his belief that the total quantity of motion in the universe is constant); the non-existence of a vacuum; and the existence of nothing real in the universe but configurations and motions.

Anticipating the methods and fundamental ideas of modern science, he examined the concepts of space, time, causality, free will and natural laws in an holistic attempt to comprehend the entire universe in all its manifestations. Thus, his system advances rational claims for a definite beginning in cosmic time and for a cosmic evolution ["*There was no time or duration before the creation*"; "*We are aware of external things only in relation to each other. All sense experience and all deductions based on them are inadequate*".]

His ideas are present in the writings of Berkeley and Mach, and in Einstein's General Relativity.[101]

Spinoza's teachings also echo in Quantum Mechanics, since he argued that *no system can be understood in isolation*, and we are forced to treat the observer as part of the physical system he is observing. Indeed, according to modern interpretation, both *free will* and *causality* are only rendered meaningful in the presence of external, and thus indeterminate, perturbations. Thus an observer, who himself should ultimately be considered part of the system, must perturb the subsystem that he is focusing on, and measure the result, in order to imbue *causality* with meaning. Likewise, Spinoza would argue that our observer's self-perceived "free will" is an illusion, stemming from his ignorance of external influences acting on *him*.

Only two of Spinoza's writings were published during his life time: *Renati des Cartes Principiorum* (1663), and the *Tractatus Theologico-Politicus* (1670). Three additional works appeared in the year of his death (1677): the *Ethics*, which brought him universal fame in the annals of philosophy; his treatise *On the Improvement of the Understanding*, and his *Political Treatise*.

Baruch Spinoza, or, as he later called himself, Benedict de Spinoza, was born in Amsterdam on the 24^{th} of November, 1632. His forefathers fled from Spain to Portugal in 1492, but in 1498 were forced to convert by the Inquisition (yet remained Jews in spirit). After the establishment of the Union of Utrecht in 1579, his grandfather's family sought refuge in the emancipated Netherlands (1593), and returned there to Jewish orthodoxy. The name, variously written Espinoza, de Spinoza, and Despinoza, probably is derived from the city of *Espinoza de los Monteros* in Leon, not far from the city of Burgos. Baruch's father, Michael de Spinoza, a respectful merchant, was one of the leaders of the Sephardic community of Amsterdam. He married thrice and Baruch was the third of the four children born to him from his second wife (this wife and her sister, which became his third wife, were also from the Espinoza family).

Spinoza was six years old when his mother died (1638), and his father died in 1654. He was trained at the communal school and at the Pereira Yeshivah, over which Manasseh ben Israel and Saul Morteira presided. There he studied Hebrew, Bible, Talmudic literature, and, toward the end of his course, some of the Jewish philosophers: Maimonides, Gersonides, Hisdai

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[101] For further reading, see:

- *Reflections and Maxims* by B.Spinoza (with *Introduction* by Albert Einstein), Philosophical Library, New York, 1965, 92 pp.
- Nadler, S., *Spinoza, a Life*, Cambridge University Press, 2001, 407 pp.

Crescas, Avraham Ibn Ezra (bible commentaries) and other representatives of Jewish medieval thought, who aimed at combining the traditional theology with ideas gotten from Aristotle and his Neoplatonic commentators. The amount of his Kabbalistic knowledge is somewhat doubtful, but both of his teachers were adepts in Kabbala.

During his studies Spinoza had shown early promise of becoming an excellent rabbinic scholar and the Amsterdam Sephardic community had high hopes for him. However, the study of Jewish philosophers of former days led him to turn to the study of philosophy in general. At that time, philosophy was abandoning its interest in theology (which had been its main concern in the Middle Ages), and was turning to the study of natural sciences and the human mind.

Spinoza was attracted by the atmosphere of free thought characteristic of the Dutch Capital. He associated himself with a number of freethinking friends and teachers, both Jews and Christians. It is also likely that his heretical views developed out of heterodox controversies *within* the Amsterdam Jewish community.

Latin, still the universal language of learning, formed no part of Jewish education, and Spinoza, after learning the elements of the language from a German master, resorted to further instruction from Franz van der Ende, an adventurer and polyhistor, under whom he also studied mathematics, physics, mechanics, astronomy, chemistry, and the medicine of the day. The mastery of Latin opened up to him the whole world of modern philosophy and science, both represented at that time by the writings of **Descartes**. Spinoza likewise acquired a knowledge of the scholasticism developed in the school of Thomas Aquinas.

His acquaintance with the works of Descartes (who led Europe in an attempt to establish a philosophy based upon reason, not tradition), accelerated his estrangement from the tradition of the synagogue and finally led to his break with Jewish orthodoxy.

Shortly after leaving the *yeshiva* (Jewish academy), rumors became persistent that young Spinoza had given utterance to heretical views. There was danger in this for the newly established Jewish community, whose enemies might now point out that Judaism was fostering irreligion and disbelief in God.

Desirous to avoid public scandal, the chiefs of the community offered him a yearly pension if he would outwardly conform and appear now and then in the synagogue. His refusal put him on a direct collision course with the congregation, and on the 27^{th} of July 1656 Spinoza was solemnly cut off from the commonwealth of Israel. While negotiations were still pending, he had

been set upon one evening by a fanatical ruffian, who thought to expedite matters with the dagger.

Spinoza was thus cast out at the age of 23 from all communion with men of his own faith and race, and there is no evidence of his coming into communication with a single Jewish soul from that time to his death.

Spinoza, however, did not mind. He was an individualist who could find no place in *any* organized religion. In the free environment of Holland he could live peacefully without being a member of any religious group. Socially, he was not alone: he had already formed a circle of friends and disciples, mainly of the *Mennonite* sect known as *Collegiants*, whose doctrines were similar to those of the Quakers; and he attended a philosophical club, with membership drawn mainly from this sect.

During 1656–1661 Spinoza took his abode with a Collegiant friend near Amsterdam, and started his research in optics through the grinding and polishing of lenses for the newly invented microscope and telescope (in which his mathematical knowledge was valuable). He also took pupils in philosophy, Latin and Hebrew.

The five years which followed the excommunication, were devoted to concentrated thought and study. Before their conclusion Spinoza had parted company from Descartes, and the main tenets of his own system were already clearly determined in his mind. He wrote what was later extended into the "*Tractatus Theologico-Politicus*", and a short tractate on "*God, Man and his Well-Being*" (afterwards developed into his *Ethics*).

In 1661 Spinoza removed to Rhijnsburg, near Leyden, then the center of the Collegiants activity. Here he spent the two most fruitful years of his life. In 1663 he removed to Voorburg (a suburb of The Hague), to be near the de Witt brothers[102], then at the height of their power. From Voorburg Spinoza used to send portions of his *Ethics*, written in Dutch, to his band of disciples in Amsterdam, who translated them into Latin. The "*Tractatus Theologico-Politicus*" was published in 1670, without the author's name, and it brought such a storm of opprobrium that it was formally proscribed by the Synod of Dort and by the States General of Holland, Zealand and West Friesland.

Spinoza's reputation as a thinker, however, had by this time been fully established by his two published works, and he was consulted both in person and by letter by many important scientific men of the day, including **Oldenburg**

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[102] Spinoza's income was supplemented by a small pension given to him by **John de Witt**. This arrangement assured his independence, and left him sufficient time to pursue his philosophical writings and correspond with the leading scientists of the day.

(secretary of the London Royal Society), **Huygens**[103], **von Tschirnhausen** and **Leibniz**[104]. In 1670 Spinoza settled at the Hague itself where he spent the remaining years of his life in the state of frugal independence which he prized. In 1673 he received an invitation to become a professor of philosophy in the University of Heidelberg, but he declined because it required from him "*that he will not misuse his freedom of speech to disturb the established religion*".

Early in 1677 Spinoza became seriously ill. He had a hereditary tendency to consumption derived from his mother, and this was aggravated by the inhalation of particles of crystal incidental to his work as lens grinder. He died on the 21^{th} of February 1677, with his friend Dr. Meyer as the only witness of his last moments. He was little more than 44 years of age.

Many mourned him; for the simple folk had loved him as much for his gentleness as the learned had honored him for his wisdom. Philosophers and magistrates joined the people in following him to his final rest, and men of varied faiths met at his grave[105]. In 1678, the Dutch government confiscated

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[103] It was as an optician that he first came into contact with Huygens. In fact, Huygens and his brother tried to spy on Spinoza's own techniques of grinding lenses. An optical "*Treatise on the Rainbow*" written by Spinoza was discovered in 1862.

[104] Those of Leibniz's works that have been published give little evidence of any connection with Spinoza other than in the latter's calling as optician, and his public utterances on Spinozism were in every case hostile and derogatory; but more recent evidence shows that during the critical period of his development, from 1676 to 1686, he took a more favorable attitude toward both Spinoza and Spinozism, and this has been traced to an intimate personal association of the two philosophers during a whole month in 1676, not long before Spinoza's death. It was during this period that Leibniz developed from a pure Cartesian into an opponent of Descartes, chiefly as regards the definition of body and the principles of motion; it is known that Leibniz discussed both subjects with Spinoza. When, however, a strong outcry broke out against Spinoza's "atheism", Leibniz devoted himself to finding an escape from Spinozism, and it took him nearly ten years before he arrived at his theory of the monads, which he declared to be the only solution of the difficulty. Bertrand Russell's analysis of the philosophy of Leibniz proved that in his views on soul and body, on God and ethics, he "*tends with slight alterations of phraseology to adopt* (*without acknowledgment*) *the views of the derided Spinoza*".

[105] He was buried in the yard of the New Church, the Hague, in a grave that his friends rented for 20 years. In 1697, upon the termination of the grave-contract, his remains were removed to an unknown location. In 1953, a tombstone was erected, in that yard, by **David Ben-Gurion**, then Israel's prime minister. The

all of Spinoza's writings and his name became a symbol of heresy.

Spinoza lived in the Netherlands at the time when scientific discovery, religious division, and profound political changes has revolutionized the nature and application of philosophy. Philosophy, for Spinoza, was not a weapon, but a way of life – the adoption of truth as one's master and one's goal. But every orders requires a sacrifice, and that demanded by philosophy is neither easily undertaken nor readily understood by those who refuse it. To the mass of mankind, therefore, the philosopher may appear as a spiritual saboteur and a subverter of things lawfully established. So Spinoza appeared to his contemporaries, and for many years after his death he was regarded as the greatest heretic of the 17^{th} century.

Indeed, for more than a century after Spinoza's works were published, their author was bitterly denounced by Catholics, Jews, Protestants and free-thinkers alike. Even **David Hume**, in general a man of kindly disposition, branded him as 'infame', and **Moses Mendelssohn**, the affable advocate of tolerance, was scandalized when he heard that his friend **Lessing** had adopted Spinoza's doctrine[106].

However, a radical change took place in Germany and England: Spinoza was rediscovered by of all people, the poets! This was possibly due to the influence of **Johann Gottfried von Herder** (1744–1803) and **Goethe** (1749–1832), who had both given utterance to great admiration for Spinoza's life and thought. The wide influence of Goethe, whose philosophical views were entirely Spinozian and were expressed in some of the profoundest of his poems, was perhaps the chief influence which drew to Spinoza the attention of such men as **Samuel Taylor Coleridge** (1772–1834), **Matthew Arnold** (1822–1888) and **Joseph Ernest Renan** (1823–1892).

Post-Kantian philosophers and romantic poets in Germany were deeply influenced by Spinoza's conception of nature. In modern times, Spinoza is universally recognized as a philosopher of unsurpassed sublimity and profundity. Even his critics agree that Spinoza had a most lovable personality, one

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stone is adorned by the inscription of a *rose surrounded by thorns*. Two words are inscribed on it too: the Hebrew word 'AMCHA' (meaning: the common folk) and the Latin word *caute* (beware), which was engraved on Spinoza's ring. The philosopher himself, however, does not rest under that stone.

106 Mendelssohn may have realized that Spinoza had first shown how a critique of Judaism could be used to reach radical conclusions about the world. His example has been indeed followed by the French enlightenment, though their treatment of Judaism was far more hostile, and racial, in tone. Two centuries later, the personal anti-semitism of **Karl Marx** would play a similar role in his socio-economical theory

of the purest characters in the history of mankind. His delicate feelings, his benevolence and fondness of plain people never hampered the boldness of his thoughts and the sternness of his will to draw conclusions logically and without any deference to personal inclinations. Philosophical thinking was, to Spinoza, self-education and improvement of the mind of the thinker. His aim was to obtain, by means of reason and science, the same trust in the rules of human behavior that religious traditions claimed to grant their believers.

Spinoza affirmed that God does not exist in the way religion preaches, only as an impersonal and spiritual 'principle', as a substance which constitutes the *reality* of the universe. Nothing exists save the one substance – the self-contained, self-sustaining, and self-explanatory system which constitutes the world. This system may be understood in many ways: as God or Nature; as mind or matter; as creator or created; as eternal or temporal. And to understand it in its totality, is also to know that everything in the world exists by necessity, and that it could not be other than it is[107]. A single stuff, obedient to a single set of laws gives rise to all that we observe. The task of science is to provide the complete description of that substance and the laws which apply to it.

Contrary to Descartes, he denied the possibility of harmonizing reason with Biblical revelation, and in that, Spinoza, not Descartes, became the symbol of the end of medieval philosophy. The *scientific method* offered to Spinoza not only the measure of moral evaluation but a means of gaining

[107] The God of Spinoza is not a *personal* deity with whom man can communicate. In this sense man is *devoid of free will*, unable to change the course of his life. The deity of the Bible activates Nature and as such is *beyond* Nature. God of the Bible negotiates with man (Abraham, Moses, Job, the prophets) and sometimes changes his will in accordance with man's actions.

The conflict between the teachings of Spinoza and Judaism is that of *monotheism* and *pantheism*. i.e. the idea of *oneness* against that of *unity*: Judaism is based upon the notion that the creator (God) is above both nature and man, each authority being subjected to its own set of laws which 'he' made. But in contradiction to inanimate nature, man has a free will.

According to Spinoza, creator and creation are one and the same entity and cannot be separated; there is but one authority and one set of laws. Man is an integral part of nature with no privileges. Man is not a transcendence of nature but an immanence of it, subjected to nature's laws with no free choice whatsoever. Furthermore, while Judaism believes that God is purposeful and created man without the ability to comprehend his purposefulness, Spinoza maintains that God = nature is totally unpurposeful. Hence man is unable to comprehend the purpose of nature not because he is not capable of *knowing* it, but because it *did not exist* ab initio.

eternal bliss. To win supreme happiness or 'unceasing joy', Spinoza said, man has to attain knowledge of his union with the whole of nature.

Worldview X: Baruch Spinoza

* * *

Sed omnia praeclara tam difficilia quam rara sunt.
All excellent things are as difficult as they are rare.

* * *

Nothing in Nature is random... A thing appears random only through the incompleteness of our knowledge.

* * *

He who loves God cannot endeavor that God should love him in return.

* * *

In the language of philosophy, it cannot be said that God desires anything of any man, or that anything is displeasing or pleasing him: all those are human qualities and have no place in God.

* * *

The bees, in all their work and the orderly discipline, which they maintain among themselves, have no other end in view than to make certain provisions for themselves for the winter, still, man who is above them, has an entirely different end in view when he maintains and tends them, namely to obtain honey for himself. So also [is it with] man, in so far as he is an individual thing and looks no further than his finite character can reach; but, in so far as he is also a part and tool of the whole of Nature*, because she is infinite, and must make use of him, together also with all other things, as an instrument.*

* * *

Whenever anything in nature seems to us ridiculous, absurd, or evil, it is because we have but a partial knowledge of things, and are in the main ignorant of the order and coherence of nature as a whole, and because we want everything to be arranged according to the dictate of our own reason; although, in fact, what our reason pronounces bad, is not bad as regards the order and laws of universal nature, but only as regards the laws of our own nature taken separately.

* * *

Whatsoever is contrary to nature is also contrary to reason, and whatsoever is contrary to reason is absurd, and, ipso facto, to be rejected.

* * *

Be not astonished at new ideas; for it is very well known to you that a thing does not therefore cease to be true because it is not accepted by many.

* * *

The highest endeavor of the mind, and the highest virtue is to understand things by the intuitive kind of knowledge.

* * *

All bodies are surrounded by others, and are mutually determined to exist and operate in a fixed and definite proportion, while the relations between motion and rest in the sum total of them, that is, in the whole universe, remain unchanged. Hence it follows that each body, in so far as it exists as modified in a particular manner, must be considered as a part of the whole universe, as agreeing with the whole, and associated with the remaining parts.

* * *

If the foundations of their religion have not deserted their minds they may even, if occasion offers, *so changeable are human affairs, raise up their empire afresh, and that God may a second time elect them.*

* * *

Do not weep; do not wax indignant. Understand.

* * *

On Spinoza

* * *

"There is no other philosophy then that of Spinoza"

Ephraim Gotthold Lessing (1729–1781)

* * *

"I feel a deep spiritual affinity between me and Spinoza, albeit his soul is more profound than mine. His doctrine inspires tranquility and calm; it brings the tranquility of God or the tranquility of nature upon me. Yet, I do not dare to claim a thorough apprehension of the ideas of one who ascended to the pinnacle of reason".

Johann Wolfgang von Goethe (1749–1832)

* * *

"To be a philosopher one must first be a Spinozist".

Georg Wilhelm Friedrich Hegel (1770–1831)

* * *

"Whose dwelling is the light of setting suns,
And the round ocean, and the living air,
And the blue sky, and in the mind of man —
A motion and a spirit, which impels

All thinking things, all objects of all thought,
And rolls through all things".

William Wordsworth (1770–1850)

* * *

"I can hardly imagine how one can be a poet, and yet not admire Spinoza".

Karl Wilhelm Friedrich von Schlegel (1772–1829)

* * *

"All the contemporary philosophers, perhaps unknowingly, observe the world through lenses grinded by Baruch Spinoza".

"When we read Spinoza, we have the feeling that we are looking at all-powerful Nature in liveliest repose — a forest of thoughts, high as heaven, with green tops ever in motion — while below the immovable trunks are deeply rooted in the eternal earth. It may be that the spirit of the Hebrew prophets hovered over their remote descendant.

Benedict Spinoza teaches that there is but one substance, God. This one substance is infinite and absolute. All finite substances are derived from it, are contained in it, emerge from it or sink into it; they have only relative, transitory, accidental existence. Absolute substance manifests itself to us in the form of infinite thought as well as infinite extension. We know only these two attributes. But God, absolute substance, may posses other attributes which we do not know.

Only stupidity and malice could term this doctrine 'atheism'. No one has ever expressed himself in more sublime terms regarding the Deity. Instead of saying that he denies God, we should rather say that he denies Man. All finite things are contained in God; the human intellect is only a ray of infinite thought; the human body only an atom of infinite extension; God is the infinite cause of souls and bodies — natura naturans".

Heinrich Heine (1797–1856)

* * *

"In Spinoza is contained the fullness of modern science".

Ernest Belfort Bax (1854–1926)

* * *

"No modern writer is altogether a philosopher in my eyes, except Spinoza".

George Santayana (1863–1952)

* * *

"The system of Spinoza remains one of the outstanding monuments of Western philosophy. Though the severity of its tone has a certain Old Testament flavor, it is one of the great attempts, in the grand manner of the Greeks, to present the world as an intelligible whole".

Bertrand Russell (1872–1970)

* * *

"I believe in Spinoza's God who reveals himself in the orderly harmony of all that exists, not in a God who concerns himself with the fate and actions of men".

"I would not think that philosophy and reason itself will be man's guide in the foreseeable future; however, they will remain the most beautiful sanctuary they have always been for the select".

Albert Einstein

* * *

"This man, from his granite pedestal, will point out to all men the way of blessedness which he found; and ages hence, the cultivated traveler, passing

by this spot, will say in his heart: 'The truest vision ever had of God came perhaps, here'".

Ernest Renan, at the unveiling of Spinoza's statue, The Hague (1882)

* * *

"Original ideas are exceedingly rare and the most that philosophers have done in the course of time is to erect a new combination of them".

George Alfred Léon Sarton (1884–1956)

* * *

1660 CE End of the *English Civil War*(also known as the *Puritan Revolution*, 1642–1646) and the beginning of the *Restoration* (1660). The biggest revolution in English history; the last and grandest episode in Europe's age of religious wars. It was mainly a contest over religious and political principles — Puritanism versus Anglicanism and parliamentary self-government versus royal absolutism.

After Cromwell beheaded Charles I (Jan. 30, 1649), England was a *Puritan republic* (1649–1660); Cromwell conquered Ireland (1649) and Scotland (1651) and compelled them to accept union with England, welding Great Britain into a single political unit. During the reign of Cromwell and the Puritans, Anglicans and Catholics were persecuted, but after Cromwell's death (1658) the Puritan Republic collapsed and Charles II was invited to return (1660).

With the restoration of the Stuart monarchy, the Church of England became again the state Church and the persecution of the Puritans was resumed: Puritans were not allowed into government, army, teaching positions, or parliament, which made it difficult for many of them to get a job. The only two places open for them were trade and industry, where most of them did indeed go. As it turned out, all of the great scientists, technologists and innovators of the next hundred years in England were Church of England rejects! (e.g. **James Watt**). Thus, *religious intolerance* kicked off the *industrial revolution* and brought the chemistry it needed.

Marranos, Huguenots and Puritans — three versions of the same history.

1660–1679 CE **Marcello Malpighi** (1628–1694, Italy). Physiologist and anatomist. Founder of microscopic anatomy; the first to apply the microscope to the study of animal and vegetable structure. Malpighi was born near Bologna. He studied medicine there (1649–1653). Professor of medicine at Pisa (1656–1659), at Messina (1662-1666) and again at Bologna (1666–1691). He then moved to Rome to become the private physician to Pope Innocent 12^{th} (1691).

Malpighi studied structure of secreting glands; discovered capillary circulation in the lung of the frog (1660); the deeper portion of the epidermis is known as the *Malpighi layer*; loops of capillaries in the kidney are known as *Malpighi tufts*; masses of adenoid tissue in the spleen are called *Malpighi corpuscles*. He described structure of human lung, development of the chick (1673), structure of the brain and spinal cord, and the metamorphosis of the silkworm (1669). He published *Anatome plantarium* (1675–1679).

1662 CE **John Graunt** (1620–1674, England). Published a book, "*Natural and Political Observations made upon the Bills of Mortality*", in which he laid the foundation to the science of statistics. 43 years later, the first successful life insurance company was established in England.

1662 CE **Lorenzo Bellini** (1643–1704, Italy). Physician and anatomist. Discovered the complex of tubules composing the *kidney* (*Bellini's tubules*) and described the mechanical theory of excretion (1662). Investigated the *sense of taste*.

1662–1687 CE **William Petty** (1623–1687, England). Physician, political economist and statistician. A founder of the Royal Society and a pioneer in the field of *vital statistics*[108]

His "*Treatise of Taxes and Contributions*" contains the first clear statement of the doctrine that price depends on the labor necessary for production. Petty was a professor of anatomy at Oxford (1651).

Evolution of the Calculus[109]

It took over 2500 *years for the calculus to progress from the early notions on the subject to the form we study today. For most of this period the concepts of differential and integral calculus were considered distinct.*

It was not until the latter part of the 17^{th} *century that mathematicians, led by* **Isaac Newton** *in England and* **Gottfried Wilhelm von Leibniz** *in Europe, discovered the connection between these fundamental ideas.*

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108 Records of the most basic human events – birth, marriage, divorce, sickness and death. This data is essential for legislators, health authorities, sociologists, school administrators, insurance statisticians and market researchers. State bureaus of vital statistics and state health departments maintain files of vital records and compile statistics.

109 *Calculus* means "pebble" in Latin (hence the word *calculation*). Indeed, in the civilizations of Egypt and the Asian river valleys, numbers were represented by means of pebbles arranged in heaps of ten. This in turn led to the development of the *abacus*, or counting frame in which a number is represented by pebbles put in grooves.

Over the past 300 years, calculus has been put on a firm mathematical foundation and refined to the point that it now follows logically from a few basic notions and principles.

The underlying concept of integral calculus was used by Greek mathematicians at least as early as the time of ***Antiphon the Sophist*** *(fl. 450 BCE),* ***Eudoxos of Cnidos*** *(408–355 BCE) and* ***Archimedes of Syracuse*** *(287–212 BCE), employing the so-called* method of exhaustion *to approximate areas and volumes. Archimedes determined the area of a circle by computing the area of the inscribed and circumscribed polygons of increasing numbers of sides. In this and similar applications he adumbrated the concept behind the Riemann integral. Archimedes also derived laws for determining the tangent lines to certain curves, including parabolas and the curve that bears his name, the 'spiral of Archimedes'. In some sense, Archimedes could be considered the founder of calculus. He did not, however, have a notion of a unified theory that could be applied to more than a few specific cases, nor did he recognize a connection between the differential and integral concepts of calculus*[110]*.*

Little progress was made toward the discovery of the unified theory of calculus until the beginning of the 17^{th} *century. Then, in the course of a mere 64 years, a formidable group of precursors, pioneers, inventors and co-inventors succeeded in creating the basic mathematical framework of the calculus familiar to us today (***Kepler***, 1615;* ***Galileo***, 1619;* ***Fermat***, 1629;* ***Roberval***, 1634;* ***Cavalieri***, 1635;* ***Wallis***, 1656;* ***Barrow***, 1669;* ***Newton***, 1671;* ***Leibniz***, 1673;* ***Huygens***, 1679, 1695).*

The first method of differentiation was introduced by ***Fermat*** *(1629) based on an earlier idea of* ***Kepler*** *(1615).*

Kepler had observed that the increment of a function becomes vanishingly small in the neighborhood of an ordinary maximum or minimum value. Fermat translated this fact into a process for determining such extrema. His method is equivalent to setting

$$\lim_{h\to 0} \frac{1}{h}[f(x+h) - f(x)] = 0,$$

i.e setting the derivative of $f(x)$ *equal to zero. Fermat also devised a general procedure for finding the tangent at a point of a curve whose Cartesian equation is given.*

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110 In beginners college courses, it is customary to begin with *differentiation* and later consider *integration*. Historically, however, the idea of integration, in connection with finding certain areas and volumes, was created earlier then that of differentiation, which was associated with problems of tangents to curves and with the question of finding maxima and minima of functions.

Fermat, however, did not *know that the vanishing of the derivative of $f(x)$ is only a necessary, but not a sufficient, condition for an ordinary extremum. He also did* not *distinguish between a maximum and a minimum value.*

It was again ***Kepler*** *(1615) who applied crude integration procedures to evaluation of areas which he needed in connection with his Second Law of planetary motion and volumes of solids of revolution.*

Thus, Kepler regarded the circumference of a circle as a regular polygon with an infinite number of sides. If each of these sides is taken as a base of an isosceles triangle whose vertex is at the center of the circle, then the area of the circle is divided into an infinite number of thin triangles, all having an altitude equal to the radius if the circle. Since the area of each thin triangle is equal to half the product of its base (Δl) and the radius (r), and since $\sum(\Delta l) = 2\pi r$, the total area is πr^2. Similarly, the volume of a sphere can be imagined to consist of infinitude of small cones, each of volume $\frac{1}{3}r(\Delta s)$ where $\sum(\Delta s) = 4\pi r^2$.

Cavalieri *(1635) was influenced by this work by Kepler when he carried the refinement of the infinitesimal calculus a stage further in his method of* indivisibles.

The problem of constructing tangents to curves was also taken up by ***Roberval*** *(1634),* ***Descartes*** *(1637) and* ***Barrow*** *(1669). With such active research, it was only a matter of time until the discovery of the notion that differentiation and integration are inverse operations.*

The first published statement concerning this Fundamental Theorem of Calculus appears in Lectiones geometricae, *a treatise published by Barrow in* 1670. *The theorem, however, is believed to have been recognized intuitively by* ***Galileo*** 50 *years earlier in connection with his study of motion.*

This brings us to the time of ***Newton****, a young student at Cambridge in the* 1660*'s, and to* ***Leibniz****, who was born in Leipzig and was self-trained in mathematics. These two men systematically unified and codified the known results of calculus, giving, in essence, algorithmic procedures for the use of these results. Each gave a proof of the Fundamental Theorem of Calculus and each clearly demonstrated the importance of this new theory.*

Newton developed most of his calculus, called the "method of fluxions and fluents", during a period 1664 through 1671, and compiled his results in the tract De analysi per equationes numero terminorum infinitas *in* 1669. *Although this manuscript was circulated and studied by a number of his English contemporaries, it did not appear in print until* 1711, *over* 40 *years later. In fact, Newton probably used his calculus to develop many important*

discoveries regarding gravitation and the motion of objects[111]*, but his treatise on this subject, Philosophiae naturalis principia mathematica (1687), contains only classical geometric demonstrations.*

It is difficult to determine precisely when Leibniz first became interested in the calculus, but it was probably shortly before he traveled to France and England in 1673 *as a political envoy. While visiting the London home of* ***John Collins*** *(1625–1683), he saw Newton's* 1669 *tract. He probably did not have a sufficient mathematical background to follow Newton's arguments at this time, but he was nevertheless excited by the result, particularly those dealing with series. After studying Descartes' fundamental work La géométrie, he communicated with Newton regarding the discoveries the latter had made. The two exchanged several letters during* 1676–1677*, by which time Leibniz had developed his own theory of calculus. The letters generally describe the extent of their work, but often omit crucial details necessary for the methods of discovery.*

Leibniz understandably expected that Newton would soon publish a treatise on calculus. When it became obvious that this work was not forthcoming, Leibniz began in 1682 *to publish his own discoveries in a series of papers in the Acta eruditorum, a journal published in Berlin with a wide circulation in Europe. In* 1684 *Leibniz published the first work on differential calculus and in* 1686*, the first on integral calculus. His articles are often vague and sketchy and were never collected in a definitive treatise.*

Because of Leibniz's prior publication, his calculus became the version known to the mathematical public of the time, particularly the European scientific community. We use his differential notation, dy/dx*, for differentiation and his elongated* S *symbol,* $\int$*, to represent integration. He called his integral calculus* calculus summatorius*; the term* integral *was introduced by* ***Jakob Bernoulli*** *in* 1690*. Newton's notation was generally more cumbersome, although his symbol* $\dot{y}$ *to denote the derivative of* y *is still commonly used to indicate differentiation with respect to time.*

Many reasons have been suggested for Newton's failure to capitalize on his discovery of calculus: his reticence, his preoccupation with other research, and his lack of interest in publishing. Certainly he had a complex personality

[111] The Newtonian calculus enabled mathematicians and physicists, for the first time, to solve more complex problems of motion, which up to his time seemed insoluble. This modern branch of mathematics, having achieved the art of dealing with infinitely small entities (infinitesimals), was unknown to the ancients. It corrected the inevitable errors which the human mind cannot avoid (such as Zenon's paradoxes) when dealing with *discrete* elements of motion instead of *continuous* motion.

and a sensitivity to criticism. Nevertheless, spurred on by their friends and colleagues, Newton and Leibniz were locked in a bitter controversy for nearly 20 *years over who deserved the credit for discovering the differential calculus. This is one of the saddest chapters in the history of mathematics: Leibniz complained that Newton's attitude was the malicious interpretation of a man who was looking for a quarrel, while Newton said that second inventors count for nothing! Of course, there was more than enough honor to go around, and the effect of the quarrel has been only to tarnish the images of both these mathematical giants.*

It should be kept in mind that neither Newton nor Leibniz established their results with anything resembling modern mathematical rigor. An example is the limit concept, so basic to the study of both the differential and the integral calculus. Although this concept is intuitively clear, its definition is quite sophisticated. It was not until 1870 *that* **Eduard Heine** *(1821–1881, Germany) published the definition for the limit of a function that we use today. Heine's work was strongly influenced by that of* **Karl Weierstrass** *(1815–1897, Germany), who was one of the leaders in the movement to place function theory on a firm and rigorous basis.*

Another calculus timeline involves the calculation of arc-lengths and area of surfaces: mathematicians of the early 18^{th} *century became interested in the problem of finding paths of shortest length on a surface using the methods of the calculus. The brilliant and prolific mathematician* **Leonhard Euler** *(1707–1783) presented the first fundamental work on the theory of surfaces in* 1760 *with "Recherches sur la courbure des surfaces", and it may have been in this work that a surface was first defined as a thee-dimensional graph* $z = f(x, y)$. *In* 1771 *Euler introduced the notion of parametric representation of surfaces.*

After the rapid development of calculus in the earth 18^{th} *century, formulas for the lengths of curves and areas of surfaces were developed. The underlying concepts of the length of a curve and the area of a surface were understood intuitively before this time, and the use of the formulas from calculus to compute areas were considered a great achievement.*

The subject of the calculus has played a special role in the history of modern science: Most of physics and engineering, and important parts of astronomy, chemistry and biology, would be impossible without it.[112]

[112] We list below a number of excellent calculus textbooks for self-study, on a number of levels:

- Kline, Morris, *Calculus – An Intuitive and Physical Approach*, Dover: New York, 1998, 943 pp.

- Zeldovich, Ya.B., *Higher Mathematics for Beginners*, Mir Publications:

1662–1677 CE **Isaac Barrow** (1630–1677, England). Versatile scholar, classicist and mathematician. Developed a method of determining tangents that closely approach the methods of calculus. Prolific writer on theology, mathematics and poetry. Translated Euclid, Archimedes and Apollonios into English and Latin. His book *Lectiones Geometricae* (1669) contains the foundations of the calculus in geometrical form. It presents, for the first time, differentiation and integration as inverse processes, integration as a summation, and nomenclature and methods which were direct forerunners of the algorithmic procedures of the calculus. His presentation of the differential triangle, clearly indicates *the mutual influence of Barrow and Newton upon each other*!

Barrow's influence upon **Leibniz**, too, may be inferred from the fact that Leibniz is known to have purchased a copy of Barrow's *Lectiones Geometricae* in 1673.

Barrow was born in London. He entered Trinity College, Cambridge in 1644 and received his B.A. degree in 1648. The next four years were spent in travel, at times highly adventurous, over Eastern Europe. He returned to England in 1659. In 1662 he was elected professor of geometry in Gresham College and in 1663, he became the first Lucasian professor of mathematics at Cambridge. In 1669 he resigned this chair to his great pupil and friend Isaac Newton. In 1675 he was chosen vice-chancellor of the university. He died suddenly of a fever and was interred in Westminster Abbey.

1663–1665 CE First scientific newspapers:

- *Erbauliche Monaths Unterredungen* ('Monthly edifying discussions'). Issued (1663) in Germany.

Moscow, 1972, 494 pp.

- Granville, W.A. et al., *Elements of the Differential and Integral Calculus*, Gin and Company, 1941, 556 pp.
- Piskunov, N., *Differential and Integral Calculus*, P.Noordhoff: Groningen, 1962, 895 pp.
- Fikhtengol'ts, G.M., *The Fundamentals of Mathematical Analysis*, 2 Volumes, Pergamon Press: Oxford, 1965, I, 491 pp; II, 516 pp.
- Smirnov, V.I., *A Course of Higher Mathematics*, Addison-Wesley, 1964, Vols I+II (543 pp.+630 pp.)
- Khinchin, A., *A Course of Mathematical Analysis*, Hindustan Publishing Corporation: Delhi, India, 1960, 668 pp.

- *Journal des Savants* ('Scientists newspaper'). Issued (1665) in France, by **Denis de Sallo**.

1663–1671 CE **James Gregory** (1638–1675, Scotland). Mathematician and astronomer. One of the first to distinguish between convergent and divergent series. Expanded (1667) the infinite series $\tan^{-1} x$, $\tan x$ and $\sec^{-1} x$ and showed in 1671 that

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

yields for $x = 1$,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

The $\tan^{-1}$ series was used in 1699, with $x = \sqrt{\frac{1}{3}}$, to evaluate π to 71 correct decimal places. In 1671, Gregory preceded **Brook Taylor** (1712) in series expansion of a function about a point.

Gregory, a Reverend's son, was born and educated at Aberdeen. In 1665 he went to the University of Padua where he studied for some years. In 1674 he became a professor of mathematics at the University of Edinburgh.

Many members of the Gregory family attained eminence in various departments of science. During 1650–1850, *fourteen* of them held professorships in mathematics or medicine.

The Reflecting Telescope

Gregory, a Reverands son, was born at Aberdeen. In 1665 he went to the University of Padua, Italy, where he studied for some years. He became a professor of mathematics at the University of St. Andrews(1669) but left that school in 1674 (his salary was not paid!) and was appointed a professor at the University of Edinburgh. In Oct. 1675, while showing the satellites of the

planet Jupiter to some of his students through one of his telescopes, he was suddenly struck with blindness, and died of the stroke a few days afterwards.

In 1663 Gregory published his famous treatise Optica promota *in which he made known his invention (1661) of the Gregorian reflecting telescope.*

In the Gregorian arrangement, a concave ellipsoidal secondary mirror *reinverts the image, returning the beam through a hole in the primary image. It was first successfully constructed (with some modification) by* ***Newton*** *(1668), and only became important research tool in the hands of* ***Frederick William Herschel*** *a century later.*

The Gregorian system was incorporated into the design of 1997 Arecibo reflecting radio telescope, *having a spherical mirror with a diameter of 10 meters.*

Gregory also discovered a diffraction grating *(using a birds 'feather'). In a letter to a friend dated May 12, 1673 Gregory pointed out that sunlight passing through a feather would produce a colored pattern and he asked that his observations be conveyed to Mr. Newton .*

Cassegrain *(1625–1650, France), physician and inventor, improved the Gregory–Newton reflecting telescope by utilizing a* convex hyperboloidal secondary mirror *to further increase the angular magnification. A century later it was noted by* ***Ramsden*** *(1735–1800) that this system also partly eliminated* spherical aberration.

The system gives an inverted image of any distant object and is superior to Gregory's in two points: first, the spherical aberrations of the two mirrors tend to correct instead of reinforcing each other, thus promoting good definition of the image; secondly, the necessary radius of aperture of the convex mirror is less, so that the proportion of light stopped is less in this instrument than in Gregory's telescope.

Cassegrain's system of mirrors is used today within many modern reflecting and large refraction telescopes. Nothing is known for certain about Cassegrain's life – not even his first mane. Believed to have been a professor at the College Chartres.

1664 CE, Mar. 06 Appearance of the first issue of the *Philosophical Transactions of the Royal Society*. By 1750, 46 volumes had been published[113].

In 1887 the Phil. Trans. was divided into two series, labeled A and B respectively, the former containing papers of a mathematical or physical character, and the latter papers of a biological character.

In 1832 appeared the first volume of *Abstract of papers printed in the Phil. Trans. from the year* 1800. This publication developed in the course of a few years into a *Proceedings of the Royal Society*.

1664–1672 CE **Thomas Willis** (1621–1676, English). Physician, anatomist and physiologist. Through his studies of the anatomy of the central nervous system and the circulation of the blood he extended the concepts proposed by the Roman physician **Galen**. In his *Cerebri Anatome,* (1664) the most complete and accurate account of the nervous system to that time, he rendered the first description of the hexagonal continuity of arteries (the "circle of Willis") located at the base of the brain and ensuring that organ a maximum blood supply, and of the 11^{th} cranial nerve (spinal accessory nerve) responsible for the motor stimulation of major neck muscles. Willis attempted to correlate the knowledge of anatomy, physiology and biochemistry with chemical findings in neuropathology.

He was a member of the iatrochemistry school, which believed that chemistry was the basis of human function, rather than mechanics, as was the main belief of the time.

Willis was born in Great Bedwyn, Wiltshire. An Oxford professor of natural philosophy (1660–1675). Opened a London practice that became the most profitable and fashionable of the period. Died in London.

1664 CE The Great Plague in London. Ca 100,000 perished.

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[113] 15 million scientific papers were written since modern science began. It was written by 4 million authors, most of them are alive today. Papers are being published *now* (2000) at a rate of one million per year in 40,000 journals. The mean 'life' of most of the journals is 25 years. But half of the reading is confined to only 200 journals.

The number of scientific journals $N(t)$, has doubled every 15 years for the past 200 years. Approximately $N(t) = 4e^{\lambda t}$, where $\lambda \approx 0.0461$, and $t = 0$ corresponds to the year 1794.

1665–1687 CE **Isaac Newton** (1642–1727, England). Physicist, mathematician and astronomer. One of the greatest names in the history of human thought. Discovered the calculus, established the fundamental laws of mechanics and stated the universal law of gravitational attraction, unifying terrestrial and celestial mechanics.

In 1664 Newton began to work on his "Calculus of fluxions", the principles and methods of which were developed by him in three tracts entitled: *De analysi per aequationes numero terminorum infinitas* (1666); *Methodus fluxionum et serierum infinitarum* (1671); and *De quadratura curvarum* (1676). None of these was published until long after they were written (printed 1711, 1736, 1704, respectively).

The infinitesimal calculus was 'almost' discovered by **Fermat** (1629) and **Isaac Barrow** (1630–1677, England). Newton was Barrow's pupil, and he knew to start with, in 1664, all that Barrow knew and that was practically all that was known about the subject at that time.

The discovery of the infinitesimal calculus seems to consist of three parts:

(1) The recognition that differentiation, known to be a useful process, could always be performed, at least for the functions then known. Thus, the problem of tangents could be solved once and for all.

(2) The recognition that the operation of integration is the inverse of differentiation and could be rendered systematic.

(3) The introduction of a suitable notation through which the discovery could be rendered accessible to mathematicians in general.

During the years 1664–1666 Newton started to wonder whether the earth's gravity could account for the motion of the moon and whether the sun's gravitation could account for Kepler's laws. On the second issue, when Kepler's third law is substituted into the expression for the centripetal acceleration of a planet in its orbit about the sun, there results the dependence of the centripetal acceleration on the inverse square average distance from the sun[114]. When he used this law for the earth-moon system, the moon's acceleration

[114] It is possible that this was independently deduced by **R. Hooke**, **C. Wren** and **E. Halley** working together in 1679, using **Huygens**' 1673 law of centripetal force, and Kepler's third law. The idea of inverse-square-law was "*in the air*" when Newton made his calculations. Other scientists were speculating on a cause for Kepler's laws and asking whether planetary motions could be explained by an attraction spreading from the sun. Newton rescued the question from mere speculation and extended the guess to universal gravitation.

toward the earth was found to be equal to $g\left(\frac{r}{R}\right)^2 \simeq \frac{g}{3600}$, where g is the acceleration of gravity on the earth's surface, r the earth's radius, and R the earth-moon distance[115].

Newton then calculated the moon's centripetal acceleration $\frac{v^2}{R}$ in its orbit around the earth. Finding that it was actually equal to $\frac{1}{3600}$ of the value of gravity on earth, he knew that universal gravitation could indeed supply the force needed to maintain the moon in its orbit around the earth. All these preliminary calculations were done by Newton without invoking his own calculus. Later in 1687, however, he used the calculus to justify the assumption that the earth and the moon can be treated as point masses located at their respective centers[116].

During 1665–1672, Newton laid the foundation for the science of optical spectrum analysis. He passed a beam of light through a glass prism and studied the resulting separation of sunlight into its various color components. He was then led in 1668 to construct the first reflecting telescope, in which a reflecting mirror is used instead of a system of lenses to avoid chromatic aberration. Newton's telescope was 15 cm long with a magnification of 38. Through it he saw the satellites of Jupiter.

He believed that light behaves as if it were a stream of tiny particles, such that red light was composed of the largest particles and violet of the smallest. He then showed that Snell's law can be derived from his own principles of mechanics. **Huygens**, on the other hand, argued that light has the nature of a wave propagating in a vacuum. [Modern quantum physics has shown that *both* were right.]

In 1671 Newton introduced new coordinate systems, such as polar coordinates and bipolar coordinates.

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115 Virtually, the moon falls radially toward earth with acceleration $g' = \frac{GM}{R^2}$. On the other hand, for the earth itself $g = \frac{GM}{r^2}$; Therefore $g' = g\left(\frac{r}{R}\right)^2$, where $g' = \frac{v^2}{R} = \frac{GM}{R^2}$. Newton knew that the radius of the moon's orbit was about 60 times the radius of the earth itself, as the ancient Greeks had first shown [**Hipparchos**, ca 130 BCE].

116 Newton could have, and probably did, deduce the inverse-square distance dependence of the law of universal gravitation by amalgamating *Kepler's third law* $[R^3/T^2 = K]$, the expression for *centripetal acceleration* $[a = \omega^2 R]$ and his own *second law* $(F = ma)$.
Indeed, the force F that accelerates a mass m in a circular orbit of radius r with angular velocity ω, is explicitly expressible in the form

$$F = ma = m(\omega^2 R) = mR\frac{4\pi^2}{T^2} = (4\pi^2 K)\frac{m}{R^2}.$$

In 1687, there appeared his book: "*Principia mathematica philosophiae naturalis* ("Mathematical principles of natural philosophy").[117] In this treatise he defined the concepts of *mass* and *force*, stated the three basic laws of mechanics and postulated the universal law of gravitation through the force F between any two point masses m and M at a distance r apart: $F = G\frac{mM}{r^2}$, where G is the universal constant of gravitation. He then showed how the empirical Kepler's laws follow from his laws (1680), but made no hypothesis as to how the gravitational force is transmitted[118].

Newton then applied his theory to explain the ocean *tides* as resulting from the combined attraction of the moon and sun. He also showed that the *precession of the equinoxes* resulted from the earth's equatorial bulge and the attractions of the sun and the moon. In addition he obtained detailed corrections to Kepler's elliptical orbits. In order to make the proper calculations related to all these phenomena, Newton had to develop many mathematical techniques *in addition* to the differential calculus. The phenomenal success of

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117 For further reading, see:

- Newton, Isaac, *Mathematical Principles of Natural Philosophy*, University of California Press: Berkeley, CA, 1960, 680 pp.
- Maury, J.P., *Newton — The Father of Modern Astronomy*, Harry N. Abrams: New York, 1992, 143 pp.
- Westfall, R.S., *Never at Rest: A Biography of Isaac Newton*, Cambridge University Press: Cambridge, 1980, 908 pp.
- Rankin, W., *Newton for Beginners*, Icon Books, 1993, 176 pp.

118 With a stroke of genius, Newton created here, without knowing it, one of the fundamental concepts of modern physics — the *field*.
His *action at a distance* force acts with no apparent physical contact between interacting objects, yet this action is ubiquitous, pervading the entire space surrounding the masses.
The enormity of this step can be vividly illustrated by the fact that a steel cable of radius $d = 50$ km would not be strong enough to hold the earth in its orbit. Yet the gravitational force which hold the earth in its orbit is transmitted from the sun across a hundred and fifty million kilometers of space without any material medium to carry that force! Indeed, equating the gravitational force between the two point-mass models of these bodies, we obtain per unit cross-section of the cable: $\{GM_{\oplus}M_{\odot}R^{-2}\}/\pi d^2 = 2.1 \times 10^{12}$ dyn/cm^2, which is above the value of Young's modulus for steel. [In this calculation we took $G = 6.67 \times 10^{-11}\ \frac{\text{Newton}\times\text{meter}^2}{\text{kg}^2}$; $R = 1.5 \times 10^{11}$ meters; $M_{\odot} = 2 \times 10^{30}$ kg; $M_{\oplus} = 6 \times 10^{24}$ kg; 1 Newton $= 10^5$ dyn.]

his efforts owed much to his unusual mathematical skills and superb physical insights.

In his *Principia* (1687) Newton devised a simple way to estimate the distance of the stars nearest to the sun: Assuming that the sun is a typical fixed star, one may estimate the distance to a star by comparing its apparent brightness with that of the sun — in the same manner that a distance to a candle may be judged by comparing its brightness with that of an identical candle nearby. Newton then calculated that our nearest stars are about a million times further than the sun, in good agreement with later measurements.

One of Newton's great achievements was the formulation of the fundamental laws of mechanics (1687). These constitute a codification of observation, experience and theory into 3 propositions:

1. *Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus illud a viribus impressis cogitur statum suum mutare*.
   [Every body persists in its state of rest or uniform motion straight ahead, except in so far as it be compelled to change that state by forces impressed upon it.]

2. *Mutationem motus proportionalem esse vi motrici impressae, et fieri secundum lineam rectam qua vis illa imprimitur*.
   [The change of motion is proportional to the motive force impressed and it takes place along the right line in which that force is impressed.]

3. *Actioni contrariam semper et aequalem esse reactionem*: *sive corporum duorum actiones in se mutuo semper esse aequales et in partes contrarias dirigi*.
   [To an action there is always an equal and contrary reaction: or, the mutual actions of two bodies upon each other are always equal and directed to contrary parts.]

These propositions, while somewhat vague, have been the foundation of much of the science and technology developed to the present time. They are stated in terms of undefined *primitive* concepts such as *mass* (implicitly) and *force* (explicitly) which need to be separately quantified. Notwithstanding, they provide the infrastructure for a precise analysis of a wide range of complex and seemingly unrelated systems and phenomena at a vast range of spatiotemporal scales.

In the 18^{th} and 19^{th} centuries, **L. Euler** (1758–1765), **J. d'Alembert** (1742), **J.L. Lagrange** (1788), **S.D. Poisson** (1813), **C.G.J. Jacobi** (1837), **W.R. Hamilton** (1828) and **J.H. Poincaré** (1889) have put Newton's propositions on a much firmer analytical basis, which led to the birth

and development of special branches of continuum mechanics such as hydrodynamics, aerodynamics, gas dynamics, theory of elasticity etc. The spirit of mechanics in all of its manifestations, however, is still easily traceable to Newton.

The first law is an important special case of the 2^{nd} law, since when $\mathbf{F}^{(e)} = 0$ (vanishing of total external force) the acceleration (of a point-particle or of the center of mass of a composite system) vanishes and thus the relevant velocity vector is fixed, both in direction and magnitude.

One may ask, then, why include the first law at all? The answer is that it helps to give *meaning* to the concept of force, by *defining* the case where it vanishes in a suitable class of reference frames! Of course, $\mathbf{F}^{(e)}$ must be better defined than that in order to solve for the motion in terms of initial positions and velocities (which is the aim of classical physics). Thus, it is necessary to use some theoretical form or semi-empirical ansatz (exemplified by the Law of Universal Gravitation on the one hand, and Hooke's law on the other).

Finally, we note that the third law may be replaced by *momentum conservation*. Thus, for a closed system of two interacting masses, let $\mathbf{F}_{12}(\mathbf{F}_{21})$ be the force exerted by 1 on 2 (2 on 1), respectively; we have by the second law

$$\mathbf{F}_{12} = \frac{d}{dt}(m_2\mathbf{V}_2), \qquad \mathbf{F}_{21} = \frac{d}{dt}(m_1\mathbf{V}_1),$$

so the third law $\mathbf{F}_{12} = -\mathbf{F}_{21}$ is equivalent to $m_1\mathbf{V}_1 + m_2\mathbf{V}_2 = \text{const.}$, which is just the statement that the momentum of the overall closed system is conserved. This readily generalizes to a system of n masses, or even a continuous medium, such as an elastic solid or a liquid.

Yet *another* way of expressing this law is that the *center of mass* (COM) position vector of the closed system,

$$\mathbf{r}_{COM} \equiv \frac{1}{\sum_{i=1}^{n} m_i}\Big(\sum_{i=1}^{n} m_i\mathbf{r}_i\Big),$$

moves in a straight line at constant speed. Consulting the first law again, we see that we can think of the n-mass system as a single mass located at its *COM*. When the system is *not* closed, it obeys the second law as if it were a mass acted upon by the total force $\sum_{i=1}^{n}\mathbf{F}_i^{(e)}$, where $\mathbf{F}_i^{(e)}$ is the total external force acting on the i^{th} mass.

The important thing to remember is that the main difference between Newton's second and third laws concerns *the definition of the system under study*: If one wishes to examine the motion of a single object (however defined) that is being acted by forces that originate *outside* the object, one uses

$$\mathbf{F}^{(e)} = \frac{d(m\mathbf{V})}{dt},$$

where $m\mathbf{V}$ is termed the momentum vector of the object.

On the other hand, if we consider a whole closed system composed of parts that can *interact with each other*, then Newton's third law tells us that the sum of the changes of the individual momenta will be equal to zero and that, therefore, the *internal forces must act in pairs* such that each pair is equal and opposite.[119]

An extended closed system (and after all any real-life mass is such a system, made up of myriad of atoms!) obeys another vectorial conservation law — that of *angular momentum*:

$$\mathbf{L}_{\text{total}} \equiv \sum_{i=1}^{n} m_i \mathbf{r}_i \times \dot{\mathbf{r}}_i = \text{const.}$$

Arriving at this law from Newton's laws is more circuitous, and in fact an extra ingredient is required to derive it. It follows from the second law, applied to the individual masses, that

$$\frac{d}{dt}\mathbf{L}_{\text{total}} = \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{F}_i \equiv \mathbf{T}_{\text{total}},$$

where $\mathbf{F}_i$ is the overall force acting on m_i, by the other masses and by external influences. Here $\mathbf{T}_{\text{total}}$ is the total *torque* acting on the system, and is origin-dependent.

One way to derive

$$\frac{d}{dt}\mathbf{L}_{\text{total}} = 0$$

for a closed system, turns out to require an action principle with rotational symmetry, from which the second law must be derivable as Euler-Lagrange equations; this is true in particular for a closed self-gravitating system of masses, and indeed one can check the angular momentum conservation directly in this case:

$$\mathbf{T}_{\text{total}} = \sum_{i=1}^{n} \mathbf{r}_i \times \sum_{j \neq i} \frac{G m_i m_j (\mathbf{r}_j - \mathbf{r}_i)}{|\mathbf{r}_j - \mathbf{r}_i|^3} = 0.$$

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[119] If a subsystem or object is defined in such a way that no mass may enter or leave it, we have $\frac{dm}{dt} = 0$ and Newton's 2^{nd} law assumes the form $\mathbf{F}^{(e)} = m\frac{d\mathbf{V}}{dt} = m\mathbf{a}$, with $\mathbf{a}$ the instantaneous acceleration of the object's COM.

In addition to such so-called 'conservative' forces[120], one can add any types of contact forces, for which there is no net torque. Thus, angular momentum is conserved also in the presence of non-conservative forces such as *friction*[121].

Newton's laws of motion embody what is now known as *Galileo's principle of relativity*, according to which uniform motion has meaning only when referred to some other body or system and can be detected only by reference to something external to the body in motion. Hence all laws of mechanics are the same in all systems (frames) of reference that move uniformly relative to each other (two such systems are related by a *Galilean transformation*).

To explain the forces that act on bodies in non-uniform motion, Newton invented the concept of 'absolute space', regarding it as a *substance within which bodies move*, which reacts but cannot be acted upon. The reaction of absolute space back on the body produces the 'inertial force'. Thus, the fictitious centrifugal and Coriolis forces are the reaction of absolute space on the rotating body[122]. **Berkeley** (1734) and **Mach** (1872) later rejected

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[120] Besides gravity, other examples are: elastic, intermolecular and electrostatic or magnetostatic forces.

[121] In *classical mechanics* the law of conservation of angular momentum does not carry the strength of a universal principle; it applies mainly to two cases — particles interacting via *central forces*, and in continuum mechanics with only contact forces.

In a (classical or quantum) system derived from an action principle of fields, however, it *is* a universal law — of no less importance than the law of conservation of linear momentum.

In *special relativistic physics* we must raise the number of dimensions from 3 to 4; consequently energy and momentum become components of a single 4-vector, while the extension of the angular-momentum 3-vector into a skew-symmetric 4-tensor of rank 2 leads one to the uniform rectilinear motion of the center of mass.

When dealing with *fields* or other continuous matter-energy distributions, we are led to consider densities of these conserved quantities, since in each volume element one has energy, momentum and angular momentum w.r.t. some given point. Moreover, it makes little sense to consider energy density by itself, because what is energy density in one reference frame will be some combination of energy density, energy flux density, and momentum flux density as seen from another reference frame (even in Newtonian mechanics, a shift in the spatial origin mixes angular momentum with (linear) momentum, while a Galilean transformation between inertial frames admixes components of momentum into energy). Hence, all these quantities are best considered together.

[122] They are 'fictitious' only in the sense of arising in non-inertial frames — frames in which Newton's 1^{st} law is violated. These reference frames are accelerated relative to *inertial* frames.

this simplistic 'solution', but it was not until the advent of Einstein's general relativity (1915) that a more satisfactory interpretation emerged.

Newton also concerned himself with the equivalence of inertial and gravitational mass. The roots of this problem are to be sought in Galileo's result that *in a given gravitational field, all nearby pointlike particles fall with the same acceleration*. This statement implies that to some extent gravitational forces behave in the same way as inertial forces. Galileo's statement can be called *Galileo's principle of equivalence*.

Newton sharpened this principle by combining it with his own second law of motion. This fusion then yielded the statement that the gravitational force is proportional to the mass on which it acts, ergo: the inertial mass and the gravitational mass are proportional. Newton then assumed that these two measures of mass were *equal* and he set forth to determine the precision of this determination. *Newton's principle of equivalence* then states that *gravitational mass* (m_g) *and inertial mass* m_F (resistance to change of motion under action of forces) *are equal*.

Comparing periods of pendula of fixed length but with different masses and composition, he found[123] that

$$\frac{|m_I - m_g|}{m_I} \ll 10^{-3}.$$

Clearly, Newton's principle implies equal accelerations only for bodies of *sufficiently small size* placed in a *sufficiently homogeneous gravitational field*.

Newton's laws of motion have the inherent property that they are covariant under the Galilean transformation, i.e., they retain their form when viewed by different inertial-frame observers — those attached to frames of reference in which no inertial forces are observed; any two such frames move with fixed velocity with respect to each other $(\mathbf{r}' = \mathbf{r} - \mathbf{v}t,\ t' = t)$. [A physical law which retains its form under a particular transformation is said to be covariant w.r.t. that transformation.] Frames of reference in which a test particle moves with constant velocity unless acted on by a force are known as *inertial frames*. [One may define such a frame in a picturesque way as one in which it is possible to play three-dimensional billiards.]

Newton's laws of motion do not distinguish between past and future, in the sense that they are *symmetrical w.r.t. reversal of motion*. Indeed, with

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[123] In the period 1891–1908 **Roland von Eötvös** (1848–1919, Hungary) used a torsion balance, which he had developed, to lower Newton's equivalence bound to ca 10^{-9}. This number was further reduced to 10^{-12} in experiments performed at Princeton and Moscow in the 1960's.

the exception of a rare kind of decay among a certain type of elementary particles, all experimental facts known to date about the actual world (even at the quantum level and even in GTR) are consistent with the following symmetry postulate: *to any state or process encountered in the actual world there corresponds a time reversed state or process that is again a possible state or process in the actual world.*

Newton's views on time were clearly stated in his 'Principia' of 1687:

"*Absolute, true and mathematical time, of itself and from its own nature, flows equably without relation to anything external*".

Thus according to Newton, time would continue just the same even if the universe were completely empty.

The work of Newton cannot be properly understood without a knowledge of the science of antiquity. Newton did not create in a void. Without the stupendous work of **Ptolemy** (which completed and closed ancient astronomy) and **Kepler**'s '*Astronomia Nova*', the mechanics of Newton would have been impossible. Without the conic sections of **Apollonios**, which Newton knew thoroughly, his development of the law of gravitation is equally unthinkable. And Newton's integral calculus can be understood only as a continuation of **Archimedes**' determination of areas and volumes. The history of mechanics as an exact science begins with the laws of the lever, the laws of hydrostatics and the determination of mass centers by Archimedes. In short, all the developments of mathematics, mechanics and astronomy which converge in the work of Newton, began in Greece.

And one must not forget **Galileo Galilei**; There is an appropriate symbolism in the fact that the death of Galileo and the birth of Isaac Newton occurred in the same year, 1642. Galileo and Newton are to be considered as the parents of modern science. In the words of Alfred North Whitehead:

"*There would have been no Newton without Galileo; and it is hardly a paradox to say, that there would have been no Galileo without Newton*".

Newton was born at Woolsthorpe, a hamlet in the parish of Colsterworth, Lincolnshire. His father (also Isaac Newton) who farmed a small property of his own, died before his son's birth, a few months after his marriage to Hannah Ayscough. When Newton was two years old his mother remarried and had three more children, to the descendants of which Newton eventually left most of his property. At the age of 12 he was sent to a grammar school at Grantham. At the age of 14 his stepfather died and his mother took him

away from school, since she intended him to be a farmer. He was sent back to school, however, and admitted to Trinity College, Cambridge, in 1661.

During the next 3 years he studied Euclid's *Elements*, Descartes' *Geometry* and Wallis' *Arithmetic Infinities*. In 1665 Newton took the B.A. degree. During his college career, Newton showed no exceptional ability and was graduated without any particular distinction. In the years 1665 and 1666, Trinity College was closed on account of the Great Plague in London (ca 70,000 people died). Newton retired to the countryside at his mother's home in Lincolnshire, where he made his preliminary discoveries of the binomial theorem, the method of fluxions, universal gravitation and the light spectrum. He returned to his college in 1667, and took his M.A. degree early in 1668. In 1669 (at age 26), his teacher, Isaac Barrow, resigned the Lucasian chair in favor of Newton, who thus became a professor of mathematics. In 1672 he was elected fellow of the Royal Society and during 1689–1690 represented Cambridge University in Parliament. To afford him a substantial salary, his friends secured for him the vacated mastership of the mint in 1699, where he later drew up the English monetary reform.

In 1701 he resigned his Cambridge professorship and moved to live in London. He was knighted in 1705, and was a very popular visitor at the court of George I. Newton added very little to his achievements in physics and mathematics after 1687 and spent most of the next 40 years of his unmarried life on public activities, experiments in *alchemy* and problems of *theology* and *Biblical chronology*. He died in 1727 and was buried in Westminster Abbey.

Newton was a man of deep religious convictions, bordering on mysticism. From an early period of his life he paid great attention to theological studies. [In fact, he spent *little* of his time studying mathematics, physics and astronomy.] The preoccupation with these matters served as the driving force behind his scientific work, which he considered as the deciphering of nature's code. Most of his life was spent on investigations of the Scriptures and the writings of the Christian saints, where he hoped to find hints to the secrets of the creation. He was a Unitarian and kept it a secret.

We know that Newton adhered to the philosophical view that time is cyclic and was convinced that the world was coming to an end. He believed that the comet of 1680 had just missed hitting the earth, and in his commentaries on '*Revelations*' and the '*Book of Daniel*', unpublished in his lifetime, he indicated that the end of the world could not be long delayed. A particularly striking example of his cyclical philosophy occurs in a letter that he wrote to **Henry Oldenberg**, secretary of the Royal Society, in December 1675:

"*For nature, is a perpetual circulatory worker, . . . so perhaps may the Sun imbibe this Spirit copiously to conserve his shining, and keep the Planets from receding further from him*".

Newton's discoveries had been so impressive for nearly two hundred years that they had the hallmark of being the last word. No refinement of his laws had been suggested. His law of gravitation had successfully explained every astronomical observation (with the tiny exception of a wobble in the orbit of the planet Mercury around the sun).

In fact, during his own lifetime the success of his mechanics had led to speculations that his approach might provide a panacea for the investigation of all questions. The impressive completeness of Newton's *Principia* (1687) and the deductive power of his mathematics led to a bandwagon effect with thinkers of all shades aping the Newtonian method. There were books on Newtonian models of governments and social etiquette, and Newtonian methods for children and 'ladies'.

Nothing was imagined to be beyond the scope of the Newtonian approach. Nor was Newton himself entirely divorced from this enthusiasm. His later work on alchemy and biblical criticism reveals a deep-rooted belief in his ability to unveil all mysteries for the human race. Having first revealed the truth about God's design of the physical world, he seems to have seen himself as having a similar commission to fulfill in the realm of the spiritual and the mystical.

Newton is a deeply paradoxical figure when viewed through the lens of modern scientific attitudes. A mathematical genius who possessed the most penetrating physical intuition of any recorded scientist, he nevertheless had one foot in the Middle Ages and displayed a magician's belief in his ability to solve all problems and overcome all barriers. His achievements must have made his contemporaries believe that the end of the seventeenth century was indeed the completion of science.

Newton was basically a very religious person, deeply influenced by the Bible and the religious philosophy of the ancient Hebrews. He believed in the unity of nature and the universality of natural laws — hence the motive for his discovery of the law of universal gravitation which applies to *all* stars in the universe. This was *not* a Greek heritage. This he received directly from the Bible.

Many years of Newton's life were embittered by the professional controversies which his *Principia* evoked. A long time elapsed before his ideas became part of the equipment of the ordinary educated man. It has been said that there were comparatively few scientists in the 20^{s} and 30^{s} of the 20^{th} century who comprehended Einstein's GTR. But there have been far fewer in Newton's day who could appreciate the reasoning of the Principia.

In 1692 and 1693 Newton seems to have a serious illness characterized by insomnia, withdrawal from close friends, headaches, nightmares, loss of hair.

It is now believed that there were symptoms of *mercury poisoning*, caused by his preoccupation with *alchemy* (transmutation of mercury into silver and gold).

Newton maintained that the corpuscles of light associated with various colors excited the ether into characteristic vibrations, where the sensation of red corresponds to the longest vibration of the ether and violet to the shortest. Perhaps the main reason for rejecting the wave theory as it stood then was the blatant problem of explaining rectilinear propagation in terms of waves which spread out in all directions.

Worldview XI: Isaac Newton

* * *

"If I have seen further it is by standing on the shoulders of Giants."

In a letter to Robert Hooke, February 5, 1675

* * *

"I know not what I may appear to the world, but to myself I seem to have been only like a boy playing on the sea-shore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me."

* * *

"The light of the fixed stars is of the same nature (as) the light of the sun."

Mathematical Principles of Natural Philosophy, 1687

* * *

"Physics, beware of metaphysics."

* * *

"Nature is pleased with simplicity, and affects not the pomp of superfluous causes."

* * *

"Absolute space, in its own nature, without relation to anything external, remains always similar and immovable. . . . Absolute motion is the translation of a body from one absolute place into another."

Mathematical Principles of Natural Philosophy, 1687

* * *

"Are not gross bodies and light not convertible in to one another?"

* * *

"Amicus Plato, amicus Aristoteles, magis amica veritas.*"*
(Plato is my friend, Aristotle is my friend, but my best friend is truth*)*

* * *

"Absolute, true and mathematical time*, of itself, and from its own nature, flows equally without relation to anything external."*

* * *

"Truth is ever *to be found in* simplicity*, and not in the multiplicity and confusion of things . . . He is the God of order and not of confusion."*

* * *

"Whence it is that nature *does nothing in vain; and whence arises all that* order *and* beauty *which we see in the world."*

* * *

On Newton

* * *

"Newton, forgive me. You found the only way that was possible for the man of the highest powers of intellect and creativity. The concepts that you created still dominate the way we think in physics."

"Let no one suppose, however, that the mighty work of Newton can easily be superseded by reality on any other theory. His great and lucid ideas will retain their unique significance for all the time as the foundation of our whole modern conceptual structure in the sphere of natural philosophy."

Albert Einstein

* * *

"Nature, and nature's laws lay hid in night.
God said, Let Newton be! and all was light."

Alexander Pope

* * *

"Nearer to the Gods no mortal may approach."

* * *

“Newton, with his prism, and silent face: The marble index of a mind for ever voyaging through strange seas of thought alone.”

William Wordsworth
(on seeing Newton’s statue in the chapel at Trinity College by moonlight)

* * *

“One had to be a Newton to notice that the moon is falling when everyone sees that it does not fall.”

Paul Valery (1871–1945)

* * *

Time and Tides

The periodic rise and fall of the ocean level, termed simply the Tide, was suggested by ancient peoples to be in some way related to the celestial bodies, long before a word for gravity existed. But the theories advanced were fantastic. It is natural that the writings of the classical authors of antiquity should contain but few references to the tides, for the Greeks and Romans lived on the shores of an almost tideless sea.

Pytheas of Massilia *(fl. ca 310 BCE) was familiar with the tides in the region of the British Isles and North Sea and is said to be the first to have actually* measured *the rise and fall of the tide. The Greek geographer and historian* ***Strabo*** *(ca 64 BCE–20* CE*) quotes from* ***Poseidonios*** *(135–51 BCE) a clear account of the tides on the Atlantic coast of Spain. He also gives the law of* diurnal inequality[124] *of the tide in the Indian Ocean as observed by* **Seleucus the Babylonian**. *Seleucus was the first known commentator to offer a rational (thought incorrect) mechanism of tide-generation.*

The Roman historian **Pliny the Elder** *(23-79* CE*) described the variation in the tidal range accompanying the moon's phases and changes in declination. In his* Historia Naturalis *he writes:*

"Much has been said about the nature of waters; but the most wonderful circumstance is the alternate flowing and ebbing of the tides, which exist, indeed, under various forms, but is caused by the sun and the moon. *The tide flows twice and ebbs twice between each two risings of the moon, always in the space of* 24 *hours. First, the moon rising with the stars swells out the tide, and after some time, having gained the summit of the heavens, she declines from the meridian and sets, and the tide subsides. Again, after she has set, and moves the heavens under the earth, as she approaches the meridian on the opposite side, the tide flows in; after which it recedes until she again rises to us. But the tide of the next day is never at the same time with that of the preceding".*

Julius Caesar *and his officers were totally ignorant of the connection between moon and tide: in the year* 55 *BCE, his first assault-landing on the Kentish coast of Britain, near Dover, failed as a direct result of a devastating spring tide; high water came about an hour before midnight, and driven by*

[124] The difference in the amplitudes (at locations in the middle latitudes of either hemisphere) of the two diurnal tides due to the periodicity in the moon's declination.

the gale, the beach galleys were swept by the breakers, all sustaining heavy damage. As a result, Caesar and his army returned to France.

The next 1500 *years did not advance the tidal lore beyond the observations of* ***Pytheas*** *and* ***Pliny****. The occurrence, at many places, of high tide at about the time of the moon's passage across the meridian may have prompted the idea that the moon exerts some* attraction *on the water, but the occurrence of a second high tide when the moon is on or near the opposite meridian was a great puzzle to the few* philosophers *who thought about it.*

We know about two Englishmen who were occupied with this problem: ***Alexander Neckam*** *(1157–1217), a learned monk, who was for a time a professor at the University of Paris, wrote a book of general knowledge,* De Naturis Rerum *in which he remarked that he was unable to resolve the vexed question as to the cause of the tides, but that the common belief was that they are due to the moon.* **Wallingford** *(d. 1213) made recordings of tidal observations for the purpose of prediction and tabulated the occurrences of floods at London Bridge in ca* 1210*.*

The Franciscan friar ***Roger Bacon*** *of Oxford attempted, at about* 1250 CE*, a rational (though wholly false) solution to the problem of the 'second high tide', based on the Ptolemaic conception of the universe.*

Johann Kepler *had recognized the tendency of the waters of the ocean to move towards the centers of the moon and the sun, and he wrote of some* attraction *between the moon and the earth's waters.* ***Galileo*** *then expressed regret that so acute a man as Kepler should produced a theory with occult qualities of the ancient philosophers (!!) His own explanation referred the phenomenon to the rotation and orbital motion of the earth, and he considered that it afforded a principal proof of the Copernican system.*

It was not until **Newton** *published the consequences of his law of universal gravitation in the* Principia *(1687) that the basic mechanics of tidal behavior could be understood.*[125] *It is a fact of the history of science that during the entire period from Pliny to Newton, nobody had the conviction that ocean tides are caused directly by the moon or the sun.*

But even if someone could believe it, and even if that someone knew about gravitational attraction between masses, he still could not have accounted quantitatively for the tide generating forces without stating clearly the three fundamental laws of dynamics discovered by Newton! It is the association of these two categories that made Newton see the light. The essence of his reasoning is this: When an astronomical body is moving under the action of

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[125] For further reading, see:

- Cartwright, D.E., *Tides*, Cambridge University Press, 2000, 292 pp.

the gravitational forces of other astronomical bodies, the orbit of the moving body is that of its center of mass. The finite extent *of the body introduces two new effects:*

(1) A precession *of the axis of* rotation *in case of a rotating body (provided the body is not a sphere with homogeneous or concentric distribution of mass).*

(2) Appearance of tidal forces: Under the assumption that the body is rigid, the forces of inertia *due to the orbital motion (revolution without rotation) are uniform for each mass element of the body. They have the same magnitude and opposite direction to that of the* gravitational *force acting on the body's center of mass. The gravitational forces, however, are not uniform throughout the body so that outside the center of mass, differences appear between gravitational forces and the inertial forces. These differences appear as tidal forces: Over the earth's hemisphere nearer to the moon, the gravitational attraction is* greater *and the* centrifugal[126] *acceleration is* less *so that over this hemisphere there is a distribution of unbalanced upward force directed moonward and acting* against *the earth's own gravitation force.*

Over the opposite hemisphere the gravitational force of the moon is less *and the centrifugal force is* greater *than at C, so that there, too, a distribution of unbalanced force results, directed obliquely upward. Particles free to move, like those of the sea and air, will do so under the actions of these unbalanced forces. The mathematical formulation of this concept, on the basis of the law of gravitational attraction, does not require more than elementary algebra.*

A tide-producing force (on the surface of a hypothetical yielding, spherical earth) can be resolved into a tangential (horizontal) component and a normal (vertical) component. Assuming this sphere to be covered with an oceanic layer, the motion *of the water on its surface will be governed by the horizontal component of the tidal force. This motion will lead toward an accumulation, or a heaping up, of water at the sublunar centers (points nearest and farthest from the moon), with an attendant rise in sea level.*

On the other hand, a withdrawal of water will tend to take place along the great-circle zone on a plane normal to the earth-moon line, where the sea level must fall. In general if the whole spherical earth were made of a yielding material and made to respond to the tidal forces by deforming freely, it would assume the shape of an ellipsoid with its major axis coinciding with the earth-moon direction. An equilibrium shape would be reached when the inequalities of the earth's own gravitational attraction (resulting from the development

[126] An earthbound observer co-revolving with the earth prefers to see it as a *centrifugal* force, while an outside observer will see it as a *centripetal* force. Both views are equivalent.

of its prolated form), exactly counterbalance the tide-producing forces at all points.

Thus, if the earth were not rotating about its own axis, the heaping up of water at the two tidal centers, and the lowering of the sea level along the great-circle zone would quickly reach a state of equilibrium, becoming permanently fixed in geographic coordinates. We would then have no tidal fluctuations due to the moon's attraction. As the earth turns on its axis, however, the direction and magnitude of the tidal force acting at any given place on the earth's surface changes periodically. This causes the two tidal centers (bulges) to move westward as a tidal wave around the earth. At any given geographical point, the period of the lunar tide is 12 *moon-hours, i.e.* 12^h24^m *(the time*[127] *elapsed between successive meridian passages of the moon).*

When Newton developed his theory of tides he assumed, for the sake of simplicity, that the earth was covered by an ocean of uniform depth and that the flow of water to the two centers of tidal rise would quickly bring about an equilibrium form of the sea surface in which pressure differences would exactly balance the horizontal forces. Thus, the water are devoid of inertia in this approximation, whereas its gravitational properties are kept. This is Newton's theory of the equilibrium tide. The observed tides are generally much greater than those derived from the equilibrium theory. The oceans are unable to respond instantly and completely to the rapidly moving system of horizontal forces. Nevertheless, the equilibrium theory is valid as a fundamental explanation of the tide.

Newton was well aware of this discrepancy between theory and fact, but pursued it no further. The quasi-static theory was completed in 1741 by ***Daniel Bernoulli****,* ***L. Euler*** and ***C. Maclaurin****. In* 1774*,* ***P.S. Laplace*** *presented his dynamic theory of tides. He considered tides as waves induced in a uniform ocean layer by periodic forces, taking into account Coriolis forces and friction. But even this theory could not account for local observations. The nature of actual tides is complicated due to the presence of land masses stopping the flow of water, the unknown friction in the oceans and between oceans and the ocean floors, the rotation of the earth, the variable depth of the ocean, winds and other factors*[128]*.*

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[127] During the semi-diurnal period of 12^h, the moon advances in its orbit about the earth. Since its speed is 30 times slower than that of the earth, the earth must spend $\frac{12^h}{30} = 24^m$ more to overtake the moon.

[128] The numerical integration of the *Laplace tidal equation* (1774) for realistic models of the world oceans was undertaken by **C.L. Pekeris** (1908–1993, Israel) during 1969–1978.

Both the times and the heights of high tide vary considerably from place to place on the earth. The earth's rapid rotation causes the tide-raising forces within a given mass of water to vary too rapidly for the water to adjust completely to them. These forces however, recurring periodically, set up forced oscillations *in the ocean surfaces, so that the water over a large area rises and falls in step. Consequently, the highest water does not necessarily occur when the moon is highest in the sky (or lowest below the horizon).*

Sometimes shallow coastal seas have such shapes and sizes that the natural frequency of the basin waters is very nearly the same as that of the tidal period in the adjacent ocean. Then the ocean tide can set up resonance oscillations in the basin (e.g. the Bay of Fundy between New Brunswick and Nova Scotia and the Gulf of Maine; under favorable conditions, the tidal range at the head of the Bay of Fundy can exceed 15 *meters).*

*Tides also occur in the atmosphere (**Laplace**, 1825) and the solid earth (**Lord Kelvin**, 1863).*

The sun too produces tides on the earth, although it is less than half as effective a tide-raising agent as the moon. Actually, the gravitational attraction between the sun and the earth is about 180 *times as great as that between the earth and the moon, but the earth-sun distance is about* 390 *times larger than the earth-moon distance. The moon's tides, therefore, dominate. On the other hand, when the sun and the moon are lined up at new or full moon, both tides reinforce each other* (spring-tides)*. In contrast, when the moon is at first quarter or last quarter, the tides produced by the sun partially cancel out the tides of the moon, and the tides are lower than usual* (neap-tides)[129].

Consider first the lunar effect, and neglect the rotation of the earth about its axis. The centripetal acceleration of the earth and the moon, required for revolution of the earth-moon system about their common mass-center, is provided by their mutual attraction*, but only at the mass center of each body is the gravitational force precisely equal to the centripetal force.*

Put the centers of the earth and the moon at points O and M respectively (a distance d apart on a z-axis). At an arbitrary point P in the earth with coordinates $\{r, \beta\}$ relative to the z-axis, the centripetal acceleration is also in the z-direction. The difference between this acceleration and the acceleration due to the moon's attraction, gives the tidal acceleration at P.

To see this quantitatively we perform a preliminary calculation for points on the earth nearest and farthest from the moon. The moon's attraction

[129] The unit of acceleration in the cgs system is 1 cm s^{-2} and this unit is called 1 *gal*, in honor of **Galileo Galilei** (1564–1642) who was a pioneer in the study of motion of bodies under gravity. One thousandth of a *gal* is called 1 milligal (mgal). One millionth of a *gal* is called 1 microgal (μgal).

on unit masses located at these points, and at the earth's mass center, are $f_1 = GM(d-r)^{-2}$, $f_2 = GM(d+r)^{-2}$ *and* $f_0 = GMd^{-2}$ *respectively, with* M *the lunar mass. Hence*

$$f_1 - f_0 = GM\big[\frac{1}{(d-r)^2} - \frac{1}{d^2}\big] = GM\frac{(2d-r)r}{d^2(d-r)^2};$$

$$f_2 - f_0 = GM\big[\frac{1}{(d+r)^2} - \frac{1}{d^2}\big] = -GM\frac{(2d+r)r}{d^2(d+r)^2}.$$

Since $d \gg r$, *we have to a good accuracy*

$$f_1 - f_0 \cong \frac{2GMr}{d^3}, \qquad f_2 - f_0 = -\frac{2GMr}{d^3}.$$

For points which are not necessarily on the mass-center line, the above calculation involves vector subtraction. It is then more convenient to write the gravitational potential Φ *of the moon, acting on a unit mass at* P *at distance*

$$R = \sqrt{d^2 - 2dr\cos\beta + r^2} = d\sqrt{1 - 2\frac{r}{d}\cos\beta + \frac{r^2}{d^2}}$$

from the moon's center: $\Phi = -\frac{GM}{R} + C$. *Clearly,* C *is a constant that must be assigned the value* $\frac{GM}{d}$ *in order to secure* $\Phi(r=0) = 0$.

Thus

$$\Phi = \frac{GM}{d}\big[1 - \big(1 - 2\frac{r}{d}\cos\beta + \frac{r^2}{d^2}\big)^{-1/2}\big].$$

Expanding the inverse square root in a series of Legendre polynomials we find

$$\Phi = -\frac{GM}{d^2}z - \frac{GMr^2}{d^3}\big[P_2(\cos\beta) + \frac{r}{d}P_3(\cos\beta) + \cdots\big],$$

where $z = r\cos\beta$. *The first term on the r.h.s. is just the uniform centripetal acceleration. The second term dominates the moon's tide generating potential. Since* $\frac{r}{d}$ *is about* $\frac{1}{60}$, *this term is very often sufficient. The force associated with this potential, namely* $\mathbf{f} = -\operatorname{grad}\Phi$, *has the radial (vertical) component*

$$f_v = -\frac{\partial\Phi}{\partial r} = \frac{3GMr}{d^3}\big(\cos^2\beta - \frac{1}{3}\big)$$

and the azimuthal (horizontal) component

$$f_h = -\frac{1}{r}\frac{\partial\Phi}{\partial\beta} = -\frac{3GMr}{d^3}\sin\beta\cos\beta.$$

These expressions show that the amplitudes of the radial and the azimuthal tidal accelerations on the earth's surface are $2g\big(\frac{M}{E}\big)\big(\frac{r}{d}\big)^3$ *and* $\frac{3}{2}g\big(\frac{M}{E}\big)\big(\frac{r}{d}\big)^3$

respectively, where $g = G\frac{E}{r^2}$ *is the surface gravity acceleration and* E *is the mass of the earth. Taking the values* $\frac{M}{E} = \frac{1}{81.5}$, $\frac{r}{d} = \frac{1}{60}$, *one obtains a vertical acceleration of* 0.112 *mgal against a centripetal acceleration of*

$$\frac{GM}{d^2} = g\Big(\frac{M}{E}\Big)\Big(\frac{r}{d}\Big)^2 = 3.38 \ mgal.$$

Thus, the vertical tidal acceleration acting on a mass of 1 *kg at sea-level is only* 0.11 *milligram, a mere* 10^{-7} *g! [This will make the weight of the "Queen Elizabeth" ocean liner (83,673 tons) lighter by ca* 9 *kg as she passes under the zenith of the moon, compared to a location where* f_v *vanishes.] However, being an unbalanced force, it may nevertheless cause large displacements.*

The tide-generating potential which incorporates the rotation of the earth about its axis, is obtained directly from the above expression for Φ *if we go over from the intrinsic earth-moon coordinate system to the celestial sphere coordinate system. Applying the trigonometric identity*

$$\frac{1}{2}(3\cos^2\beta - 1) \equiv \frac{1}{2}(3\sin^2\delta - 1)\frac{1}{2}(3\cos^2\theta - 1)$$

$$+\frac{3}{4}\sin 2\delta \sin 2\theta \cos H + \frac{3}{4}\cos^2\delta \sin^2\theta \cos 2H$$

(δ = moon's declination; (φ, θ) = spherical coordinates of observatory; $H = \varphi + t$; $t = -\varphi_{_M}$ = hour angle of moon at Greenwich], the three terms on the r.h.s. of this identity represent respectively the lunar fortnightly tide M_f, *the principal lunar diurnal component* O_1 (25.82^h) *and the principal lunar* M_2 (12.42^h). *The explicit expression for the latter potential is*

$$\Phi_2 = \frac{3}{4}gr\Big(\frac{M}{E}\Big)\Big(\frac{r}{d}\Big)^3 \cos^2\delta \sin^2\theta \cos 2(\varphi + t).$$

For $\theta = 90^\circ$, r *= mean equatorial radius, and* $\langle\cos^2\delta\rangle = 0.722$, *the height of the tide at the equator follows the expression*

$$\eta = \frac{1}{g}\Phi_2 = 25.6\cos 2(\varphi + t) \ cm.$$

The corresponding expression for the sun's tide is $11.8\cos 2(\varphi + t)$ *cm. Since* δ *and* d *both depend on time, the tide at any given location, even in the framework of equilibrium tidal theory, is a combination of a great number of Fourier components, each with its own period, amplitude and phase.*

Of special interest in the earth sciences is a rare event, occurring approximately every 1600 years, when perigee (moon closest to earth) coincides with

syzygy (centers of sun, earth and moon are collinear) as well as with perihelion (sun closest to earth); the moon's nodes are on the line connecting the earth and the moon (moon on the ecliptic); and the declination between the moon and the sun is zero. These conditions, which give the greatest possible tide-raising force, have the following schedule of occurrence: 3500 BCE, 1900 BCE, 250 BCE, 1433 CE, *3300* CE.

Newton's Calculus of Fluxions (1664–1671)

When Newton received his B.A., at the age of 23, in June 1665, his examiner, Professor Barrow was of the opinion that Newton did not even know his basic Euclid. Newton had indeed sorely neglected the syllabus. What Barrow did not realize was that Newton was already advancing beyond Descartes, who in his turn had already advanced beyond Euclid. Newton was entirely self-taught — in the sense that he worked largely alone, from books. All his work was confined to his notebooks — which nobody else had seen. However, despite the gaps in his knowledge, he was allowed to continue studying for an M.A. degree.

Newton seemed to thrive in isolation, and events now conspired to make sure this continued. By August 1665[130] *Cambridge University had effectively closed down on account of the Great Plague in London, causing Newton to return to Woolsthorpe, where he remained for about a year.*

Newton's first major breakthrough was the development of the differential calculus – a method for finding the tangent to a point on a curve[131]. *This*

[130] This was to result in an *annus mirabilis*, the like of which was **Einstein**'s 1905, 240 years later.

[131] **Pierre de Fermat** (1638) beat him to that and was considered by **Lagrange** to be the true originator of the differential calculus.

method, which he named the *method of fluxions*, was communicated to Barrow in 1669, written in 1671, but was not published until 1736.

In this work, Newton considers a curve as generated by the continuous motion of a point. Under this conception the abscissa and the ordinate of the generating point are, in general, changing quantities. A changing quantity is called a *fluent* (a flowing quantity), and its rate of change is called the *fluxion* of the fluent. If a fluent, such as the ordinate of the point generating a curve, be represented by y, then the fluxion of this fluent is represented by $\dot{y}$. In another standard notation we see that this is equivalent to $\frac{dy}{dt}$, where t represents time.

In spite of this introduction of time into geometry, the idea of time can be evaded by supposing that some quantity, say the abscissa of the moving point, increases constantly. This constant rate of increase of some fluent is called the *principal fluxion*, and the fluxion of any other fluent can be compared with this principal fluxion. The fluxion of $\dot{y}$ is denoted by $\ddot{y}$, and so on for higher ordered fluxions.

On the other hand, the fluent of y is denoted by the symbol y with a small square drawn about it, or sometimes by $\overset{|}{y}$ (these notations are no longer used).

Newton also introduces another concept, which he called the *moment* of a fluent; it is an infinitely small amount by which a fluent such as x increases in an infinitely small interval of time o. Thus, the moment of the fluent x is given by the product $\dot{x}o$.

Newton remarks that we may, in any problem, neglect all terms that are multiplied by the second or higher power of o, and thus obtain an equation between the coordinates x and y of the generating point of a curve and their fluxions $\dot{x}$ and $\dot{y}$.

As an example he considers the cubic curve

$$x^3 - ax^2 + axy - y^3 = 0.$$

Replacing x by $x + \dot{x}o$ and y by $y + \dot{y}o$, we get

$$\begin{aligned} x^3 &+ 3x^2(\dot{x}o) + 3x(\dot{x}o)^2 + (\dot{x}o)^3 \\ &- ax^2 - 2ax(\dot{x}o) - a(\dot{x}o)^2 \\ &+ axy + ay(\dot{x}o) + a(\dot{x}o)(\dot{y}o) + ax(\dot{y}o) \\ &- y^3 - 3y^2(\dot{y}o) - 3y(\dot{y}o)^2 - (\dot{y}o)^3 = 0. \end{aligned}$$

Now, using the fact that $x^3 - ax^2 + axy - y^3 = 0$, dividing the remaining terms by o, and then rejecting all terms containing the second or higher power of o, we find

$$3x^2\dot{x} - 2ax\dot{x} + ay\dot{x} + ax\dot{y} - 3y^2\dot{y} = 0.$$

Newton considered two types of problems. In the first type, we are given a relation connecting some fluents, and we are asked to find a relation connecting these fluents and their fluxions. This is what we did above, and is, of course, equivalent to differentiation. In the second type, we are given a relation connecting some fluents and their fluxions, and we are asked to find a relation connecting the fluents alone. This is the inverse problem and is equivalent to solving a differential equation. The idea of discarding terms containing the second and higher powers of o was later justified by Newton by the use of primitive limit notions.

Newton made numerous and remarkable applications of his method of fluxions. He determined maxima and minima, tangents to curves, curvature of curves, points of intersection, convexity and concavity of curves, and he applied his theory to numerous quadratures and to the rectification of curves.

Newton's awkward notation led him into long and complex calculations. But he eventually derived simple rules for differentiation of the elementary polynomial, algebraic, trigonometric and exponential functions.[132]

This process of the differential calculus provided the new mathematics with one of its most powerful tools — allowing the calculation of all kinds of rates of change. This included, for instance, the determination of maximum and minimum points in any curve – which occur when the rate of change, $\frac{dy}{dx}$, *is equal to zero.*

Problems of maximum and minimum were solved by great mathematicians long before Newton; in Euclid's Elements VI, 27 we read:

> *"Of all the parallelograms applied to the same straight line and deficient by parallelogrammic figures similar and similarly situated to that described on the half of the straight line, that parallelogram is greatest which is applied to half of the straight line and is similar to the defect"*

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132 The rules of thumb for differentiation, in modern notation, are put in very simple terms, and can be mastered even by high-school kids who know what to do, but don't really know what they are doing. Thus, for example, who was not amazed to learn that the function $y = e^x$, like a phoenix rising again from its own ashes, is its own derivative. Indeed, by Newton's own method:

$$\dot{y} = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = e^x \lim_{h \to 0} \left\{ \frac{e^h - 1}{h} \right\} \to e^x.$$

After a lapse of 1900 years, problems of maxima and minima were taken up again by ***Fermat*** *(1638). Fermat's method was made more general by Newton, who called it 'the method of fluxions'.*

Early Applications of the Calculus

(1) Numerical Approximations

Today, a student of mathematics pushes a little key on his or her pocket calculator, and there appears π*, correct to* 8 *decimal places, ready to be used. Whenever higher accuracy is required, a simple computer subroutine can produce* π *to hundreds decimal places in a matter of seconds. In the pre-calculus era,* π *was still being calculated through the old Archimedean method of regular polygons. The peak of this endeavor was reached in* 1596*, when* ***Ludolph van Ceulen****, after devoting years of effort to the task, calculated* π *to* 35 *significant figures, using polygons with* 2^{62} *sides!*

This inefficient classical method of approximating π *had carried mathematicians far. But in the* 17^{th} *century came a mathematical explosion of epic proportions, one of whose advances at last supplanted Archimedes' approach and pushed the search for* π *into a new phase. In* 1665*, young* ***Isaac Newton*** *applied his generalized binomial theorem and his newly invented method of fluxions — that is, calculus — to get a very accurate estimate of* π *with relative ease.*

He considered a circle having its center at $C\left(\frac{1}{2}, 0\right)$ *and radius* $r = \frac{1}{2}$*. Since its equation is* $y = \sqrt{x - x^2}$*, the area of the circular segment* ABD*, with a base* $A(0,0)$ *to* $B\left(\frac{1}{4}, 0\right)$*, is*

$$S = \int_0^{1/4} \sqrt{x}\,\sqrt{1-x}\,dx,$$

where D *is a point on the circle at* $x = \frac{1}{4}$*,* $y = \frac{1}{4}\sqrt{3}$*. Replacing* $\sqrt{1-x}$ *by its binomial expansion and integrating term by term, Newton obtained*

$$S = \frac{2}{3 \cdot 2^3} - \frac{1}{5 \cdot 2^5} - \frac{1}{28 \cdot 2^7} - \frac{1}{72 \cdot 2^9} - \cdots$$

On the other hand, the segment ABD equals the sector ACD less the triangle BCD, and since $CD = \frac{1}{2}$, $BD = \frac{\sqrt{3}}{4}$, Newton found $S = \frac{\pi}{24} - \frac{\sqrt{3}}{32}$. On comparing the two expressions for S, he obtained

$$\pi = \frac{3\sqrt{3}}{4} + 24\left[\frac{1}{12} - \frac{1}{5 \cdot 2^5} - \frac{1}{28 \cdot 2^7} - \frac{1}{72 \cdot 2^9} - \cdots\right].$$

Here, 22 terms were sufficient to give him 16 decimal places (the last term was incorrect because of the inevitable round-off error). Since $a_n/a_{n+1} \to 4$, the method is not suitable for the calculation of many *significant figures.*

Newton also devised and ingenious algorithm for approximating the roots of a numerical equation, known today as the Newton-Raphson method. *To find a root x of an algebraic or transcendental equation $f(x) = 0$, one starts with a given approximation x_n and seeks an improved approximation x_{n+1}.*

Let e_n, e_{n+1} be the respective errors in x_n, x_{n+1} so that

$$x_n = x + e_n, \qquad x_{n+1} = x + e_{n+1}.$$

Expanding by Taylor's series we get

$$0 = f(x) = f(x_n - e_n) = f(x_n) - e_n f'(x_n) + \frac{1}{2}\,e_n^2 f''(x_n) - \cdots .$$

If $f'(x_n) \neq 0$ and if we ignore e_n^2 and higher powers we get

$$e_n \approx \frac{f(x_n)}{f'(x_n)}; \qquad x \approx x_n - \frac{f(x_n)}{f'(x_n)}.$$

It follows that

$$x_{n+1} \approx \left[x_n - \frac{f(x_n)}{f'(x_n)}\right] + e_{n+1}.$$

Discarding e_{n+1} and setting x_{n+1} equal *to the first r.h.s. term, one can easily show that $e_{n+1} = ke_n^2$ where $k = \frac{1}{2}\frac{f''(x)}{f'(x)}$, and therefore negligible, within the limits of the claimed accuracy. All told, an improved approximation to x is then*

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

which is the Newton-Raphson formula.

It has a simple geometrical *interpretation: draw $y = f(x)$ and construct a tangent to the curve at $x = x_1$. This tangent intersects the x-axis at $x = x_2$. At $x = x_2$ erect a line normal to the x-axis and let the line intersect the curve. Draw a new tangent to $f(x)$ at this point. It intersects the x-axis at $x = x_3$,*

etc. The process can be repeated and the root of $f(x) = 0$ is approached with great rapidity.

Joseph Raphson *(1648–1715) published (1690) a tract, Analysis aequatorium universalis which essentially describes the method. Newton's earliest printed account appeared in Wallis' Algebra (1685).*

The chief contribution of ***Wallis*** *to the development of the calculus lay in the theory of integration. The first to realize in full generality that differentiation and integration are reverse operations was* ***Isaac Barrow*** *(1670). He developed a method of determining tangents that closely approached the methods of calculus.*

At this stage of the development of differential and integral calculus many integrations had been performed, many cubatures, quadratures, and rectifications effected, a process of differentiation had been evolved and tangents to many curves constructed, the idea of limits had been conceived, and the fundamental theorem recognized.

What more remained to be done? There still remained the creation of a general symbolism with a systematic set of formal analytical rules, and also a consistent and rigorous redevelopment of the fundamentals of the subject. It is precisely the first of these, the creation of a suitable and workable calculus, *that was furnished by Newton and Leibniz, working independently of each other. The redevelopment of the fundamental concepts on an acceptably rigorous basis had to outwait the period of energetic application of the subject, and was the work of the French analyst* ***Augustin-Louis Cauchy*** *(1789–1857) and his nineteenth-century successors.*

With the invention of calculus, the history of elementary mathematics had essentially terminated. There remained, however, one special preoccupation which still kept haunting the spirit of mathematicians – the mystique of π!

Ever since the ancient Greek circle-squarer's cult, this number held mathematicians in wonder and awe and each generation devised new representation for it. The calculus served as a novel tool to this end:

John Wallis *started from the equation*

$$\int_0^1 \sqrt{1-x^2}dx = \frac{\pi}{4}$$

to obtain

$$\frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdots} \qquad \textit{(1655)}$$

William Brouncker *then followed with a novel expression of his own*

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cdots}}}}. \qquad \textit{(1660)}$$

Newton *used*

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x$$

to obtain

$$\frac{\pi}{3} = 1 + \frac{1}{(3 \cdot 2^3)} + \frac{1 \cdot 3}{4} \frac{1}{(5 \cdot 2^5)} + \frac{1 \cdot 3 \cdot 5}{4 \cdot 6} \frac{1}{(7 \cdot 2^7)} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{4 \cdot 6 \cdot 8} \frac{1}{(9 \cdot 2^9)} + \cdots \qquad \textit{(1665)}$$

James Gregory *(1638–1675) appeared on the scene in 1663. A Scott mathematician and astronomer, he was one of the first to distinguish between convergent and divergent series. He expanded (1671) the infinite series*

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots ,$$

which for $x = 1$ *yields*

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

In 1699, the $\tan^{-1} x$ *series was used by him, with* $x = \sqrt{\frac{1}{3}}$, *to evaluate* π *to 71 correct decimal places. Thus Gregory preceded* ***Brook Taylor*** *(1712) in series expansion of a function about a point. Note that the* $\tan^{-1} x$ *series can be alternatively written with* $x = \tan\theta$, *as*

$$\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta + \cdots$$

and as such, has some computational advantages.

Another important series was discovered by ***Nicolaus Mercator-Kaufmann*** *(1650)*

$$\log_e(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$

It is sometimes referred to as the Mercator series. It was independently discovered by ***G. Saint-Vincent*** *(1584–1667).*

The arithmetic, algebra, geometry, and trigonometry ordinarily taught in the schools today, along with college algebra, analytic geometry, and the calculus usually taught during the freshman or sophomore year in college, constitute what is generally called "elementary mathematics".

At this point, then, we have virtually concluded the historical treatment of elementary mathematics in the form that we have it today. It is interesting to note, without carrying the generalization too far, that the sequence of mathematics courses studied in the classroom follows quite closely the evolutionary trend of the subject.

(2) Mathematical Astronomy: orbits and gravitation

Orbits were discussed already by the Greeks, and their method of epicycles is, in fact, an early application of Fourier series. ***Copernicus*** *(1543) proposed openly that the planets and the earth were in circular orbit round the sun. However, astronomical observations soon began to show that his proposal was not strictly accurate.*

In 1609, Kepler showed that a planet moved around the sun in an elliptical orbit which has the sun in one of its focii (First Law). He also showed that a line joining the planet to the sun sweeps out equal areas in equal times (Second Law). These laws were not accepted with enthusiasm by Kepler's peers: the first was given a cool reception and was thought to require further work to confirm it. The second was ignored (!) by scientists for about 80 years. Kepler's Third Law, that the squares of the periods are proportional to the cubes of the mean radii of their orbits (1619), was, however, widely accepted from the time of its publication.

Newton *(1665–6; 'Principia' 1687) suggested, for the first time, that planetary motion is a result of a central force, proportional to the inverse-square distance from the centers of the sun and the planet. He then associated this force with a* universal *law of gravitation. It is possible that* ***Hooke*** *(1679) independently deduced the inverse-square-law, using Huygens' law of centripetal acceleration (1673) and Kepler's Third Law.*

In his Principia *(1687), the problem of two attracting bodies is completely solved, showing that an inverse-square law must produce elliptical, parabolic*

and hyperbolic orbits. The observed parabolic orbit of a bright comet (visible 14 Nov 1680 – 05 Dec 1680) confirmed Newton's theory.

For more than two point masses, only approximations to the motion of a body could be found and this line of research led to large efforts by mathematicians to develop methods to attack this 3-body problem. *But even if the earth-moon system were considered as a 2-body problem, the orbits could not be simple ellipses; neither the earth nor the moon is a perfect sphere and so does not behave as a point-mass. This was to lead to the development of mechanics of rigid bodies, but even this would not give a complete accurate picture of the 2-body problem since* tidal forces *mean that neither the earth nor the moon are rigid.*

***Halley** (1682) calculated the perturbations of Jupiter and Saturn on the orbit of a comet which appeared in 1378, 1456, 1537, 1607 and 1682, and used the Newtonian theory to predict its return on 13 April 1759, giving an error of one month on either side of this date. The comet was indeed observed, reaching perihelion on 12 March 1759.*

Note that the notion that bodies fell to earth owing to some form of attraction exerted by the earth did not originate with Newton. His genius, however, showed itself in extending this idea to the whole universe, formulating his result in a single law, and verifying it by examination of the motion of the planets, comets, the earth and the moon.

Newton was worried that his model of the solar system could become gravitationally unstable in the long run. (He was correct, but this was not proven until 1989). To compensate for the instability, he suggested a cyclic process whereby the planets would be assisted by God when they were periodically perturbed from their orbits by their mutual gravitational action.

*The great mathematical physicists of the 18^{th} century, such as **Euler**, **Laplace**, and **Lagrange**, showed that the solar system was in fact stable to first order, the perturbations which worried Newton leading merely to a* cyclic *oscillation of the planetary orbits. The periods of the oscillations were of the order of a few thousand years, and the astronomers of the 19^{th} century concluded that the solar system was stable for at least this length of time.*

More than 300 years after the publication of the Principia, *the full implications of Newton's deceptively simple law of gravity, with its surprisingly complicated consequences, still elude us. One has only to look at the strangeness of a chaotically tumbling satellite like* Hyperion *or at the intrinsic difficulties of calculating the moon's itinerary or delving into the solar system's origin, to sense the dynamical mysteries that confront us. Apparently, even in the classical world God, after all, 'plays dice'. Yet, recent studies of the dynamics of the solar system assure us that its past history certainly suggests that it probably remains stable for geologically significant periods.*

Newton, Shakespeare and the Law of Gravitation

*A sign of **Shakespeare's** (1564–1616) many-sided genius is his anticipation of a scientific vision of later times: **Kepler's** Third Law was discovered in 1618 and Newton's law of universal gravitation was stated by him in 1687. Yet, in Troilus and Cressida (1609) the heroine thus expressed herself (iv.2):*

"Time, force, and death,
Do to this body what extremes you can,
But the strong base and building of my love
Is as the very centre of the earth,
Drawing all things to it."

*Indeed, **Newton** cannot rightly be said to have discovered the law of gravitation; he only applied it to the movements of members of the solar system. Even **Aristotle** had defined weight as "the striving of heavy bodies towards the centre of the earth". Among men of classical culture in England in Shakespeare's time, the knowledge that the centre point of the earth attracts everything to it was quite common. It seems that several of the men whose society Shakespeare frequented were among the most highly-developed intellects of the period. That his astronomical knowledge was not, on the whole, in advance of his times is proved by the expression, "the glorious planet Sol" (Troilus and Cressida i,3). He never got beyond the Ptolemaic system.*

*Another example of this kind concerns the field of geology: **Steno** (1669) first systematized geological conceptions; but he was by no means the first to hold that the earth has been formed little by little, and that it was therefore possible to trace in the record of the rocks the course of the earth's evolution. His chief service lay in directing attention to* stratification*, as affording the best evidence of the processes which have fashioned the crust of the earth.*

In the second part of Henry IV (iii, 1), composed in 1597, King Henry says: –

"O God! that one might read the book of fate,
And see the revolution of the times
Make mountains level, and the continent,
Weary of solid firmness, melt itself
Into the sea! and, other times, to see

The beachy girdle of the ocean
Too wide for Neptune's hips; how chances mock,
And changes fill the cup of alteration
With divers liquors!"

The purport of this passage is simply to show that in nature, as in human life, the law of transformation reigns; but no doubt it is implied that the history of the earth can be read in the earth itself, and that changes occur through upheavals and depressions.

There is nothing in these lines that presupposes any special or technical knowledge; Shakespeare's knowledge was not of a scientific cast. He learned from men and from books with the rapidity of genius. Not, we may be sure, without energetic effort, for nothing can be had for nothing; but the effort of acquisition must have come easy to him, and must have escaped the observation of all around him. There was no time in his life for patient research; he had to devote the best part of his days to the theater, to uneducated and unconsidered players, to entertainments, to the tavern. We may fancy that he must have had himself in mind when, in the introductory scene to Henry V (1598) he makes the Archbishop of Canterbury thus describe his hero, the young king: –

"Hear him but reason in divinity,
And, all-admiring, with an inward wish
You would desire the king were made a prelate:
Hear him debate of commonwealth affairs,
You would say, it hath been all-in-all his study:
List his discourse of war, and you shall hear
A fearful battle render'd you in music:
Turn him to any cause of policy,
The Gordian knot of it he will unloose,
Familiar as his garter; that, when he speaks,
The air, a charter'd libertine, is still,
And the mute wonder lurketh in men's ears,
To steal his sweet and honey'd sentences;
So that the art and practice part of life
Must be the mistress to this theoric:
Which is a wonder, how his grace should glean it,
Since his addiction was to courses vain;
His companies unletter'd, rude, and shallow,
His hours fill'd up with riots, banquets, sports;
And never noted in him any study,
Any retirement, any sequestration
From open haunts and popularity."

To this the Bishop of Ely answers very sagely, "The strawberry grows underneath the nettle." We cannot but conceive, however, that, by a beneficent provision of destiny, Shakespeare's genius found in the highest culture of his day precisely the nourishment it required.

1665 CE First mathematical journal appeared.

1665–1684 CE The Italian family of Cassini produced four generations of astronomers who succeeded each other in official charge of the observatory of Paris. The first was **Giovanni Domenico Cassini** (1625–1712), who first determined the rotation periods of Jupiter, Mars and Venus (1665–1667). During 1671–1684 he discovered 4 Saturnian satellites and in 1675 he found the division in Saturn's ring named after him. Made the earliest sustained observations of the zodiacal light.

Cassini was also the first person to see the Martian polar caps, which bear a striking resemblance to the arctic and antarctic polar caps on earth. (More than a century elapsed, however, before **William Herschel** first suggested that the Martian polar caps are made of ice.)

G.D. Cassini was born near Nice. Educated by Jesuits at Genoa, he was nominated in 1650 professor of astronomy at the University of Bologna. In 1657 he was appointed director of waterways in the papal states by Pope Alexander VII. Louis XIV of France applied for his services in 1669. He died at the Paris Observatory. A partial autobiography was published by his great-grandson, Count Cassini, in 1810.

As the quality of telescopes improved, details of the ring and of Saturn's cloud cover (Huygens, 1655) became visible. In 1675 G.D. Cassini discovered a dark division in the ring that looks like a gap about 5000 km wide. Afterwards, astronomers began to view the ring as a system of rings, known today as the *Cassini division*.

Cassini was also the first to make an indirect measurement of the *solar parallax* by measuring Mars' distance from us, at its nearest approach to earth. This he achieved by obtaining measures of the *parallax of Mars* at the same time from two stations (Paris and Cayenne, South America), widely separated on the earth's surface. A value of $9.5''$ was obtained for the solar parallax.

The *mean distance* of the earth from the sun is the average of major and minor axes of the earth's orbit. This distance is defined by the *solar parallax*

which is the *angular size* of the earth's radius as seen from the sun. Since the earth is not quite spherical, the *equatorial radius* is the one used, and because its distance from the sun varies, the *mean radius*, corresponding to the mean distance, is employed. The measured quantity is called the sun's *mean equatorial horizontal parallax* (*horizontal*, because it is the angle between the direction of the sun on the horizon and the direction it would have if viewed from the earth's center), or simply a *geocentric parallax*, on the baseline of the earth's radius.

Unfortunately, this parallax is hard to measure directly on account of the great distance to the sun, which makes it less than $9''$. **Aristarchos**, **Kepler** and **Huygens** tried to measure it directly, but obtained very inaccurate results.

Cassini's geometrical method of triangulation was an ingenious way to circumvent a frontal attack: measure first the parallax of Mars at a smaller distance from the sun: every 15 or 17 years, Mars comes to a point where it is nearest to us. At its nearest, Mars' distance from us is little more than $\frac{1}{3}$ of our distance from the sun, and its geocentric parallax is $23''$.

Minor planets are even better subjects than Mars; they have smaller images, and some of them approach nearer than Mars. From all geometrical measures of the solar parallax, the mean value is calculated to be $8''.803 \pm 0.001$. In combination with the best value for the earth's equatorial radius, this gives for our mean distance from the sun the value 1 AU = 149,459,000 $\pm$ 17,000 km. Other methods, based on dynamical and spectroscopic determinations of the solar parallax are in close agreement.

His son **Jacques Cassini** (1677–1756) was born at the Paris Observatory, as was Jacques' son, **César Francois Cassini** (1714–1784), as well as the fourth Cassini, **Jacques Dominique Cassini** (1748–1845). He succeeded in 1784 to the directorate of the observatory, but his plans for its restoration and re-equipment were obstructed in 1793 by the animosity of the National Assembly. He resigned in that year and was thrown into prison in 1794, but released after several months. He then withdrew to his estate at Thury and died there at the age of 97.

1666 CE *Foundation of the French Academy of Sciences in Paris*. The academy arose from an association of a group which used to meet at the cell of **Mersenne** (1588–1648), a man active in spreading the teaching of Galileo. The original members included **Descartes**, **Pascal**, **Fermat** and **Gassendi** (whose commentaries on Epicuros revived the atomistic speculations of the early Greek materialists).

The Paris Academy, like the English Royal Society, was actively interested in all problems related to navigation, then the cornerstone of mercantile supremacy. Under its auspices, the Paris observatory was inaugurated and completed 3 years before the one at Greenwich. A rich harvest of discoveries followed immediately. To Paris came Cassini from Italy and **Römer** from Denmark. Cassini undertook the calculation of tables forecasting eclipses of Jupiter's satellites for use in determining longitude at sea (the project was undertaken in accordance with a suggestion made by Galileo himself).

The determination of longitudes by eclipses of Jupiter's satellites merely depend on the known fact that the same event does not occur at the same solar time in two places on different meridians of longitude. The tables that Cassini prepared for calculating longitude by observations of the satellites of Jupiter, were used by the French Navy during the first half of the 18^{th} century.

The academy sponsored several expeditions, notably one to French Guiana with a view to simultaneous observations on the parallax of Mars from the Paris observatory and Cayenne (Lat. $4°46'$N). This expedition, which gave the first relatively satisfactory scale of the solar system, ushered in a new era in clock technology.

1666–1686 CE **Thomas Sydenham** (1624–1689, England). Physician. A founder of clinical medicine and epidemiology. Often called "the English Hippocrates". Believed and taught that medicine could be learned only at the bedside of the patient. He was a keen observer and gave excellent descriptions of gout, scarlet fever, measles, influenza, smallpox, malaria and hysteria. He had great faith in the healing power of nature, and he felt that fever was nature's way of fighting the injurious matter that caused disease.

Sydenham introduced *opium* into medical practice and adopted *quinine* for the treatment of fevers at the time when many doctors opposed this new drug. He was one of the first to use iron in treating anemia. Studied epidemics in relation to different seasons, years, and ages. Insisted on clinical observations instead of theory.

Sydenham was born at Wynford Eagle, Dorset, and studied medicine (1642–1663) at the universities of Oxford, Cambridge and Montpellier. Served in parliamentary forces in the Civil War. Success came slowly to him, but eventually he gained recognition as one of the great doctors of his time.

1667 CE, July 21 *Treaty of Breda* to end the 'Musk-Seed War' between England, Holland, France and Denmark. It ended a long war over Far-East spice routes between the East-India companies of the respective countries. England retained New Amsterdam (later, *New York*) and Holland got Surinam.

1667–1704 CE **Francis Willughby** (1635–1622, England) and **John Ray** (1627–1705, England). Naturalists. Their plant and animal classification were the first significant attempts since **Aristotle** (334 BCE) to produce systematic taxonomy based on a variety of structural characteristics, including internal anatomy.

From 1663 to 1666 they toured Europe to study flora and fauna and collect specimens.

After the death of Willughby (1672), Ray completed the three-volume *Historica Generalis Planetarium* (1704) in which he attempted to produce an extensive botanical classification based on a scheme of Aristotle but incorporating many of the new plant forms discovered on the 16 th and 17 th century voyagers of discovery. Altogether, 18,600 European species were covered.

Although it was not possible to devise a natural classification system until **Charles Darwin** and **Alfred Wallace** formulated evolutionary theory (1859), Ray's system approached that ideal more closely than those of any of his contemporaries and remained the best attempt at classification until superseded by **Linneaus**' taxonomic work (1735).

Ray was born in Black Notley, near Braintree, Essex. He was educated at Cambridge and was appointed lecturer in Greek (1651), mathematics (1653) and humanity (1655). He was elected FRS in 1667.

1668 CE **John Pell** (1611–1685, England). A scholar whose contributions to mathematics were worthless, but who had the good fortune to propagate his name through the "*Pell* (*or Pellian*) *equation*" erroneously named after him by **Euler** (1759) [some claim that Pell never saw his equation].

Pell was a professor of mathematics at the University of Amsterdam (1643–1646) and a fellow of the Royal Society (1663) [his output is still carefully preserved in the form of 40 folio volumes in the British Museum]. Both **Newton** and **Leibniz** were happy to discuss their latest researches with him, and Oliver Cromwell made him his political emissary to the Protestant cantons of Switzerland. It is not clear today how he earned his reputation as a mathematician. It is known however that for a time he was confined as a debtor in the king's bench prison. Pell died in abject poverty at the College of Physicians in London.

The so-called *Pellian* is the non-linear Diophantine equation[133]

$$x^2 - Ny^2 = 1,$$

[133] For further reading, see:

- Beiler, A.H., *Recreations in the Theory of Numbers* (The Queen of Mathematics Entertains), Dover Publications: New York, 1964, 349 pp.

where N is not a perfect square natural number and (x, y) are integers. References to individual cases of this equation occur scattered throughout the history of mathematics. The Greeks and the Hindus of ca 400 BCE realized that a/b was a good approximation to $\sqrt{2}$ when $a^2 - 2b^2 = \pm 1$, an equation which, unlike $a^2 - 2b^2 = 0$, is solvable by integers.

The most curious of these occurrences is the so-called Cattle-Problem (Problema Bovinum) of **Archimedes** (ca 250 BCE). It contains eight unknowns (numbers of cattle of various kinds) which satisfy 7 linear equations together with 2 conditions which assert that certain numbers are perfect squares. After some elementary algebra, the problem reduces to that of solving the equation $x^2 - (4,729,494)y^2 = 1$, the least solution of which is a number y of 41 digits. The least solution of the original problem, deduced from this, consist of numbers with hundreds of thousands of digits[134].

By 130 *CE*, **Theon of Smyrna** had shown how to find infinitely many approximations to $x^2 - 2y^2 = 1$. The Hindu mathematicians of about 800 *CE* also claimed to have known how to solve equations of this type. Later, ca 1150, a completely general method was given by the Hindu mathematician **Bháskara**. In modern times, **Fermat** seems to have been the first to state categorically that there are infinitely many solutions to the Pellian. In fact, in 1657 he challenged all European mathematicians to solve $x^2 - 109y^2 = 1$ [there was a slim chance that anybody at that time could have found even the least solution to that equation, since we know now that it is $y = 15,\ 140,\ 424,\ 455,\ 100$].

Nevertheless, **William Brouncker** (1620–1684, Ireland) discovered a method to solve $x^2 - 313y^2 = 1$, which is essentially the continued fraction method. **Euler** showed in 1759 how to obtain infinitely many solutions of the *general* Pellian by using the continued fraction[135] expansion of $\sqrt{N}$,

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134 A sphere with the radius of the Milky way could not contain all the cattle even if they were of the size of electrons.

135 The theory of continued fractions shows that a particular solution of

$$x^2 - Ny^2 = 1$$

is $x = p_n$, $y = q_n$, where p_n/q_n is a certain convergent of $\sqrt{N}$. Moreover, from one solution an infinite number of solutions may be found. Thus from the least-values $x = 3$, $y = 2$ of $x^2 - 2y^2 = 1$ one derives

$$x_n = \frac{1}{2}[(3 + 2\sqrt{2})^n + (3 - 2\sqrt{2})^n];$$

$$y_n = \frac{1}{2}[(3 + 2\sqrt{2})^n - (3 - 2\sqrt{2})^n]\frac{1}{\sqrt{2}}.$$

without giving a proof. This was left to **Lagrange** in 1768.

1668 CE **Francesco Redi** (1626–1697, Italy). Physician, naturalist and poet. One of the first to test scientifically *the theory of spontaneous generation*; showed that no maggots developed in meat protected so that flies could not lay their eggs on it.

1668–1674 CE **John Mayow** (1641–1679, England). Chemist, physician and physiologist. His studies on *respiration* show him to have been an investigator much ahead of his time. In *Tractatus quineue medicophysici* he noted the similarities between combustion and respiration in particular that both use only a small proportion of the available air. He suggested that in respiration the volume of air is reduced, but air must consist of two different gases.

Mayow was born in London and studied Medicine and Law at Oxford, graduating in 1670. He practiced medicine in the City of Bath and was chosen a fellow of the Royal Society (1678). The following year, after his marriage, he died in London.

Mayow, who also gave a remarkably correct anatomical description of the mechanism of respiration, preceded **Priestley** and **Lavoisier** by a century in recognizing the existence of *oxygen* (under the guise of his *spiritus nitro-ereus*) as a separate entity distinct from the general mass of the air; he perceived the part it played in combustion and in increasing the weight of the calces of metals as compared with metals themselves.

Rejecting the common notion of his time that the use of breathing is to cool the heart, or assist the passage of the blood from the right to the left side of the heart, he saw in inspiration a mechanism for introducing oxygen into the body, where it is consumed for the production of heat and muscular activity. His extent of influence on **Lavoisier** is the subject of debate.

1668–1692 CE **Anton van Leeuwenhoek** (1632–1723, Holland). Amateur scientist who, in retrospect, deserves to be called the father of *microbiology*. He was the first in his field to leave written records of his findings. Through tedious hand-grinding techniques, Leeuwenhoek prepared hundreds of lenses with magnifying powers up to 300 and resolutions of about 10^{-3} millimeters. With these lenses he studied such diverse materials as stagnant water (protozoa, 1676), blood cells (he discovered and described the *human*[136]

[136] Red blood cells in *frogs* were observed (via microscope) and described in 1658 by **Jan Swammerdam** (1637–1680, Holland). He also described the *metamorphosis of insects* (1669).

red blood corpuscles[137] in 1673), muscle fibers, and spermatozoa. His *animalcules* (tiny animals) were described as being 1000 times smaller than the eye of a louse (which he used as a standard measurement because its size is remarkably constant). He opposed the popular notion that living things can arise from dead matter.

Leeuwenhoek[138] was among the first to estimate the maximal number of people that the earth could support: he argued that if the whole planet were inhabited with the same population density as that of Holland, the number would reach about 13.4×10^9 individuals. This coincides with one of the present estimates.

Revival of the Medical Sciences

Many of the medical works of the ancient Greeks and ***Galen*** *reached Western Europe by a roundabout route that took centuries to complete.*

In 431 CE*, the Church banished Nestorius, the heretical patriarch of Constantinople, with his followers. Their descendants fled to Persia where, at Jundi Shapur, they made the university and its medical school and hospital a leading intellectual center. There the Nestorians translated into Syriac all*

[137] **Leeuwenhoek** had a very good idea of the size of a human blood corpuscle (ca 7.5×10^{-6} m). He also was aware of the fact that the blood corpuscle's size was not growing with the size of the animal. It was later found that it is related to the animal's *activity*, namely to its oxygen intake. We know today that the shape of the platelets evolved to meet three criteria: maximum volume, maximum diffusion-rate and maximal flexibility; hence the special disk-like shape which looks like a cross between pancake and doughnut.

[138] There is a fascinating biography of Leeuwenhoek by **Clifford Dobell**. As a young bacteriologist, Dobell was especially interested in studying the microbial flora of the mouth. However, each time he presented his professor with what he thought was the discovery of a new type of microbe, his professor would shake his head and respond, "No, no, Leeuwenhoek already discovered that one." Finally, motivated by a mixture of curiosity and skepticism, he decided to find out more about this man Leeuwenhoek. After 25 years of painstaking research, Dobell published a truly inspiring biography of Leeuwenhoek in 1932.

the Greek books they could find, including the Hippocratic Corpus and the works of ***Galen****.*

With the rise of Islam in the 7^{th} *century, medical schools spread. In the Eastern Caliphate of Baghdad, Muslim scholars and physicians continued to translate Greek works, adding their own commentaries.*

Islam spread through North Africa and into Spain and south-west France, until it was stopped by the Christians at the Battle of Tours (732 CE*). The Western Caliphate was centered on the Spanish city of Cordova, which had* 50 *hospitals,* 70 *public libraries and the most renowned university of the* 10^{th} *century.*

Abu al-Qasim, Khalaf *(Abul Kasim,* ***Albucasis*** *(936–1013), Spanish-Arab physician, was one of the greatest surgeons of the Middle Ages. Born in Cordova. Wrote al-Tasrif, a medical compendium partly based on earlier authors, but containing new material including remarkable illustrations and surgical instructions; This work greatly influenced European surgery for* 500 *years.*

Ibn Zuhr *(****Avenzoar****, 1090–1162), Muslim physician and greatest clinician of Western Caliphate. Born in Seville; his at-Taysir was influential throughout Europe in Latin and Hebrew translations.*

Maimonides *(1135–1204) became physician to the Saracen sultan, Saladin, whose crusader foe, Richard Coeur de Lion of England, tried in vain to secure the Jewish doctor's services. Maimonides studied the patient, not the disease; he also rejected astrology and attempted to separate medicine from religion.*

The works of these men and the works of ***Galen*** *were first translated from Arabic into Latin already in the* 11^{th} *century. The translation was continued in the* 12^{th} *century, when Galen was translated from the Greek. It marked the beginning of the Western rediscovery of the original ancient texts, to be later continued by the humanists of the Renaissance.*

In the Middle Ages, medical scholars were again carrying out human dissection (with ecclesiastical permission), but only rarely — and in circumstances hardly conductive to learning. The professor, in long robes, sat on high in a great chair reading his anatomy lecture, with the cadaver on a table below him. A junior colleague pointed out the line of incision, and a third — the menial demonstrator — did the actual cutting.

The new teaching methods of ***Mondino dei Liucci*** *(1270–1326, Italy), who taught at the University of Bologna, were a major advance and soon spread to other medical schools. His anatomy textbook, the Anathomia (1316), although full of inaccuracies, passed down from Galen and Avicenna,*

is considered the first modern work on the subject and remained authoritative until the appearance of Vesalius' anatomical work (1543).

In Italy, **Leonardo da Vinci** *(1452–1519), among others, spearheaded the new interest in anatomy. Dissecting in the secret of the night, he reproduced exactly what he saw, in* 750 *anatomical drawings. Unfortunately, his pioneering work remained hidden for more than* 300 *years, and others had to forge on without knowledge of his discoveries.*

In 1543, **Andreas Vesalius**, *a* 29*-year-old Flemish professor of anatomy at Padua, published* De Humani Corporis Fabrica *(The Fabric of the Human Body). Like Mondino, he dissected personally, and his work showed, for the first time, how nerves penetrated muscles, the nutrition of bones, the true relationship of the abdominal organs and the structure of the brain.*

Two of Vesalius' assistants also made major findings and are among the founders of modern anatomy: **Gabriele Fallopio** *(1523–1562) described the internal working of the ear, the anatomy of bones and muscles, and the sex organs: the tubes leading from the ovaries to the uterus are named after him.*

Bartolomeo Eustachi *(1520–1574) studied the kidneys and the head, describing the anatomy of the teeth and, in particular, the '*Eustachian tubes*' from the throat to the middle ear and the '*Eustachian valve*' in the heart.*

The true circulation of the blood continued to elude these pioneers. However, **Miguel Serveto** *(known as Michael Servetus; also used pseudonyms Michael de Villeneuve and Villanovanus, 1511–1553), a Spanish theologician and physician. Lectured on geography and astronomy; practiced medicine at Charlieu and Vienna (1538–1553). He included the first description of the circulation of the blood in the lungs in a theological work entitled* Christianismi restituto *(1553). For that he was arrested and brought to trial before the Inquisition at Lyons; he escaped, but was apprehended at Geneva; imprisoned at Calvin's request and burned at the stake as heretic.*

Andrea Cesalpino *(1519–1603, Italy), physiologist and botanist, stumbled across not only the pulmonary circulation but the systemic circulation as well (1583). Cesalpino was a professor of materia medica and director of the Pisa botanical gardens, physician to Pope Clement VIII and professor at Rome. [He wrote the first true textbook of botany and created first coherent system of taxonomy, to which Linnaeus acknowledged indebtedness.]*

During the Renaissance, surgery made great progress, while medicine remained the province of book-oriented physicians. Many fine surgeons of the Middle Ages had gained experience and knowledge on the battlefields of Europe; for example, the British surgeon **John Arderne** *(1307–1390), who served in the* Hundred Year's War *— and dealt not only with slashes and*

punctures from sword and lance, but also with gaping, dirt-filled wounds caused by bullets from the newly invented guns and artillery.

As the Black Death swept through Italy in 1347 *and* 1348*, taking its terrible toll, the pretensions of the physicians and barber-surgeons were stripped bare*[139]*. But within* 75 *years or so, there was a new air of inquiry in medicine, as the Renaissance began.*

To bolster up their status, physicians created professional structures for themselves, in order to prevent anyone not properly trained (and, in effect, women and Jews) from practicing. Thus, in 1518*, six prominent physicians in London were granted a charter by the King to form the Royal College of Physicians; it could license doctors, and prosecute, fine and imprison, unlicensed practitioners.*

***Philipp Aurelus Theophrastus Bombastus von Hohenheim** (1493–1541) was a German alchemist and physician who had styled himself **Paracelsus** — implying that he was greater than the great Roman encyclopedist **Celsus**. He had earned his niche in medical history as standard bearer for freedom of scientific inquiry, the central position of the patient in medicine and above all a forerunner of pharmaceutical chemistry. His opinion of the state of contemporary medicine is reflected in his conclusions:*

- *"When I saw that nothing resulted from doctor's practice but killing and laming, that they deemed most complaints incurable... I determined to abandon such a miserable art and seek truth elsewhere".*
- *"The best of our popular physicians are the ones who do the least harm. But unfortunately some poison their patients with mercury, and others purge or bleed them to death. There are some who have learned so much that their learning has driven out all their common sense".*

To emphasize his point, he pitched the books of Galen, Avicenna and other masters of medieval medicine on to a bonfire in a public square.

Paracelsus, in many ways ahead of his time, believed in the power of nature and the imagination to cure the body and the mind. The patient had to be treated as a whole: diet, surroundings, the behavior of doctor and carers — all these and more could have a profound effect on recovery. In his own words:

- *"Medicine does not consist of compounding pills and drugs of all kinds, but it deals with the processes of life, which must be understood before they can be guided".*

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139 Small wonder that Shakespeare wrote in *Timons of Athens* (1607) — "*Trust not the physician, His antidotes are poison*".

The 15^{th}- and 16^{th}-centuries 'Rebirth', or Renaissance, in medicine centered on the rediscovery of the ancient Greek and Roman works in their original form and the discovery of the fabric of the body.

On this more solid basis, medicine grew into the age of enlightenment — two centuries that saw Britain's Glorious Revolution (1688), America's Declaration of Independence (1776) and the French Revolution (1789). But these events did not occur in a vacuum. They were reflections of an attitude of mind: a rejection of social and religious constraints; a belief that progress in science and technology would lead to a utopian existence; and, as far as medicine is concerned, a determination that one day all diseases would be conquered — or so they were convinced.

Scientific and technological progress certainly played its part in medicine, as Galen's erroneous theories were finally overthrown, the circulation of the blood was understood, microbes were revealed by the microscope, and smallpox vaccination was introduced.

Galen believed that the body daily manufactures and eliminates large quantities of blood. In 1628 *this theory was overthrown by the British doctor* ***William Harvey*** *(1578–1657). Actually, Harvey made his discovery already in* 1603, *but delayed the publication of his results because he was not sure about the reaction of the medical establishment and needed more time to design a striking experimental proof.*

In 1661, ***Robert Boyle*** *(1627–1691) rejected Aristotle's* 4*-elements and instead proposed an experimental theory of the elements, thus transforming alchemy into scientific chemistry. In addition he revealed that air was necessary for life. In* 1667, *Boyle's former assistant* ***Robert Hooke*** *(1635–1703) demonstrated that the key to respiration was the alteration of blood in the lungs.*

At the end of the 16^{th} *century, the compound microscope was discovered by the Dutch spectacle makers* ***Hans*** *and* ***Zacharias Jansen****. Using one, Robert Hooke first described cells, and in* 1660, *an Italian,* ***Marcello Malpighi*** *(1628–1694), discovered the missing link in Harvey's theory: the tiny capillaries that connect the arteries and veins. However, it was a Delft draper,* ***Anton van Leeuwenhoek*** *(1632–1723), who popularized the medical use of the microscope, describing spermatozoa, red corpuscles and stripped voluntary muscles, as well as protozoa and bacteria.*

In 1709, ***Gabriel Daniel Fahrenheit*** *(1686–1736), a German physicist, invented the alcohol thermometer and, five years later, the mercury thermometer and a temperature scale that stood medicine in good stead for almost three centuries.*

*Gradually, scientific and medical research ceased to be an activity of isolated men of genius, but more organized in academies — such as the Royal Society in London (1660) [which grew out of informal tavern meetings of a group Boyle called the '*Invisible College*'; the Accademia dei Lincei in Rome (1661) and the Académie Royale des Sciences in Paris (1666).*

1669 CE **Erasmus Bartholinus** (1625–1698, Denmark). Physician, mathematician and physicist. Discovered the phenomenon of *double refraction* of light in an anisotropic crystalline substance. Bartholinus obtained some beautiful crystals from a sailor who collected them in Iceland, and when he viewed small objects through them, he found that the objects appeared double.

Bartholinus discovered the origin of this phenomenon[140]; if one sends a narrow beam of light — a light ray — into an ordinary transparent medium such as a piece of glass, it is refracted and then proceeds as a single beam. However, when it is refracted at the face of the Iceland spar (calcite, $CaCO_3$; an *anisotropic* trigonal system) a second beam is generated, and this is the reason for the appearance of a second image. Bartholinus suggested that one of the rays, which resembles the usual one in some ways, be called the *ordinary* ray and the other one, which behaves in a somewhat unusual fashion, be called the *extraordinary* ray.

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140 **Huygens** contributed to the understanding of double refraction. In his book *Traité de la Lumière* (1690) he assumed that when light is incident on the Iceland spar, each element of it produces secondary waves surfaces which are no longer spherical but rather consist of two geometrical surfaces (sheets); one of the sheets is again spherical and is associated with the ordinary rays. The other sheet has the form of an ellipsoid and is associated with the extraordinary rays. Huygens' treatment is rather incomplete, and while of appealing form, is deceptively simple. It presupposes that a diverging bundle of rays which originates from a point source behaves in the same way as a system of mutually independent plane waves. **Lamé** (1852) was the first to recognize that this presents a mathematical problem of wave propagation in an *anisotropic* medium, which is by no means simple. In addition, the existence of double refraction posed difficulties for contemporary theories of light, and was not explained until the early 1800's.

Bartholinus noted that rotating the crystal will cause one image to remain stationary while the other appears to move in a circle about it, following the motion of the crystal.

He was born at Roskilde. His father Gaspard (1585–1629) was a well-known physician and a professor of medicine at Copenhagen. Bartholinus spent 10 years visiting England, Holland, Germany and Italy, and later filled the chairs of mathematics and medicine at Copenhagen. He was **Römer**'s father-in-law.

1669 CE **Nicolaus Steno** (Niels Stenson, 1638–1686, Denmark, Italy). Danish-born naturalist and physician. The first scientist to notice that the horizontal stratification of rocks holds the key to their history. Made the first clear statement that layered rocks show sequential changes and thus laid the foundation to the time-stratigraphic record.

His fame rests on *De solido intra solidum naturaliter contento*, published at Florence in 1669. From his work on the mountains of Western Italy, Steno realized that the principle of *superposition* in the stratified rocks was the essential key. Steno also realized the importance of another principle — *original horizontality* — namely, that strata are always initially deposited nearly horizontally although they may be found dipping steeply. In his book, Steno described various gems, minerals and *fossils* enclosed within solid rocks. He found that the angles between the faces of quartz crystals were the same even though the crystals had different shapes.

Steno was born in Copenhagen and studied medicine and anatomy there and in Paris. After a period of travel he settled in Italy (1666), at first as professor of anatomy at Padua, and then in Florence as house-physician to grand-duke Ferdinand II of Tuscany. He returned to his native city in 1672, but left again for Florence and was ultimately made apostolic vicar of Lower Saxony. He died at Schwerin in Mecklenburg.

1670 CE **Gabriel Mouton** (1618–1694, France). Mathematician. First to suggest the metric system, the decimal system, and the treatment of series by the method of finite differences ahead of Leibniz (1673).

Mouton was born in Lyon, took the holy orders and spent his whole career as the vicar of the Church of St. Paul in Lyon. His most famous work *Observationes* (1670) studied *interpolation*. His methods of interpolation were similar to those used by Briggs in the construction of his log tables. He produced 10 place tables of logarithmic sines and cosines and an astronomical pendulum of remarkable precision. He suggested (1670) a standard unit of length based on the length of the arc of one minute of longitude on the earth's surface and divided decimally.

1671 CE **Jean Richer** (1630–1696, France) and **Giovanni Domenico Cassini** (1625–1712, France-Italy) measured the scale of the solar system (earth-sun distance) from the parallax of Mars at Cayenne and Paris. Their result was about 10 million km short of the actual figure.

Richer's second important work was to examine the periods of pendulums at different point on the earth. He examined the period of a pendulum at Cayenne and found that it beat more slowly than in Paris. From this he deduced that gravity was weaker at Cayenne, so it was further from the center of the earth than was Paris.

History of measurements of absolute distances on earth and inside the Solar System (ca 585 BCE–1671 CE)

Thales *(fl. 585 BCE) measured the height of the Pyramid of Cheops (146 m) by measuring the length of its shadow at a time when the height of a nearby stick of known length was equal to the length of its shadow.*

Eratosthenes *(ca 235 BCE) estimated the length of the earth's circumference to be 160 km in excess of the present accepted value: he found that when the sun was overhead at Syene (Aswan), it was about 7° from the vertical in Alexandria, about 800 km away. Assuming the sun rays to arrive almost parallel to both places, the circumference is* $\approx \frac{360}{7} \times 800 = 40,000$ *km. From the circumference, the diameter of the earth can be calculated by the familiar formula* $D = \frac{40,000}{\pi}$*. Taking Euclid's value* $\pi = 3\frac{1}{7}$*, one finds* $D = 12,700$ *km.*

There is a beautiful simplicity about the method which Eratosthenes used. It invokes no mathematical principles which had not been current in the Greek-speaking world two centuries before his time; and its importance to posterity lies less in any direct impetus to theoretical inquiry than to the fact that it provided an indispensable basis for any successful attempt to measure the distance of the earth from the sun or the moon, as attempted by his contemporary ***Aristarchos*** *(ca 280–240 BCE) with inadequate information.*

At about 150 BCE, Alexandrian astronomy and geography received an enormous impetus from the work of three contemporaries: ***Hypiscles***, ***Hipparchos*** *and* ***Marinos***. *The first introduced the Babylonian system of angular measurement (360° to a full circle) and the sexagesimal fractions.*

Hipparchos introduced a new system of mapping the position of stars by guide lines comparable to our familiar system of terrestrial latitude and longitude. To this end he needed, and had indeed constructed, a table of trigonometrical ratios (probably by the half-angle formulae $\cos\frac{A}{2} = \sqrt{\frac{1+\cos A}{2}}$*).*

Marinos is notable as the first to introduce the use of circles of latitude and longitude to map the habitable globe as then known.

Now, in 130 BCE, Hipparchos made an observation at Rhodes, from which he obtained a remarkable accurate estimate of the earth-moon distance. His method has been suggested by ***Aristarchos****, about 150 years later.*

The method involves a clear understanding of the positional relationship of sun, earth, and moon. First, he knew that sun and moon subtended almost exactly the same angle α *at the earth. Hipparchos measured this angle to be* 0.553° *(* $\approx \frac{1}{103.5}$ *radian); he also knew what Aristarchos before him had found — that the sun is far more distant than the moon. Hipparchos used this knowledge in an analysis of an* eclipse of the moon by the earth: *Assume that centers of the sun, earth and moon (in this order) are collinear, and that the rays coming from the extreme edges of the sun and tangent to the earth, cut the moon's circular orbit at two points* A *and* B*. Let the angle subtended between these two boundary rays be* α*. The moon passes through the shadow from* A *to* B*, and from the measured time that passage took, Hipparchos deduced that the angle subtended at the earth's center by the arc* BA *was* $2.5\ \alpha$*. The rest is simple geometry: if the distance from the earth's center to the moon is* D*, the length of the arc* AB *is* $\overline{AB} = 2R_E - \alpha D$ *(*$R_E =$ *earth's radius). Also* $\overline{AB}/D = 2.5\alpha$*. With* $\alpha = \frac{1}{103.5}$*, Hipparchos found* $D/R_E \approx 59$*.*

Nothing new happened in the field of distance measurements in the solar system until 1671 CE. In that year, ***Domenico Cassini*** *and* ***Jean Richer*** *measured for the first time an absolute earth-sun distance from the* parallax of Mars *at Cayenne and Paris. Using a* Galilean telescope *and the theory of* ***Kepler****, their result was 10 million km short of the actual figure of about 150 million km.*

1672–1715 CE **Gottfried Wilhelm von Leibniz** (1646–1716, Germany). Mathematician, logician, scientist, philosopher, theologian, jurist and

diplomat — a Universal man with a wide range of interests, who deliberately ignored boundaries between different disciplines and believed in cross-fertilization of ideas, which he saw as essential to the advance both of knowledge and wisdom. Leibniz's most immediate influence was as a mathematician. His philosophical influence was rather less direct. As a logician he was far ahead of his time.

His contributions to the various disciplines are:

Mathematics (1672–1700)

- Discovered the equations for the curves known as *catenary*, *isochrone* and *brachistochrone*. Laid (1694) the foundations of the theory of *envelopes*[141].
- Discovered the basic principles of topology, for which he coined the Latin name: *analysis situs*. He saw it as complementing the analytic geometry of Fermat and Descartes. His ideas in this field remained dormant until the 19^{th} century.

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[141] Previously, **Huygens** (1673) originated the idea of *evolutes* of plane curves (envelope of normals to a given curve). However, the concept may be traced to **Apollonios** (ca 200 BCE) where it appears in the fifth book of his *Conic Sections*.

While **Leibniz** (1694) and **B. Taylor** (1715) were first to encounter singular solutions of differential equations, the geometrical significance of envelopes was first indicated by **Lagrange** (1774). Particular studies were made by **A. Cayley** (1872) and **G.W. Hill** (1888).

The envelopes of a family of plane curves $f(x, y; \alpha) = 0$ are determined by the elimination of α between the simultaneous equations $f(x, y; \alpha) = 0$ and $\frac{\partial}{\partial\alpha} f(x, y; \alpha) = 0$.

If a family of curves is given in terms of *two* parameters by the equations $f(x, y; \alpha, \beta) = 0 \qquad g(\alpha, \beta) = 0$, the envelopes of this system are determined with the aid of the third equation $\frac{\partial f}{\partial\alpha}\frac{\partial g}{\partial\beta} - \frac{\partial g}{\partial\alpha}\frac{\partial f}{\partial\beta} = 0$.

Examples:

- The family of straight lines $x\cos\alpha + y\sin\alpha - p = 0$. Differentiation of this equation w.r.t. α yields $-x\sin\alpha + y\cos\alpha = 0$. The elimination of α between them reveals that the envelope is the circle $x^2 + y^2 = p^2$.
- The trajectory of shells fired from a gun at velocity v_0 at angular elevation α with the horizon, is given by the parabola $y = x\tan\alpha - \left(\frac{g}{2v_0^2\cos^2\alpha}\right)x^2$. The envelope of all trajectories is the *safety parabola* $y = \frac{1}{4a} - ax^2$, where $a = \frac{g}{2v_0^2}$. No point outside it is within reach of shells fired from a gun with velocity v_0.
- Consider the envelope of a line of constant length moving with its ends upon the two coordinate axes: $\frac{x}{a} + \frac{y}{b} = 1$ where $a^2 + b^2 = 1$. The resulting envelope is the *astroid* $x^{2/3} + y^{2/3} = 1$.

- Improving on Pascal's calculating machine, he devised one which performed the four fundamental operations and also extracted roots (1673).
- Invented *binary arithmetic*[142] (1700). He failed, however, to generalize it into a theory of modular arithmetic with its own special theorems; nor did he try to design a calculating machine which used it[143]. [Leibniz was not the first to discover the binary system. It has already been thought of by **Thomas Harriot** early in the century.]
- Discovered the infinite series representation for π:

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

  In his efforts to sum these series (squaring the circle!), he discovered the 'Leibniz test' for convergence.
- Discovered the infinitesimal calculus independent of Newton[144] (1673). His discovery arose from the concept of an infinite series converging to a

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[142] After thinking that he had invented binary numerals, Leibniz was astonished to find that an ancient Chinese book, the *I Ching*, contained a set of numbered figures, called *hexagrams*. Each *hexagram* consists of 6 lines, each of which is either solid or broken. The *hexagrams* are related to the binary sequence in a simple way.

[143] It may seem odd to us, in the age of the computer, that someone who invented both a calculator *and* binary arithmetic should not have put the two together, and come up with something closer in principle to the modern computer. But in the context of the technology of that time, a binary machine would only increase Leibniz's difficulties. There would have been more wheels, more friction, more carrying, and there would have had to be an extra mechanism for translating between binary and decimal, in order to make the calculator usable by ordinary people. The binary system came into its own only with the advent of electronics. As far as Leibniz was concerned, the greatest significance of his discovery was metaphysical, or indeed mystical, as showing how the whole universe could be seen as constructed out of number.

[144] Newton beat him to it by 9 years, though Leibniz was first to publish the discovery in 1684. If matters had rested with Newton and Leibniz, there would have been no quarrel between them. But early in the 1700's, their supporters on opposite sides of the channel started squabbling about the respective merits of the two systems and about the priority of their discovery. Newton and Leibniz were soon drawn into the dispute, which became unpleasantly acrimonious. In particular, Leibniz had to defend himself against charges of plagiarizing Newton's letters during the early 70's, and of subsequently tampering with the evidence. It is beyond reasonable doubt that Leibniz's discovery was in fact independent, but the nationalistic fervor aroused by the dispute, and the incontrovertible

limit: the differential calculus was a technique for determining the *limit* of such a series, and the integral calculus for finding its *sum*. But he never thought of the *derivative* as a limit.

Newton's approach was basically geometrical and his notation clumsy and difficult to work with. Leibniz's approach was algebraic, introducing such new notions as: *differential* and *function*. His notation, which we still use today, was clear and elegant. it was based on the letter for 'difference' (as in $\frac{dy}{dx}$), and the contemporary long s ($\int$) for 'sum', or integral[145].

- Made important contributions to the theory of determinants, the calculus of finite differences, and the theory of numbers[146].
- In a letter to Huygens (first published in 1833) he discussed the possibility of creating a system which would serve as a direct method of space analysis. It can be ranked as the first conceptual forerunner of vector analysis.

Mechanics (1671–1695)

While Newton proposed to measure motion by momentum, Leibniz argued for another quantity, the "*vis viva*", which — except for the factor $\frac{1}{2}$ — is identical with our "*kinetic energy*". Leibniz replaced the Newtonian equation by the equation that "the change of kinetic energy is equal to the work done by the force". The ideas of Leibniz were in harmony with later developments in analytical mechanics. Both the kinetic energy and the work of the acting forces could easily be generalized from one single particle to an arbitrary system of particles. The work of the forces could be replaced by another more fundamental quantity, the negative of the "*potential energy*" (a term coined by **W.J.M. Rankine** in 1853). Both kinetic and potential energy were quantities which could characterize a system *as a whole*, and later became essential in the variational formulation of the laws of mechanics.

Symbolic logic (1666–1696)

At the age of 20, he earned the right to teach at the University of Leipzig with his paper entitled: "*Dissertatio de Arte Combinatoria*", his first thoughts on the subject of symbolic logic. He later returned to his fundamental idea of

evidence in favor of Newton's priority, had disastrous consequences for English mathematics. While the Continental mathematicians of the 18^{th} century made great strides in the theory of the calculus, and in its applications to Newtonian physics, the English stuck loyally to Newton's own much less suitable method of fluxions, and remained in a backwater for over a century!

145 Their 'official' use started only in 1812 due to the reform of **George Peacock**.

146 There is evidence that *Wilson's theorem* [For any prime p, $(p-1)! \equiv -1 (\mathrm{mod}\ p)$] was known to Leibniz long before 1770.

'Language of Concepts' again and again, clarifying, emending and implementing it. Two fragments found among his papers contain Leibniz's introduction to symbolic logic. They clearly establish him as one of the founders of the science. It was not until the work of **George Boole** (who probably did not know of Leibniz's paper) that an algebra came into existence which can be called a realization of Leibniz's ideas.

Philosophy

His philosophical system stands at the interface between the holistic and vitalist world-view of the Renaissance, and the atomistic and mechanistic materialism that was to dominate the 18^{th} and 19^{th} centuries.

Leibniz grasped that space and time were merely phenomenal things (appearances) and not genuine realities. He called these entities '*monads*' (Greek for unity).

Leibniz was born in Leipzig. His father, Friedrich Leibnütz (1597–1652) was a professor of philosophy at Leipzig University. His mother (1621–1664) was Friedrich's third wife. Though the name Leibniz, or Lubeniecz, was originally Slavonic, his ancestors were German, and for 3 generations had been in the employment of the Saxon government.

At an early age he mastered Latin, Greek and scholastic philosophy, which formed the basis of his later massive erudition in the classics. At the age of 14 he enrolled in the University of Leipzig, following the standard 2-year arts course which included philosophy, rhetoric, mathematics, Latin, Greek and Hebrew.

He devoted the next 3 years to legal studies, and in 1666 applied for the degree of doctor of law. Refused on the ground of his youth, he left his native town forever. The doctor's degree refused him there was conferred on him at once (1666) at Altdorf — the university town of the free city of Nüremberg — where his brilliant dissertation procured him the immediate offer of a professor's chair. But by that time Leibniz had changed his mind about an academic career, and decided instead to become more involved in the outside world. It is possible that already at that stage of his mental development, he became hostile to universities as institutions because their rigid faculty structure was bent on intellectual and scientific specialization.

During 1667–1672, Leibniz stayed for some time in Nüremberg where he was associated with a secret brotherhood of alchemists[147]. He soon left them

[147] To his dying day, Leibniz retained a close interest in *alchemy*. Unlike Newton, he never did actual laboratory work. His declared motives were scientific, but in fact he hoped to make his fortune from it. Thus, in 1676 he entered into a formal profit-sharing agreement with two practicing alchemists, his side of the bargain being to provide capital and technical advice.

to become an Assessor in the Court of Appeal of the Elector of Mainz, where he spend the next five years. In the course of his work there he also applied his mind to literary[148] and political activities. Thus he devised a plan to distract Louis XIV away from Northern Europe with an enticing scheme for a French conquest of Egypt (the strategy he suggested was almost identical to the one actually carried out by Napoleon a century and a half later). He was then sent to Paris to try and lay it before the French government.

Strongly attracted to the society of the leading scientists and mathematicians in Paris, Leibniz renewed his mathematical studies under the guidance of **Huygens**. He attacked the current problems in mathematics and science[149] with characteristic gusto (1672–1676) and by the time he left Paris he had already made most of the discoveries that were to earn him his place in the history of mathematics.

In 1673 he went to London and made personal contacts with members of the Royal Society. He showed them his mechanical calculator, which impressed then considerably (at the time even educated people rarely understood multiplication, let alone division!!). His trip to London was cut short by the news of the sudden death of his patron, the Elector of Mainz. He sought a research post attached to the Paris Academy, but it was denied him. He then accepted the post of Court Councilor at the service of the Duke of Hannover. Leibniz remained in the service of the Brunswick family for 40 years to the day of his death (1676–1717).

On his way back to Germany he had 4 days of intense discussions with **Baruch Spinoza** at The Hague (1676). His work in the service of the Duke of Hannover can be divided into 3 periods: During 1676–1686 he was the chief librarian of the great Hanover Library, where his duties were onerous but fairly mundane: general administration, purchase of new books and cataloging[150].

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148 All his life he prided himself on his Latin poetry and boasted that he could recite the bulk of Virgil's *Aeneid* by heart. In 1676 he translated Plato's *Phaedo* and *Theaeteus* into Latin.

149 While in Paris, Leibniz was full of technological ideas: a device for calculating a ship's position without using a compass or observing the stars, a compressed-air engine for propelling vehicles or projectiles, a ship which could go under water to escape enemy detection and various improvements to the design of lenses.

150 **Leibniz** supported himself as a librarian (1676–1717), helping the dukes of Brunswick-Lüneburg in Hanover arrange their collection of 3000 volumes. Then he went on to organize the 30,000-volume ducal Library of Wolfenbüttel, for which he provided one of the first comprehensive alphabetical author catalogue. Leibniz signaled the transition from the royal and ecclesiastical collection for

In the second period (1687–1697) he became the historian and archivist of the House of Brunswick: his genealogical researches in Italy and elsewhere in Europe established the Hanoverian claim for a succession to the throne of Great Britain. He spent much time traveling. Although he had his own coach, it is nevertheless remarkable that he managed to write letters while on the move[151]. In this phase of his life he interacted strongly with the **Bernoulli** brothers, Jakob and Johann, exchanging mathematical challenges with them;

The ubiquitous practice of issuing challenge problems was actually inaugurated at this time by him. They were at first intended merely as exercises in the new calculus.

Thus, in 1687, Leibniz proposed the problem of the *isochrone curve*[152]: it was solved by 'the brothers', Huygens, and himself. Jakob Bernoulli returned the challenge with the *Catenary problem*[153] (1690), which was readily solved by Huygens, 'the brothers' and himself (quite an exclusive club!). During 1698–1714, Leibniz was engaged in diplomatic tasks for Hannover in Vienna, London, Berlin and Paris. He also promoted scientific societies and academies.

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the privileged few to the library serving everyone.

In 1856, **Anthony Panizzi** (1797–1879) became the Principal Librarian of the *British Museum*, establishing an egalitarian Reading Room there. **Thomas Carlyle** (1795–1881) established the *London Library* (1841), where books could be *checked out* by subscribers – the first circulating library in the world. But the *public library* "in every town", which Carlyle demanded, was yet to come. Panizzi still required users to present letters of introduction to enter the Reading Room and his books did not circulate. **Andrew Carnegie** (1835–1919) would spread public libraries across the United States of America.

[151] Leibniz was an avid letter-writer. He was corresponding with intellectuals from all over Europe, sometimes with hundreds of people at a time, on almost every subject under the sun — science, mathematics, law, politics, religion, philosophy, literature, history, linguistics, numismatics. He was obsessive about preserving his letters, and over 15,000 survived.

[152] A particle descends a smooth curve under the action of gravity, describing *equal vertical distances* in equal times, and starting in a vertical direction. Taking x as the horizontal axis and y in the vertical (downward) direction with the initial condition $\dot{x}(0) = 0$, $\dot{y}(0) = V$, the energy equation yields $\frac{1}{2}\left(\frac{ds}{dt}\right)^2 = gy + \frac{1}{2}V^2$ or $\dot{x}^2 + \dot{y}^2 = 2gy + V^2$, while the constraint is $y = Vt$. Eliminating $\dot{y} = V$ and integrating leads to $y^3 = \left(\frac{9V^2}{8g}\right)x^2$, a *semi-cubical parabola*.

[153] To find a curve formed by a chain of uniform weight suspended freely from its ends $\left[y = acosh\left(\frac{x}{a}\right)\right]$.

During his last years Leibniz was rather miserable and lonely. He was getting too infirm either to travel or to start a new life elsewhere. He died peacefully in the presence of his secretary and coachman.

1672–1682 CE **Nehemiah Grew** (1641–1712, England). Physician and botanist. A founder of *plant anatomy*. First to hypothesize in print on *sex in plants*.

Through microscopic observations he discovered sexual reproduction in plants and identified the stamen and pistil as the male and female organs respectively, as well as representing detailed drawings of plant anatomy.

Author of *Anatomy of Plants* (1682). It was the first complete account of the subject and remained the most authoritative work in this field for over 150 years.

Grew was born in Atherstone and educated at Cambridge and Leiden.

1672 CE **Regnier de Graaf** (1641–1673, The Netherlands). Physician and anatomist. One of the founders of experimental physiology. Discovered the Graafian vesicles of the *mammalian female gonad*, coining the term '*ovary*' for the organ.

He was born in Schoohoven and studied at Utrecht and Leiden.

1673–1685 CE **Phillipe de la Hire** (1640–1718, France). Mathematician, astronomer, physicist, naturalist, architect and painter. In his book *Sectiones conicae* he argued for the power and potential of *projective geometry*[154] and thus secured its place in mathematics. His work had influenced Newton, yet during de la Hire's lifetime, the mathematical community was not convinced that the synthetic methods of Desargues (1639) can match in power the analytic methods of Descartes (1637). The father of de la Hire, a well-known painter, was a student of Desargues.

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154 *De la Hire's theorem*: On a line L outside a conic (e.g., ellipse) and coplanar to it, we choose three points and draw from each of these points two tangents to the conic; connect the opposite points of tangency by lines; then these three lines meet at a point Q, dual to L.

Table 3.5: GREATEST MATHEMATICIANS OF THE 17^{th} CENTURY

NAME	NAT.	LIFE-SPAN	MAJOR CONTRIBUTION
Joost Bürgi	SW	1552–1632	Logarithms (1603); Decimal and exponential notations
John Napier	E	1550–1617	Logarithms (1614)
Thomas Harriot	E	1560–1621	Mathematical symbols
Henry Briggs	E	1561–1630	Decimal logarithms; Logarithmic tables (1624)
Johannes Kepler	G	1571–1630	Forerunner of calculus (areas and volumes, 1615); New polyhedra; Applied conic sections.
William Oughtred	E	1575–1660	Mathematical symbols; Logarithmic slide-rule (1622)
Edmund Gunter	SW	1581–1626	Cosine, cotangents (1620), slide-rule (1620)
Girard Desargues	F	1591–1661	Early development of synthetic projective geometry
Albert Girard	D	1595–1632	General algebraic equations: roots and coefficients; Fundamental theorem of algebra (conjecture).
Rene Descartes	F	1596–1650	Coordinate geometry; Topology ($v - e + f = 2$); Algebraic notation.
F.B. Cavalieri	I	1598–1647	Precursor of the integral calculus
Pierre de Fermat	F	1601–1665	Modern number theory; Modern analytic geometry; Differential calculus; Probability theory.

Table 3.5: (Cont.)

Name	Nat.	Life-span	Major Contribution
G.P de Roberval	F	1602–1675	Pre-calculus method for tangents and areas.
E. Toricelli	I	1608–1647	First notion that differentiation is the inverse of integration; envelopes of families of curves.
John Wallis	E	1616–1703	Pre-calculus integration; Concept of limit; First geometric representation of complex numbers.
Nicolaus Mercator	E	1620–1687	Mercator series.
William Brouncker	E	1620–1684	Continued fractions; 'Pells' equation; Infinite series.
Johann H. Rahn	SW	1622–1676	Mathematical symbols
Vincenco Viviani	I	1622–1703	Geometer, physicist and inventor of instruments.
Blaise Pascal	F	1623–1662	Calculating machine; Mathematical theory of probability; Projective geometry; Binomial triangle; Cycloid.
Pietro Mengoli	I	1626–1686	Infinite series; Divergence of harmonic series; Infinite product for π.
Christiaan Huygens	D	1629–1695	Probability; Rational approximation for gear-ratio by continued fractions.
John Hudde	D	1628–1704	Algebraic equations with literal coefficients standing for negative and positive numbers.

Table 3.5: (Cont.)

NAME	NAT.	LIFE-SPAN	MAJOR CONTRIBUTION
James Gregory	E	1638–1675	Convergent and divergent infinite series ($\tan^{-1} x$)
Isaak Barrow	E	1630–1677	Pre-calculus differentiation: tangents and rectification of curves; maxima and minima.
P. de la Hire	F	1640–1718	Projective geometry; pole and polars.
Seki Kowa	N	1642–1708	General determinant theory
Isaac Newton	E	1642–1727	Creation of workable calculus (1671)
G.W. Leibniz	G	1646–1716	Infinitesimal calculus (1673–1676); Calculus of finite differences; Binary arithmetics; Early topology; Theory of determinants (1698); Symbolic logic; Theory of envelopes; Theory of numbers.
Giovanni Ceva	I	1647–1734	Synthetic geometry (Ceva theorem)
Walter von Tschirnhausen	G	1651–1708	Theory of equations; "The T. Transformation"
Michel Rolle	F	1652–1719	Rolle's theorem (1691)

E = England; I = Italy; G = Germany; F = France;
D = Holland; N = Japan; SW = Switzerland.

1675 CE Scarlet fever first identified or described with accuracy.

1675 CE **Olaus Römer** (1644–1710, Denmark). Astronomer. Made the first measurement of the velocity of light — 309,000 km/sec.

Previously, **Kepler** and **Descartes** believed it to be infinite. **Galileo** failed to devise a successful method of measurement. Römer used observations based on the times of eclipses and occultations of the four large satellites of Jupiter. The four moons could be easily seen with the telescopes of that day. Römer compiled a table of their periods of revolution around the planet. He could predict the times at which each was eclipsed as it moved into Jupiter's shadow or occulted when it passes behind the planet's limb. He found that these phenomena occur sooner than expected during part of the year and later then expected during the other part, and correctly inferred that the advance or delay of the occurrence was due to the finite velocity of light.

Römer's observations showed that there is a difference of 1000 seconds at two dates roughly 6 months apart. In 1675, the diameter of the earth's orbit was known to be 309 million km; hence Römer's value for the speed of light in free space.

In 1849 **Armand Hippolyte Louis Fizeau** (1819–1896, France) made a first laboratory measurement of the velocity of light, by synchronizing the rate of a rapidly rotating toothed wheel with the reflection of a light beam so as to allow the beam to *enter* and *leave* through two adjacent slits. Again the result was close to 300,000 km/sec. Later measurements by **Michelson** (1879, 1887, 1926) improved it gradually to within an error of less than 10^{-5} percent. The meter is nowadays *defined* by means of a value of c, agreed upon by international committee to be $c = 299,792.458$ km/sec.

1675 CE Foundation of the Royal Observatory at Greenwich. This date indicates the beginning of the precise standardization of time measurements, needed primarily for navigation.

1678–1692 CE **Giovanni Ceva**[155] (1647–1734, Italy). Mathematician. Discovered one of the most important results on the synthetic geometry of the triangle between Greek times and the 19^{th} century. His geometrical treatise *De lineis rectis* (1678) contains a theorem now known by his name: "*If three concurrent lines* (*known as Cevians*), *one from each vertex of a triangle, are drawn, they divide the opposite sides into six segments such that the products*

155 A town in Piedmont, Italy, in the province of Cuneo. In the Middle Ages it was a strong fortress defending the confines of Piedmont towards Liguria, but the fortifications on the rock above the town were demolished (1800) by the French.

of three segments having no common end is equal to the product of the three other segments".

The converse of *Ceva's theorem* is also true.

Ceva's theorem is a direct generalization of the corresponding theorems in elementary plane geometry and marks a point of departure of the new European geometry away from the classical Greek tradition. Its applications lead immediately to some important properties of the triangle[156].

Giovanni was in the service of the Duke of Mantua; known also for his calculations of centers of gravity, areas and volumes of geometric figures. His brother **Tomasso Ceva** (1648–1737) was a teacher of mathematics in the Jesuit College at Milan, and wrote on the cycloid and mathematics in general.

1678–1718 CE **Edmund Halley** (1656–1742, England). Astronomer-royal. A close friend of Isaac Newton and active in many areas of astronomy. Halley is best known for his pioneering study of comets.

After what he termed a 'prodigious deal of calculations', Halley (1705) published parabolic orbital elements for 24 well-observed comets. He noted the similarities in the orbits for the comets of 1682, 1607, and 1531 and published his first correct prediction for the return of a comet. It did return on time (16 years after his death) and has been called "Comet Halley" ever since.

Other discoveries of Halley are:

(1) The proper motions of the stars on the celestial sphere (1718). Halley shattered man's ancient belief in 'fixed stars' by charting the motions of Sirius, Arcturus and Aldebaran, listed as bright stars in Ptolemy's *Almagest*.

(2) *Secular acceleration of the moon's mean motion* (1693). Halley found from a comparison of ancient and modern eclipses that the mean velocity of the moon in its orbit is gradually increasing. Nearly 100 years later (1787), Laplace showed that it is caused partly by the gradual average decrease of the eccentricity of the earth's orbit which has been going on for many thousands of years.

Indeed, due to planetary perturbations *and* the slowing down of the earth's rotation (10^{-3} second per century. First suggested by **William Ferrel** in

[156] E.g., if each side of the triangle is divided into n equal parts and the Cevian lines are drawn to the first point from each vertex in a clockwise (or anticlockwise) direction around the triangle, the central triangle has an area of $\frac{(n-2)^2}{n^2-n+1}$ of the original triangle.

1856) caused by tidal braking, the moon is speeding up at a rate that is proportional to *square* of elapsed time. This apparent change in the moon's rate is manifested in the appreciable *shift* in tracks of eclipses. When actual eclipse tracks are calculated for ancient eclipses on the basis of *current* motions of sun and moon, they are found to deviate slightly *eastward* from the observations.

(3) Determination of the solar parallax by means of the transit of Venus (1677).

(4) Explanation of the trade-winds and monsoons (1686): published a world map indicating the prevailing winds over the tropical oceans. He explained the equatorward[157] flow of the trades as resulting from a combination of the rising of air near the equator due to solar heating and the resulting surface inward flow of air toward the updraft region. [An improved explanation based on the rotation of the earth and atmosphere was given later by George Hadley (1735).]

Halley went to St. Helena island during 1676–1678 to catalog stars not visible from Northern observatories. His resulting star catalog started the systematic study of the Southern sky. It was the first study based on telescopic, rather then naked eye observations.

Before Halley made his study on comets, most people believed that comet apparitions were random. But Halley argued that comets belonged to the solar system and that their orbits are governed by Newton's law of universal gravitation.

In 1981 Tao Kiang et al. numerically integrated the orbital motion of comet Halley back to 1404 BCE, using the Newtonian equations of motion. They took into account the perturbations by the nine major planets over the past 3500 years and non-gravitational forces due to 'rocket effects' of an outgasing water ice-nucleus. The dynamic model used to compute the long-term motion of the comet successfully reproduced the ancient Chinese observations over nearly two millennia.

If Halley could comment on the wonders of the computer era and its astronomers, he would not be likely to change even one word of what he already said in his paper of 1705:

"*You see therefore an agreement of all the elements in these three, which would be next to a miracle if they were three different comets... Wherefore, if according to what we have already said it should return again about the*

[157] This basic pattern was known already to the ancient Hebrews, for we read in the Bible [*Eccl.* **1**, 6]: "*The winds blow to the south, and turn to the north; round and round it goes, ever returning on its course*".

year 1758, *candid posterity will not refuse to acknowledge that this was first discovered by an Englishman*".

1679–1709 CE **Denis Papin** (1647–1712, France). Physicist and inventor. One of the inventors of the steam-engine (1690). Papin was born in Blois. He studied medicine at the University of Angers (1662–1669). Assisted **Huygens** in Paris in his experiments with the air-pump (1674–1675) and afterwards assisted **Boyle** in his experiments in London. At this time he experimented with hydraulic and pneumatic transmission of power, improving the air-pump, inventing the condensing pump, and the "steam-digester" [a pressure cooker with which he showed that boiling point is raised or lowered as the pressure exceeds or falls below atmospheric pressure]. He also invented the safety valve and is credited with being the first (1690) to apply steam to raise a piston.

In 1687 Papin was appointed to the chair of mathematics in the University of Marburg, and there he remained until 1696. In 1707 he sailed with his family to London in an ingeniously constructed boat, propelled by paddle-wheels. He died in London in poverty and total obscurity.

1680 CE **Edme Mariotte** (1620–1684, France). Physicist. Independently discovered Boyle's law.

1682 CE A Russian Physician reportedly repaired the skull of a wounded nobleman *using bone from a dog*. The surgery was said to be successful, but the Russian Church threatened the nobleman with excommunication, prompting him to have the graft removed.

This is the first recorded case in medical history of animal tissue transplantation.

In the late 1800's *frog skin* was often grafted onto patient's skin in an attempt to heal burns or skin ulcer. Good results were reported.

1682–1683 CE **Ehrenfried Walther von Tschirnhausen** (1651–1708, Germany). Physicist and mathematician. Discovered the caustic of reflection[158] (1682). Endeavored to solve equations of any degree by removing

[158] Parallel light rays from the sun fall onto a nearly full cup of coffee. Each ray is reflected from the circular surface of the cup and these reflected rays form an *envelope* known as a *caustic*. At the caustic, the intensity of the light is theoretically infinite (according to geometrical optics) since the cross-section of the ray pencil at each point on the envelope has zero area. In fact this is not quite true, as is obvious on physical grounds and as a more accurate analysis confirms, but the intensity can indeed be very great: sufficient to burn a piece

all the terms except the first and the last (this procedure has been tried before him by the Frenchman **Francois Dulaurens** and the Scot **James Gregory**).

In 1683 Tschirnhausen published *Method of Eliminating All Intermediate Terms from a Given Equation*. Although the title exaggerated, the paper was the most important idea for the solution of algebraic equations in about 200 years. It showed that a polynomial of degree $n > 2$ can be reduced by his transformations to a form in which the coefficients of the terms of degrees $(n-1)$ and $(n-2)$ are zero.[159]

Tschirnhausen studied at Leyden, and for a while served in the Dutch army. Later he spent some time in England. He visited Paris several times, and was elected (1682) to the French Academy of Sciences. He also set up a glasswork in Italy to further his experiments on light.

Tschirnhausen was a man of wide acquaintance and interests. Everywhere he went he sought contact with leading scientists, collected observations and reported interesting discoveries to **Leibniz**. He thus met with **Spinoza**, **Huygens**, and **Wallis** and corresponded with **Newton**, **Jakob Bernoulli** and

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of paper, for example (hence the Greek name *caustic*).

Let the inner surface of the cup be represented by the unit circle, and let the incident rays be parallel to the (horizontal) x-axis. If a ray is incident on the cup at the point Q, whose coordinates we may take to be $(\cos\theta, \ \sin\theta)$, then since the angle of reflection is equal to the angle of incidence we may easily find the equation of the reflected ray from Q: $(y-\sin\theta)\cos 2\theta = (x-\cos\theta)\sin 2\theta$. Considered as a family of equations with θ as parameter, this represents *all* the reflected rays.

The caustic is the envelope of this family, i.e. the curve which is tangent to every member of the family. Now, the equation of the envelope of a one-parameter family of curves $f(x,y;\theta)=0$ is found by eliminating the parameter θ from the equation $f=0$ and $\frac{\partial f}{\partial\theta}=0$. It is shown that the parametric equations of the envelope are

$$x = \cos\theta - \frac{1}{2}\cos 2\theta\cos\theta; \qquad y = \sin\theta - \frac{1}{2}\sin 2\theta\cos\theta.$$

These are the equations of a curve known as the *nephroid*. Note that throughout this calculation we have been suppressing the z-coordinate; the equation we have derived is really that of a cylinder with the nephroid as cross section, and what we observe in the intersection of this cylinder with a plane $z=$ constant, i.e. with the surface of the coffee.

159 To dig deeper, see:

- Panton, A.W., *The Theory of Equations*, Dover: New York, 1960, Vol I (286 pp.); Vol II (318 pp).

Johann Bernoulli. He also examined unpublished and posthumous papers of **Descartes** and **Pascal**.

However, he exhausted his mathematical talents in searching for algorithms and lacked insight into the more profound mathematical ideas of his age (e.g., he considered infinitesimal symbolism to be of limited applicability). He could have achieved more, but being essentially an autodidact he lacked the guidance of experienced and strict teachers, who might have instilled in him a greater measure of self-criticism.

Algebra and the Theory of Equations

Descartes *(1637) rejected* complex roots *and termed them imaginary. Even* ***Newton*** *did not regard complex roots as significant, most likely because in his day they lacked physical meaning.* ***Leibniz*** *worked with complex numbers formally, but possessed no understanding of their nature.*

Despite the lack of any clear understanding during the 16^{th} and 17^{th} centuries, the operational procedures with real and complex numbers were improved and extended. ***John Wallis*** *(1673) was first to show how to represent* geometrically *the complex roots of a quadratic equation with real coefficients, as point in a plane. His work was ignored because mathematicians were not receptive to the use of complex numbers.*

It is remarkable that the free use of algebra provoked a host of protests. The philosopher ***Thomas Hobbes*** *(1588–1679), though only a minor figure in mathematics, nevertheless spoke for many mathematicians when he objected to the application of algebra to geometry. He characterized John Wallis' book on the algebraic treatment of conics as a scurry book and as a "scab of symbols". Many mathematicians, including* ***Pascal*** *and* ***Barrow****, objected to the use of algebra because it had no logical foundation; they insisted on* geometric *methods and proofs.*

Unlike ***Descartes****, who still regarded algebra as the servant of geometry,* ***John Wallis*** *and* ***Newton*** *recognized the full power of algebra.* ***Leibniz****, too, noted the growing dominance of algebra and fully appreciated its effectiveness.*

Albert Girard *(1629) conjectured that an equation of degree n always has n solutions in the domain of complex numbers. Attempts to prove this were made by* ***Descartes****, d'Alembert and others, but it was only* ***Gauss*** *(1799) who succeeded in giving a rigorous proof without gaps.*

Girard *(1629) was also the first to engage in nonalgebraic solutions of algebraic equations, involving an infinite number of arithmetic steps, such as infinite series or products. He had shown that trigonometric functions (which are nonalgebraic, or transcendental functions) are effective in obtaining solutions when the cubic formula yields irreducible results (3 distinct real roots). Therefore, mathematicians after Galois' day conceived the idea that the elliptic functions, which generalize ordinary trigonometric functions, might offer a means of expressing solutions of some higher-degree equations that are not solvable algebraically. The ideas of Girard were picked by* ***Lambert*** *(1757), who suggested a solution base on series.*

The search for meaning of negative and imaginary roots of equations that started with ***Cardano*** *(1545), continued in the 17th century:* ***Albert Girard*** *(1629) interpreted negative numbers as a kind of a relative orientation, which eventually paved the way toward the idea of the number-line. Girard retained all imaginary roots of equations because they show the general principles in the formation of equation from its roots. He stated clearly the relation between roots and coefficients, allowing of negative and imaginary roots of equations.*

Descartes *(1637) coined the term 'imaginary' for expressions involving square roots of negative numbers, and took their occurrence as a sign that the problem was insoluble.* ***Leibniz*** *(1670) was confused and perplexed by expressions such as*

$$\sqrt[3]{6+\sqrt{-\frac{1225}{7}}}+\sqrt[3]{6-\sqrt{-\frac{1225}{7}}}=4$$

which results from Cardano's solution of $x^3-13x-12=0$. Today such relations are considered trivial by a good high-school algebra student.

Apart from the problem of the existence of solutions of algebraic equations, there is also the problem of determining them. After the solution formulae for the cubic and quartic equations had been found during the Renaissance, the mathematicians of the 17th and the 18th centuries searched with great tenacity for the corresponding solution formulae of degree 5 and higher.

In their quest for a solution of the quintic by radicals, mathematicians were first concerned with the reduction of the general quintic into the simplest possible canonical form. The feasibility of this procedure is based on the pioneering discovery of ***Tschirnhausen*** *(1683) that a special transformation can eliminate the terms of degree $(n-1)$ and $(n-2)$ from any polynomial of*

degree $n > 3$. He started from the observation that the transformation of the equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

by the linear substitution $y = x + h$ eliminates the $(n-1)^{th}$ power by choosing $h = \frac{1}{n}a_{n-1}$. He then noticed that it is possible to remove both the $(n-1)^{th}$ and the $(n-2)^{th}$ powers by effecting a quadratic transformation $y = x^2 + ax + b$. To see this one first assumes the "existence of an equation in y of the same degree as the original in x, i.e.

$$y^n + b_{n-1}y^{n-1} + \cdots + b_1 y + b_0 = 0.$$

This is always possible since x^{n+k} can be expressed as a polynomial of degree not higher than n by a repeated use of the original equation. Thus $\{y, y^2, y^3, \ldots, y^n\}$ will be each a polynomial of degree lower or equal to n. When the equation in y is formed, the free parameters $\{p, q\}$ are chosen such that $b_{n-1} = b_{n-2} \equiv 0$.

Thus he found that by a transformation of the form $y = x^2 + ax + b$, a general cubic is reduced to the form $y^3 = K$. Another such transformation reduces a general quartic to $y^4 + py + q = 0$, and a general quintic to the form $y^5 + \alpha y^2 + \beta y + \gamma = 0$. In general, a Tschirnhausen transformation of a polynomial equation $f(x) = 0$ is one of the form $y = g(x)/h(x)$, where g and h are polynomials and h does not vanish for a root of $f(x) = 0$. The transformation by which Cardano and Viète solved the cubic were special cases of such transformations.

1683 CE **Seki Kowa** (1642–1708, Japan). Mathematician. Considered by the Japanese the greatest mathematician that their country has produced. His most original and important work is the invention of *determinants*, at least 10 years ahead of Leibniz (1693). Also, Leibniz treated only 3 equations with 3 unknowns, whereas Seki considered n equations and gave a more general treatment. Seki knew that a determinant of the n^{th} order, when expanded, has $n!$ terms and that rows and columns are interchangeable. Discovered *Bernoulli Numbers* before Jacob Bernoulli. Credited with the independent invention of the differential calculus. Seki was a great teacher who attracted gifted pupils. He discouraged divulgence of mathematical discoveries made by himself and his school.For that reason it is difficult to determine with certainty the exact origin and nature of some of the discoveries attributed to him.

1683–1699 CE *Islam vs. Christendom*; Since 1656 the Ottoman Empire had been undergoing a revival, and in the 1660's began a new thrust up the Danube Valley directed at their old enemies, the Habsburgs. Louis XIV, also an inveterate enemy of the Habsburgs, allied himself with the Turks and Hungarian rebels against his Austrian foes. This resulted in two of the greatest military confrontations of the second half of the 17^{th} century[160]: the Habsburg's successful resistance to the advance of Turkey into Europe and their subsequent counter-offensive; and the great coalition which halted Louis XIV's attempt to dominate the Continent.

The crisis came in July 1683, when a Turkish army of ca 200,000 *laid siege to Vienna*. For two months the fate of Christendom seemed to hang in the balance. Then volunteers began to flow in from all over the continent to help the emperor in his extremity: Pope Innocent XI contributed moral and material aid, and King Jan Sobieski of Poland arrived with an army that helped rout the Turks by September. This marked the beginning of the decline of the Ottoman Empire, and the end of a thousand year military conflict between Islam and Christianity. Thus failed the last (perhaps) attempt of Islam to subdue Christendom and conquer Western Europe [1^{st}, 732 at Tour, 2^{nd}, 1571 at Lepanto].

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160 *The Jewish connection*: The financier **Samuel Oppenheimer** [1630–1703; a distant relative of **Joseph Oppenheimer**, the so-called "Jud Süss" (1698–1738)] was the Imperial War Purveyor to the Austrian monarchy during 1673–1702 and played a decisive role in the above wars of the Habsburgs. He was running the finances of two-front war, marshaling the resources. Some historians believe that he was indeed the man who saved Vienna during the siege of 1683 when the emperor fled.

No one ever rendered greater services to the Habsburgs. But the Austrian Treasury never payed him back! Moreover, in 1701 his house in Vienna was "accidentally" burned down, destroying most of the financial business records. At that time the Crown owed him more than ten million florins. But Oppenheimer could no longer produce proofs of the debts due to him from the State. So the State produced its own records "proving" that he had been overpaid! All his services were then forgotten and the Jew was rewarded by being thrown into prison while his family was left penniless.

His nephew **David Oppenheimer** (1664–1736), rabbi of Prague, managed to gather a large library of rare Jewish books and manuscripts. These he kept in Hamburg, away from the reach of *Inquisition* in Catholic Bohemia. His collection was purchased by Oxford University early in the 19^{th} century, and now forms the basis of the *Bodleian hebraica*, encompassing over 7,000 books and 1,000 manuscripts.

Historians have explored the question of why the West (nearly) always wins? For 2,500 years, from ancient Greece to the present day, Western armies vanquished their non-Western adversaries in almost every war, with rare exceptions when the West was caught completely by surprise or was exceptionally outnumbered.

A comprehensive examination of important battles, from the Battle of Salamis in which the Athenians defeated the Persian fleet to the Battle of Midway in World War II in which the Americans defeated Japan refutes widespread assumptions that Western military superiority is explained by greater valor, military-technology advantage or greater economic strength. Hanson[161] argues that the secret is that Western military forces are more effective killers. This results from the "citizens' army" model created in a Western "open society," which was born in the ancient Greek tradition of storming the enemy.

While Ancient, Eastern monarchs considered war a sport, a game of balance between forces, ancient Greek democracy gave rise to an utterly unsportsmanlike perception of war. It viewed war as a fight for liberty and freedom of community and citizen, an existential fight to the death. Its primary objective was not to defend city and homeland but, to the greatest extent possible, to prevent the enemy from recovering in time for another round. While non-Western forces strive to gain points, their Western adversaries strive for a knockout.

1685 CE, Oct. 18 *Revocation of the edict of Nantes* (1598) by Louis 14^{th} of France; all religions except Roman Catholicism became forbidden by law. About 400,000 Protestants (Huguenot) fled France and emigrated to the neighboring countries and North America.

There were about a million Huguenots out a total population of perhaps 18 million in France in 1650. After Richelieu deprived them of their military and political privileges, they had become good citizens and remained loyal to the crown. Many were successful in industry and the professions. The French Catholic clergy had long tried to persuade Louis 14^{th} that the continued exercise of the Protestant religion in France was an insult to his dignity and authority, and as Louis became more concerned about his salvation, the idea of atoning for his sins of the flesh by crushing heresy became more attractive to him. The edict was "interpreted" more and more strictly. Protestant children were declared of age at seven and converted to Catholicism, and any attempt of their parents to win them back was punished by imprisonment. Money was offered to converts. Protestant chapels were destroyed,

[161] The American military historian **Victor Hanson** in his book "*Carnage and Culture*" (Doubleday, 2002).

and troops were quartered on prominent Huguenots to make life miserable for them.

Finally Louis aided and abetted by his mistress and his Jesuit advisers, announced that since all the heretics had finally been converted to Catholicism, there was no further need for the Edict of Nantes and it was therefore revoked; Protestant churches and schools were closed, and all Protestant children were baptized as Catholics. The Revocation was savagely enforced by imprisonment, torture and condemnation to the gallows.

The Huguenots escaped to England, The Netherlands, Brandenburg, and the New World, where their industry and skill contributed appreciably to the economic, technological and cultural life of their new homes. To France the Revocation 'brain-drain' brought both economic and cultural loss.

Thus, under the influence of the clergy, was committed one of the most flagrant political blunders in the history of France.[162]

The Protestant Huguenots experienced in 1685 what the Jews had already gone through during 1492–1498 in the Iberian Peninsula. In this sense the fate resemblance of the *Marranos* and the *Huguenots* is a striking example for the way in which history repeats itself from time to time.

1686 CE **Bernard (Le Bovier) de Fontenelle** (1657–1757, France). Man of letters, thinker and science-fiction writer. Made the first major attempt to present scientific knowledge to the layman in an attractive literary form.

Born in Rouen, a nephew of Corneille. Studied law, but having lost the first case which was entrusted to him, he soon abandoned law and gave himself completely to literary pursuits.

In his *Entretiens sur la pluralite des mondes habites* (1686), he popularized among his countrymen the astronomical theories of **Descartes**. It is a lucid exposition of astronomy according to Copernicus and Descartes, enlivened by speculations concerning *life on other planets*. In 1691 he was received into the French Academy, and in 1697 became its perpetual secretary.

1687–1789 CE Leading European Poets and Novelists in the Enlightenment (Age of Reason):

- Daniel Defoe 1660–1731
- Jonathan Swift 1667–1745

[162] In fact, the oppression of the Huguenots in France and the Puritans in England, during 1640–1685, accelerated the outbreak of the *Industrial Revolution* in England in the 1760's.

•	Alexander Pope	1688–1744
•	Voltaire	1694–1778
•	Johann Wolfgang von Goethe	1749–1832
•	William Blake	1757–1827
•	Friedrich von Schiller	1759–1805

1689–1695 CE **John Locke** (1632–1704, England). Empirical philosopher. Rejected the notion of the 'divine rights' of kings, as well as the infallibility (absolute truth) of religion and the dogma of the Church. Opposed the authority of the Bible and the Church in temporal affairs. Maintained that political sovereignty rests upon the consent of the governed.

His political philosophy is strongly felt in the *American Constitution* and *Declaration of Independence*. In his own words:

"No man has the right to more than others, because we are all equal, of the same species and condition, equal amongst ourselves, with equal rights to enjoy the fruits of nature".

Held that men were free to think of God in their own way, not as any religion told them to. His major works: *Two Treaties on Government* (1689); And *Essay on Human Understanding* (1690).

Locke was born in Wrington in Somerset County. He attended Oxford University. When his friend Anthony Cooper became involved in plots against the King, the suspicion also fall on Locke and he fled to Holland (1684), but returned (1689) as favorite of the court of Prince William of Orange.

1691 CE **Michel Rolle** (1652–1719, France). Mathematician; author of a theorem named after him[163], found in his '*Methode pour résoudre les egalitez*' (1691). The name *Rolle's theorem* was first used in 1834 in Germany and in 1846 in Italy (**G. Bellavitis**).

1690 CE **Jakob (Jacques, James) Bernoulli** (1654–1705, Switzerland). Among the principal contributors to mathematics in the 17^{th} century. Jakob and his brother Johann gave up earlier vocational interests and became mathematicians when Leibniz's papers began to appear in the *Acta eruditorum*. They were among the first mathematicians to realize the surprising

[163] *Rolle's Theorem*: If $f(x)$ is continuous in the interval $a \leq x \leq b$, $f'(x)$ exists in the open interval $a < x < b$ and $f(a) = f(b) = k$, then there is a point c, such that $a < c < b$, at which $f'(c) = 0$ [one may assume, without loss of generality, that $f(b) = f(a) = 0$ since one may apply the theorem to the new function, $f(x) - k$, instead of $f(x)$]. For polynomials, Rolle's theorem takes the form: between any pair of roots of $P(x) = 0$ lies a root of $P'(x) = 0$.

power of the calculus, and to apply the tool to a plethora of problems, and first to use the term *'integral'* (1690).

Jakob Bernoulli invented polar coordinates (1691). [Newton may have discovered them earlier in 1671, but this is not clear from his writings.] He wrote on infinite series, studied many special functions and introduced the *Bernoulli numbers* that appear in the power series expansion of the function $z(e^z - 1)^{-1}$ and the *Bernoulli polynomials*[164] of number theory. In 1700 he developed further the theory of probability (*Bernoulli distribution*) and rediscovered the *law of large numbers*, a theorem named after him. Was first to apply calculus to probability theory.

The solution of the *Brachistochrone* problem by him and his younger brother Johann, started an acrimonious quarrel between them that dragged on for several years. Jakob is also known for the early use of *radius of curvature* of a plane curve, discovery of the *isoperimetric figures*, the *Bernoulli equation* in the theory of ODE, and his pioneering work in the calculus of variations. In his 1690 solution to the problem of the *isochrone* [curve along which a body will fall with uniform vertical velocity], we encounter for the first time the word "*integral*" in a calculus sense. Leibniz had called the integral calculus *calculus summatorius*, but in 1696, **Leibniz** and **Johann Bernoulli** agreed to call it *calculus integralis*.

Jakob Bernoulli was struck by the way the equiangular spiral reproduces itself under a variety of transformations and asked, in imitation of **Archimedes**, that such a spiral be engraved on his tombstone along with the inscription "*Eadem mutate resurgo*" ("I shall arise the same, though changed").

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164 *Bernoulli numbers* B_n were introduced in his *Ars Conjectandi* (published posthumously, 1713) through the definition:

$$\frac{z}{2}\cot\frac{z}{2} = 1 - B_1\frac{z^2}{2!} - B_2\frac{z^4}{4!} - \cdots - B_n\frac{z^{2n}}{(2n)!} - \cdots, \quad |z| < 2\pi$$

with $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}$, $B_4 = \frac{1}{30}$, $B_5 = \frac{5}{66}$.
He also introduced the polynomials $\Phi_n(z)$ as the coefficients of $\frac{t^n}{n!}$ in the expansion of

$$t\frac{e^{zt} - 1}{e^t - 1} = \sum_{n=1}^{\infty} \Phi_n(z)\frac{t^n}{n!}, \quad |t| < 2\pi$$

Explicitly,

$$\Phi_n(z) = z^n - \frac{n}{2}z^{n-1} + C_2^n B_1 z^{n-2} - C_4^n B_2 z^{n-4} + C_6^n B_3 z^{n-6} - \cdots$$

the last term being z or z^2 and C_2^n, C_4^n, ... the binomial coefficients.

The Bernoulli family is one of the most illustrious families in the annals of science. They originally came from Antwerp. Driven from Holland during the oppressive government of Spain for their attachment to the Reformed religion, the Bernoullis sought asylum first in Frankfurt (1583) and afterwards in Basel, where they ultimately rose to the highest distinctions.

Jakob was born in Basel. He was educated at the city's public school. Upon the conclusion of his philosophical studies at the university, some geometrical figures which he chanced to see excited in him a passion for mathematics, and in spite of the opposition of his father, who wished him to be a clergyman, he applied himself in secret to his favorite science. In 1676 he visited Geneva on his way to France, and subsequently traveled to England and Holland. While in Geneva he taught a blind girl several branches of science, and also how to write; and this led him to publish *A Method of Teaching Mathematics to the Blind*. In London he was admitted to the meetings of **Robert Boyle**, **Robert Hooke** and other learned men. On his return to Basel in 1682 he devoted himself to physical and mathematical investigations, and opened a public seminary for experimental physics.

In the same year he published his essay on comets, *Conamen Novi Systematis Cometarum*, which was occasioned by the appearance of the comet of 1680. In 1687 the mathematical chair of the University of Basel was conferred upon him, and he was later made rector of his university. In 1684 he had been offered a professorship at Heidelberg; but his marriage to a lady of his native city led him to decline the invitation. He wrote elegant verses in Latin, German and French; but although these were held in high esteem in his own time, it is on his mathematical works that his fame now rests.

1694 CE **Rudolph Jakob Camerarius** (**Camerer**) (1665–1721, Germany). Physician and botanist. In *De Sexu Plantarum Epistola*, presented a conclusive demonstration of the sexuality of plants. Professor at the University of Tübingen (from 1688).

1694–1718 CE **Johann (Jean, John) Bernoulli** (1667–1748, Switzerland). One of the leading mathematicians of the 18^{th} century. A member of a remarkable Swiss family that produced 8 mathematicians — three of them outstanding — who in turn had a swarm of descendants who distinguished themselves in many fields.

In 1694 he took a doctor's degree in medicine but became fascinated by calculus and applied it to many problems in geometry, differential equations and mechanics. In 1695 he was appointed professor of mathematics and physics at Groningen in Holland, and on his brother Jakob's death, succeeded him in the professorship at Basel. In 1696 he proposed the famous '*Brachistochrone*

Problem' as a challenge to the mathematicians of Europe[165] [curve of shortest descent-time between two fixed points in a homogeneous gravitational field — the cycloid]. It was solved by **Newton**, **Leibniz**, his brother Jakob and himself [solved earlier (1673) by **Huygens** and applied by him in the construction of a pendulum clock]. Most of the calculus integration techniques were systematically worked out by the Bernoullis and **Euler**. Johann, however, pioneered the use of *substitutions*.

The so-called *L'Hopital Rule* was actually obtained by him in 1696; but L'Hopital and Bernoulli had an agreement (1692) whereby Johann sent L'Hopital some of his mathematical discoveries, to be used as L'Hopital chose, in exchange for regular salary. It was only after the death of **Guillaume Francois Antoine L'Hopital**, **Marquis de St. Mesme** (1661–1704) that Bernoulli accused him of plagiarism, an accusation that at the time was generally dismissed but now seems to be well founded[166]. The Marquis introduced this method in the *first* calculus textbook "*Analyse des infiniments petits*", published in Paris in 1696. This book had a wide circulation, and brought the differential notation into general usage in France as well as making it known throughout Europe.

Johann Bernoulli was the first to recognize the *principle of virtual work* as a general principle of statics with which all problems of equilibrium could be solved. He introduced the product of the force and the virtual velocity in the direction of the force, taken with a positive or negative sign according to the acute or obtuse angle between force and velocity (scalar product of vectors!). In 1717 he announced the general principle that *for all possible infinitesimal displacements, the sum of all these products must vanish if the forces balance each other*.

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165 He was the first to denote the acceleration of gravity by the symbol g, and first to write the relation $v^2 = 2gh$. The notation ϕx to indicate a function of x was introduced by him in 1718.

166 In 1921, Johann's manuscript on the differential calculus was discovered. Together with the correspondence of L'Hopital and Bernoulli, it proved that the Swiss mathematician was the true author of L'Hopital's calculus book.
One of Johann's contributions, made jointly with Leibniz, was the technique of *partial differentiation* The two kept this discovery secret(!) for 20 years in order to use it as a "secret weapon" in problems about a family of curves. Another startling result of Johann (1697) was

$$\int_0^1 x^x dx = 1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \cdots$$

Remarkably ingenious, too, is another observation of Bernoulli; he compared the motion of a particle in a given field of force with the propagation of *light* in an optically heterogeneous medium, and tried to give a mechanical theory of the refractive index on this basis. Bernoulli is thus the forerunner of the Hamilton-Jacobi theory, that links optical and mechanical systems and presages *quantum mechanics*.

1696–1727 CE **Stephen Gray** (1666–1736, England). Physicist and chemist. A pupil of Newton. One of the first experimenters in static electricity, using frictional methods to prove conduction (1727).

Discovered the *conduction* of electrical charges and made the distinction between conductors and insulators. Gray showed that electricity can be transmitted from one object to another and over distances through conductors and that static electrical charges reside on the surfaces of objects, not in the interiors (1729). Transmitted electrical charges, generated by electric generator, over brass wires 100 meters long. Demonstrated that anything can be charged with static electricity if it is isolated by nonconducting materials (1731). His work had a great influence on the electrical theory of **Du Fay** (1733–1740).

Gray was born in Canterbury and followed his father's trade as a dyer, but the thirst of education led him to Cambridge University. His first scientific paper (1696) described a microscope made of a water droplet, similar to the glass bead microscope made so famous by **Leeuwenhoek** (1703).

1697–1733 CE **Abraham de Moivre** (1667–1754, England). An outstanding mathematician of the 18^{th} century. Extended the pioneering ideas of **Fermat**, **Pascal**, **Huygens** and **Jakob Bernoulli** in probability theory, and originated other 'simple discoveries' which are found today in school textbooks. His achievements are:

(1) Wrote a systematic treatise on probability: "*Doctrine of chances*" (1714). In 1733 he showed the manner in which the normal distribution function arises in probability, as means of approximately evaluating probabilities associated with the binomial law[167]. He proved the central limit theorem for a special case. He is credited with the first treatment of the probability integral $\int_0^\infty e^{-x^2}dx = \frac{1}{2}\sqrt{\pi}$ and of the normal frequency curve $y = ae^{-\lambda^2 x^2}$ in statistics theory. Moivre is also noted for his work

[167] De Moivre's observation, in modern notations, is as follows: The binomial law states that

$$(py + q)^n = \sum_{k=0}^{n} P_n(k)y^k,$$

where $P_n(k) = \frac{n!}{k!(n-k)!}p^k q^{n-k}$, $(q = 1-p)$ is the probability of k successes in n independent trials. Simple manipulation via differentiation and the subsequent

substitution $y = 1$, yield the identities:

$$\sum_{k=0}^{n} P_n(k) = 1, \qquad \bar{k} = \sum_{k=0}^{n} kP_n(k) = np; \qquad \bar{k^2} = \sum_{k=0}^{n} k^2 P_n(t) = n(n-1)p^2 + np$$

It thus follows that the first and second order parameters of the binomial distribution are: mean $= k = np$; dispersion $= D = \bar{k^2} - (\bar{k})^2 = npq$; root mean square deviation$= \sigma = \sqrt{npq}$.

Introducing a new variable $x = k - \bar{k} = k - np$, the binomial probability

$$P_n(k) = \frac{n!}{(np+x)!(nq-x)!} p^{np+x} q^{nq-x}$$

is approximated by means of Stirling's formula $n! \approx n^n e^{-n}\sqrt{2\pi n}$ and becomes

$$\begin{aligned} P_n(k) &= (1 + \frac{x}{np})^{-x-np}(1 - \frac{x}{nq})^{x-nq}\left[2\pi n(p + \frac{x}{n})(q - \frac{x}{n})\right]^{-1/2} \\ &= \frac{\exp\left[-(np+x)\log_e(1+\frac{x}{np}) + (nq-x)\log_e(1-\frac{x}{nq})\right]}{\sqrt{2\pi n(pq - \frac{x}{n}(p-q) - \frac{x^2}{n^2})}} \end{aligned}$$

Using the Taylor expansions for small $\frac{x}{np}$ and $\frac{x}{nq}$, namely

$$(np+x)\log_e(1 + \frac{x}{np}) = (np+x)\left[\frac{x}{np} - \frac{1}{2}\frac{x^2}{n^2p^2} + \frac{1}{3}\frac{x^3}{n^3p^3} - \cdots\right],$$

$$(nq-x)\log_e(1 - \frac{x}{nq}) = -(nq-x)\left[\frac{x}{nq} + \frac{1}{2}\frac{x^2}{n^2q^2} + \frac{1}{3}\frac{x^3}{n^3q^3} + \cdots\right].$$

De Moivre obtained the asymptotic expressions

$$P_n(k) \approx \frac{1}{\sqrt{2\pi pqn}} e^{-\frac{1}{2}\frac{x^2}{npq}} \qquad \text{or} \qquad P_n(k) \approx \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(k-\bar{k})^2}{2\sigma^2}}.$$

With $\xi = \frac{k-np}{\sqrt{2npq}} = \frac{k-\bar{k}}{\sigma\sqrt{2}}$, the (approximate) probability of ξ lying between ξ and $\xi + d\xi$ is $p(\xi)d\xi = \frac{1}{\sqrt{\pi}}e^{-\xi^2}d\xi$. Therefore, the probability for k lying between the two values k_1 and k_2 is

$$\frac{1}{\sqrt{\pi}}\int_{\xi_1}^{\xi_2} e^{-\xi^2} d\xi$$

with

$$\xi_1 = \frac{k_1 - np}{\sqrt{2npq}}; \qquad \xi_2 = \frac{k_2 - np}{\sqrt{2npq}}.$$

"*Annuities upon Lives*", which played an important role in the history of actuarial mathematics.

(2) Was first to derive the factorial approximation $n! \simeq (2\pi n)^{1/2}(n/e)^n$, misnamed *Stirling's formula*.

(3) Announced in 1707 the keystone formula of analytic trigonometry[168]:

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx$$

for positive integer n. It was explicitly stated and proved inductively by Euler in 1748.

(4) Introduced (1730) the powerful method of *generating functions* which proved to be of great importance in combinatorics, probability and number theory. He used it to obtain a closed-form expression for the general term of the Fibonacci sequence, namely

$$F_n = \frac{1}{2\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] + \frac{1}{2}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

(It was rediscovered by Binet in 1843).

Moivre was born in Vitry, Champagne, to a French Huguenot family. His father was a surgeon. He was compelled to take refuge in England at the revocation of the edict of Nantes in 1685. He started his mathematical education in France and furthered it in London. There, he eked out a living by giving private lessons in mathematics and games of chance. He never married.

He was greatly influenced by the "*Principia Mathematica*" of Newton and was among Newton's personal friends. He became distinguished among first-rate mathematicians and was admitted in 1697 to the Royal Society of London. His merits were so well known and acknowledged that the society found him fit to decide the famous contest between Newton and Leibniz. In spite of all this he never secured a university position, perhaps because he was not British by birth.

His old age was spent in obscure poverty. A bizarre story is associated with his death: de Moivre noticed, so the story goes, that each day he required a quarter of an hour more sleep than on the preceding day. When this

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[168] Using this formula he proved that $y = \cos n\theta$ is a polynomial in $x = \cos\theta$. Indeed this is a direct consequence of the relation

$$2y = \left(x + \sqrt{x^2-1}\,\right)^n + \left(x - \sqrt{x^2-1}\,\right)^n, \qquad |x| \le 1.$$

He thus anticipated the Chebyshev polynomials $T_n(x) = \cos n(\cos^{-1} x)$.

progression reached 24 hours, de Moivre passed away, almost blind, at the age of 87.

1699 CE **William Dampier** (1652–1715, England). Buccaneer, navigator and oceanographer. In the publication "*Discourse of Winds, Breezes, Storms, Tides and Currents*", he suggested that major ocean currents might be caused by winds (they were previously explained as resulting from ocean height differences produced by evaporation and rain). Distinguished the steady low-latitude trade winds from the mid-latitude westerlies.

Dampier spent 38 years of his life at sea in various capacities: he went to sea as a boy, and joined the British navy in 1672. During 1679–1711 he was engaged in Pirating (1679–1680) and privateering[169] in the South Seas (against the Spaniards). In 1688 he sailed to Australia (then called New Holland) on a pirate ship. In 1699 he reached Australia again in a voyage financed by the British Admiralty. Dampier also reached New Britain and New Ireland, islands off the coast of New-Guinea.

In 1703–1711 Dampier commanded two privateers on an expedition to the South Pacific. Alexander Selkirk[170], the original Defoe's hero, Robinson Crusoe, was the master of one of his vessels.

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[169] *Privateer*: armed vessel owned and officered by private person holding a government commission, authorized to use it against hostile nations, especially in capture of merchant shipping. Dampier visited Jamaica (1679) and joined a party of pirates with whom he spent the year 1680 on the Peruvian Coast, sacking, plundering and burning Spanish ships.

[170] **Alexander Selkirk** (1676–1721). Scottish sailor. Ran away to sea (1695) and joined Dampier in a privateering expedition to the South Seas, going with the *"Cinque Ports"* galley (96 tons, 16 guns) as sailing master. In September 1704, he quarreled with his captain, and at his own request was put ashore on Mas Afuera, one of Juan Fernandez islets (33.45° S; 80.45° W, some 750 km west of Valparaiso, Chile). There he was marooned in complete solitude for four years and four months (Sept. 1704–Jan. 1709), until taken off by one of Dampier's ships.

He was later given command of one of the privateering vessels. Selkirk met with **Defoe** in Bristol (1711) and handed over his papers to him. Defoe then wrote his novel: *The Life and Strange Surprising Adventures of Robinson Crusoe* (1719). Defoe's narrative is an amalgamation of Selkirk's story and background material from Woodes Rogers' *"Cruising Voyage round the World"* (1712), and Edwards Cooke's "Voyage in the South sea and round the World" (1712) (both, the earliest descriptions of Selkirk's adventures). Nevertheless, most of the incidents in Defoe's masterpiece are fairly independent of his sources; thus the decidedly tropical description of Crusoes island and the whole narrative of the Cannibals' visits etc. agree rather with one of the West Indies than with

Dampier accounts of his voyages ["*New Voyage Round the World*" (1697); "*Voyages and Descriptions*" (1699); "*Voyage to New Holland*" (1699)] are famous. He had great talent for observation, especially of the scientific phenomena affecting the seaman's life. His style is easy, clear and manly. His knowledge of natural history, though not scientific, appears surprisingly accurate and trustworthy.

1701 CE **Giacomo Pylarini of Smyrna**. A Greek physician. The first immunologist. Inoculated children with smallpox in Constantinople.

1701 CE **Joseph Sauveur** (1653–1716, France). Physicist. Coined the terms '*acoustics*' (study of sound) and '*harmonics*' (multiples of a fundamental frequency) while experimenting with vibrating strings.

1701–1731 CE **Jethro Tull** (1674–1741, England). Gentleman farmer and agriculturist. Technical innovator of the agrarian revolution[171]. Introduced new agricultural machinery (1701) and new farming methods (1731) through the use of manure, pulverization of the soil, growing crops in rows and hoeing to remove the weeds.

Tull was born in Berkshire, and was educated at St. John's College, Oxford University. He traveled in France and Italy to observe farming methods. In his days, farmers sowed the seed by throwing it by hand. Tull regarded the practice both wasteful and uncertain. So he invented a drill for boring straight rows of holes into which he dropped the seed. His ideas were adopted slowly.

1702 CE First daily newspaper in England.

1703 CE French chemists **Nicolas Lémery** (1645–1715) and **Martin Lister** (1638–1712) promoted a theory that the source of an earthquake was an *explosion* produced by mixing minerals inside the earth composed of the same chemicals used for explosives (iron, sulfur, salt, water). This theory became very popular and **Isaac Newton** in his book *Optiks* (1704) adopted this idea of the *mineral explosion* in subterranean cavities.

Juan Fernandez. The best biography is the "Life and Adventure of Alexander Selkirk" by Jonh Howell (1829).

Selkirk returned to sea, and died as master's mate of H.H.S. "Weymouth" on Dec. 12^{th} 1721.

171 The English statesman **Charles Townshend** (1674–1738) introduced into England the four-course system of crop rotation (1731), which he first practiced at his Raynham estate.

Robert Bakewell (1725–1795, England) was first to introduce stock-breeding improvements and grassland amelioration by systematic irrigation.

The 'explosion theory' reigned supreme for more than 150 years. It was finally discarded[172] in the wake of the great Napolitan earthquake (1857) under the impact of the new ideas of **Robert Mallet** (1860).

The 'Little Ice Age' (LIA 1560–1850)

Current research on global climate change, drawn from tree rings and Greenland Ice cores, provides much detailed information on weather and climate history[173]*. This new information can be correlated with historical accounts on major weather events and their influence on the human condition.*

Indeed, this new knowledge provides an engaging history of Western Europe; it reveals that Europe experienced a prolonged warm period known as the Medieval Warm Period[174] *(600–1150), cooling of the climate (1150–1460), a brief warming (1460–1560), followed by dramatic cooling known as the Little Ice Age*[175] *(1560–1850).*

For people living near subsistence levels, as most European did before 1800, abrupt changes in weather could mean the difference between prosperity and

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172 As often happen in science, the theory was recently resurrected and incarnated: it has been claimed that under increasing strain, minerals undergo a phase transformation and become metastable polymorphs of higher free energy density. This instability leads to an explosion creating a shock wave with supersonic velocity, which then causes fluidization of the fault core; the fault is thus unlocked, releasing the stored elastic energy, *ergo*, an earthquake.

173 Brian Fagan: "*Little Ice Age*" Basic Books, N.Y. 2001, 246 pp.

174 The *Viking expansion* from Scandinavia through Europe and the North Atlantic (800–1050), occurred through this period.

175 It seems that the LIA affected the entire globe; Temperature data obtained from a *Peruvian* ice core whose layers date from 1600 CE to the present agree well with independent temperature histories derived for the Northern hemisphere, confirming that the LIA of 1400 to 1650 and the climatic impact of the 1815 Tambora volcanic eruption in Indonesia, were global in extent and demonstrating that some mid-latitude glacial records, too, can play important roles in studies of historical climate.

pauperhood, or even between life and death – especially if these changes lasted more than one season. Events such as the French Revolution *and the Irish Potato-famine, are now seen through the lens of weather and its effects on harvests: the colder weather impacted agriculture, health, economics, social strife, emigration and even art and literature. Increased glaciation and storms also had a devastating affect on those that lived near glaciers and the sea.*

The climate of a region is typically defined by its monthly mean temperature *(Fig. 3.2) and annual total* precipitation. *However, direct observations of these variables began only after the invention of the* barometer *(**Torricelli**) and the thermometer (**Galileo**) in the first half of the 17^{th} century.*

To determine earlier climate, investigators infer the climate record from physical and biological fossil data including, among others: oxygen isotope ratios detected in ice cores, tree-rings dating, ice flow and glacier data, and archaeological discoveries, and also from records intended for other purposes such as: weather diaries, shipping logs, tax records, crop production and pricing records, allusions to climate in art and literature, etc.

- *Oxygen Isotope Record: Measurements of the isotopes ratio $^{18}O/^{16}O$ in ice indicates the temperature of the snow at the time if was formed. Higher ratios of the heavier ^{18}O oxygen isotope indicate the snow formed at a higher temperature while lower ratios indicate the snow formed at a cooler temperature.*

- *Tree-Ring Data: individual rings represent individual years while the width of each ring shows the growth-rate during that year. The width of rings from trees found at higher altitudes and higher latitudes in generally a function of temperature, where wide rings indicate warm years and narrow rings indicate cool years. Because the pattern of rings is similar to a fingerprint, dendrochronologists are able to construct a chronology by matching similar ring patterns found in living trees, construction timbers, and fossil trees.*

- *North Atlantic Drift Ice: Drift ice is carried from the Arctic ice pack and the waters north of Iceland by ocean currents. In colder times, arctic waters carry the ice southward while in warmer times the Gulf Stream dominates the Iceland area, keeping drift ice away. Drift ice was carefully observed by Icelanders both from shore and from ships because it threatened ships and therefore affected commerce. Thus, drift ice can be considered a thermometer of the North Atlantic.*

- *Glacier Waxing and Waning: Mountain glaciers in Scandinavia and the Alps can be used to record climatic changes. Because glaciers are*

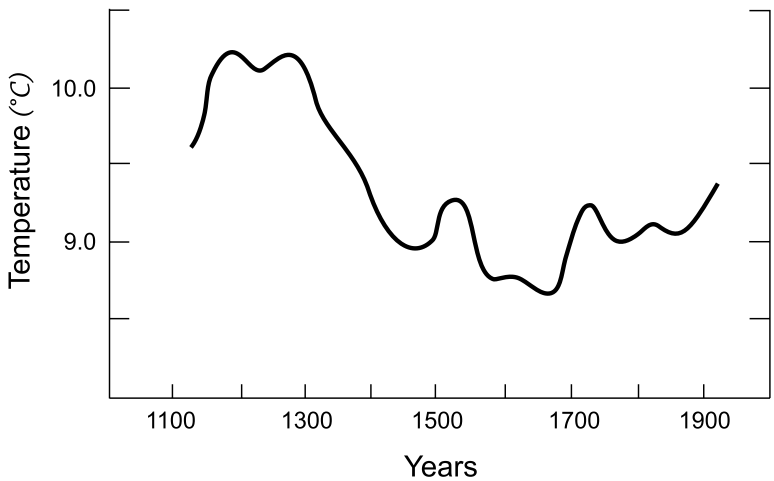

Fig. 3.2: Estimated mean yearly temperatures based on a variety of climatic, political and social indicators. (Redrawn from *Climate, History and the Modern World* **H.H. Lamb** Methuen, London, 1982.)

massive, they respond to long-term temperature and precipitation variations on a time scale of decades and centuries. Glaciers grow during winters by accumulating snowfall and glaciers decline during summers due to above-freezing temperatures. For a glacier to maintain its position, snowfall must equal snowmelt. Cooler summers result in less snowmelt and longer winters increase the number of days of potential snowfall (clearly, cooler winters may also bring drier air which could decrease snowfall, but this factor is much smaller than the former).

There are four main possible causes (forcing mechanisms) of the climatic changes experienced during the LIA. These include: sunspot variations, volcanic eruptions, changes in the large-scale ocean current conveyor belt and changes in the earth's albedo. None of these factors on their own offers conclusive evidence; it is likely that each has played a role.

- *Sunspot Variations: Because the sun is earth's greatest source of energy and is the driving force behind its atmospheric circulation, any variation in solar output will influence the weather. Scientists have observed that the number of sunspots on the surface of the sun has been determined to correspond to solar output variability. More sunspots correspond to a higher solar energy output while fewer sunspots correspond to a lower solar output. A record of sunspot numbers has been recorded through time by various indicators including naked eye observations, auroral reports, and* ^{14}C *isotope concentrations in tree rings.*

  Thus, during the Medieval Warm Period (600–1150 CE) there was a high number of sunspots referred to as the Medieval Maximum, while during the Little Ice Age (1560–1850 CE) there were two periods of very low sunspots numbers called the Spörer Minimum and the Maunder Minimum (1645–1715 CE).

- *Volcanic Eruptions: Ash and other small particulate matter injected into the stratosphere can effectively reduce incoming solar radiation received at the earth's surface. Sulfur compounds from eruptions condense into very tiny sulfuric acid droplets that form clouds which may stay suspended in the stratosphere for years, further reducing incoming sunlight.*

  Large eruptions at low latitudes can cause the greatest global climate change. Weaker eruptions only send their eruptive materials into the troposphere where weather processes quickly remove them and high latitude eruptions only send their materials into one hemisphere. The explosion of Mt. Tambora in 1815 led to the year 1816 being called "the year without a summer" across much of Europe. The eruption of Mt. Pinatubo in 1991 provided a good example of how a large low-latitude eruption can quickly influence global climate: in nine days the sulfur dioxide plume had spread into both hemispheres and around half the planet.

  The result was an estimated $1\,^{\circ}C$ *global cooling that lasted two years. It is unlikely that a single eruption can cause long-term cooling over hundreds of years such as during the LIA. But evidence has shown that there was an increase in the frequency of large eruptions during the LIA that corresponds quite well with the coolest years during this time period.*

- *Large-Scale Ocean Current Conveyor Belt: Warm waters in the upper 1500 meters flow northward to the vicinity of Iceland. Winter cooling increases the density of the water permitting it to sink to great depths. Once at depth, the water flows the length of the Atlantic and becomes mixed into the deep southern hemisphere current. Because the ocean*

and atmosphere are a coupled system, any changes in this large-scale ocean circulation could cause large-scale atmospheric changes on the order of hundreds of years. The ocean is both a heat source for the atmosphere by releasing carbon dioxide, a greenhouse gas, and a heat sink by conducting heat away from the air that rests upon it. Surface water that comes into contact with air is referred to as ventilated water. Scientists have demonstrated that very high rates of deep water ventilation occurred curing the LIA, which means the oceans were removing heat from the atmosphere at a greater rate than normal during that period. That could explain the dramatic cooling observed during the LIA.

- *Earth Albedo: Albedo is a measure of the reflectivity of a surface. Snow and ice have a high albedo because their properties allow them to reflect up to 90% of incoming sunlight. After a global cooling event has begun, it can become self-perpetuating. With increased snow cover and glaciation, the planet's surface will have a higher albedo, which in turn will cause more incoming sunlight to be reflected. With less sunlight being absorbed at the earth's surface there will be a subsequent cooling effect. This cooling effect may cause even more snow cover and glaciation that would increase the planet's albedo even more. As the climate cooled during the LIA, earth's albedo increased due to more snow and greater glaciation. The process can last for many years; however, it eventually does subside because cooler oceans experience less evaporation which leads to a decrease in cloud cover. Reduced cloud cover allows more sunlight to reach the surface which results in higher global air temperatures.*

The Historical Record

The cooling of 1.5–$2.0\,^{\circ}C$*, synchronous over broad regional areas for a span of several hundred years caused a wide gamut of accompanying phenomena, documented in Europe and North America:*

(i) Glacier movements

Glaciers in many parts of Europe began to advance about the mid-13th century, influencing almost every aspect of life for those unfortunate enough to be living in their path. It destroyed farmland and caused massive flooding. Glaciers in the Swiss Alps advanced, gradually engulfing farms and crushing entire villages. On many occasions bishops

and priests were called to bless the fields and to pray that the ice stopped grinding forward. Various tax records show glaciers over the years destroying whole towns caught in their path. A few major advances are:

***1595 CE**: Gietroz (Switzerland) glacier advances*, *dammed Dranse River, and caused flooding of Bagne with 70 deaths.*

***1600–1610 CE**: Advances by Chamonix (France) glaciers cause massive floods which destroyed three villages and severely damaged a fourth.*

***1670–1680's CE**: Maximum historical advances by glaciers in Eastern Alps. Noticeable decline of human population by this time in areas close to glaciers, whereas population elsewhere in Europe had risen.*

***1695–1709 CE**: Iceland glaciers advance dramatically, destroying farms.*

***1710–1735 CE**: A glacier in Norway was advancing at a rate of 100 m per year for 25 years.*

***1748–1750 CE**: Norwegian glaciers achieved their historical maximum LIA positions.*

In general, habitual structures which were once at high altitude in the Alps were destroyed by glacier activities: a glacier blocked the Saas valley, including its river (1589) and eventually formed a lake. Ice sheets advanced over farms, villages and valleys in Greenland. Once productive Icelandic farms were covered by advancing glaciers. So serious was the climatic change experienced by Icelanders that Denmark, the parent country, considered evacuating all the Icelanders and re-settling them in Europe.

Glaciers advances in North America occurred from 1711–1724 and 1835–1849.

(ii) Storms

During the LIA, there was a high frequency of storms. As the cooler air began to move southward, the polar jet stream strengthened and followed, which directed a higher number of storms into the region. At least four sea floods of the Dutch and German coasts in the thirteenth century were reported to have caused the loss of around 100,000 lives. Sea level was likely increased by the long-term ice melt during the MWP which compounded the flooding. Storms that caused greater than 100,000 deaths were also reported in 1421, 1446, and 1570. Additionally, large hailstorms that wiped out farmland and killed great numbers of livestock

occurred over much of Europe due to the very cold air aloft during the warmer months. Due to severe erosion of coastline and high winds, great sand storms developed which destroyed farmlands and reshaped coastal land regions.

Two great storms in the North Sea occurred in 1362 and in 1703. The first destroyed the Island of Strand and the city of Ronghold. The second (Nov 26) killed ca 8,000 people on the eastern coast of the British Isles.

(iii) Freezing

The Baltic Sea and rivers such as the Thames in England and the Tagus in Spain, currently ice-free the year around, were regularly frozen several inches thick. Winter Landscape, painted by ***Peter Brueghel the Younger*** *(1601) exhibit the frozen canals of Holland, now regularly ice-free the year around. A generation earlier,* ***Peter Breughel the Elder*** *recorded the merrymaking of Flemish peasantry in their daily lives. His artworks-started off with fairly warm sunny summer weather, but In the 1560s he suddenly switched to cold snow-swept landscapes. This change began with* Hunters in the Snow*, depicting a group of men returning from a hunt, set against a frozen lake. It was at this time that the winter of 1564–1565 struck – the longest and most severe for well over a century.*

In the winter of 1780, New York Harbor froze, allowing people to walk from Manhattan to Staten Island. Sea ice surrounding Iceland extended for miles in every direction, closing the island's nation's harbors to shipping.

(iv) Volcanic Activity

Throughout the Little Ice Age the world also experienced heightened volcanic activity. When a volcano *erupts, its ash reaches high into the atmosphere and can spread to cover the whole earth. This ash cloud blocks out some of the incoming solar radiation, leading to world-wide cooling that can last up to two years after an eruption. Also emitted by eruptions is* sulfur *in the form of SO_2 gas. When this gas reaches the* stratosphere *it turns into sulfuric acid particles, which reflect the sun's rays, further reducing the amount of radiation reaching the earth's surface. The 1815 eruption of* Tambora *in* Indonesia *blanketed the atmosphere with ash; the following year, 1816, came to be known as the* Year Without A Summer*, when frost and snow were reported in June and July in both* New England *and Northern* Europe.

There's probably no better example of the artistic weather record than ***Joseph Turner****. Because he was obsessed with the light of the sky, clouds and sea, Turner has given us a stunning insight into the climate of the early nineteenth century. His glorious red skies were a particular sign of strong atmospheric powers at work, because this was a time when volcanic eruptions in the Azores in 1811 and Tambora in 1815 had shot clouds of dust across the globe. That dust cooled the earth and scattered the light, filtering out the blues in the low sun and giving sumptuous red sunrises and sunsets.*

(v) *Agriculture, economics and health*

Crop-failures, poor harvests, increasing grain prices, lower wine-production and severe diseases marred the lives of people in Western Europe during the LIA, especially throughout the Maunder minimum.

These ere some of the many disasters impacted by the dramatic cooling of the climate. Due to the famine, storms, and growth of glaciers, many farmsteads were destroyed, which resulted in less tax revenues collected due to decreased value of the properties.

The change in climate during these years greatly affected crop production and animal husbandry. Famine became more frequent and death from diseases increased.

Each grain crop requires several conditions before a successful growing season and harvest is possible. Minimum temperatures are necessary for seed germination. Higher altitudes are more susceptible to adverse climatic cooling. Frost will occur later in the spring and earlier in the fall causing a shortened growing season. Increase cloud cover and cool weather retard the growing process and prolong the ripening of the grain. In addition, if the summer remains wetter than usual, grain crops may not be able to mature by drying out. If an early frost comes, the still-moist grain will suffer damage. A cooling trend can affect the growing plant in several ways, compounding the possibility of crop failure.

Thus, in the years of the LIA the price of grain increased over five times, imposing an obvious hardship on the poor.

It is estimated that in the coldest decades of the Little Ice Age the growing season was shortened by 3-4 weeks. This may represent an approximate reduction of 20% of the total growing season which would range from May to September in the Northern latitudes.

Exceptionally grim reports of mass deaths are frequent in the literature of this time. There were population decreases in large portions of Europe. While diseases such as bubonic plague (Black Death) definitely

had their effect, the generally weakened health of the people in years of poor harvest must certainly be considered. In fact, population declines attributed to low food levels began 40 years before the plague arrived.

In conclusion, the cooler climate during the LIA had a huge *impact on the health of Europeans*, dearth and famine killed millions. *Cool, wet summers* led to outbreaks of an illness called St. Anthony's Fire. Whole villages would suffer convulsions, hallucinations, gangrenous rotting of the extremities, and even death. *Grain, if stored in cool, damp conditions, may develop a fungus known as ergot blight* and also may ferment just enough to produce a *drug similar to LSD*. (In fact, some historians claim that the Salem, Massachusetts, witch hysteria was the result of ergot blight.)

Malnutrition led to a weakened immunity to a variety of illnesses. In England, malnutrition aggravated an *influenza epidemic of 1557–8* in which whole families died. In fact, during most of the 1550's deaths outnumbered births The Black Death (Bubonic Plague) was *hastened by malnutrition all over Europe.*

(vi) *Social Unrest*

Conditions during the LIA led to many cases of social unrest. The winter of 1709 killed many people in France. Conditions were so bad, a priest in Angers, in west-central France, wrote:

"The *cold began on January 6, 1709*, and lasted in all its rigor until the twenty-fourth. The crops that had been sewn were all completely destroyed... Most of the hens had died of cold, as had the beasts in the stables. When any poultry did survive the cold, their combs were seen to freeze and fall off. Many birds, ducks, partridges, woodcock, and blackbirds died and were found on the roads and on the thick ice and frequent snow. Oaks, ashes, and other valley trees split with cold. Two thirds of the vines died... No grape harvest was gathered at all in Anjou... I myself did not get enough wine from my vineyard to fill a nutshell."

In March the poor rioted in several cities to keep the merchants from selling what little wheat they had left.

The winter of 1739–1740 was also a bad one. After that there was no spring and only a damp, cool summer which spoiled the wheat harvest. The poor rebelled and the governor of Li told the rich to *"fire into the middle of them. That's the only way to disperse this riffraff"*.

One of history's most notorious quotes might have been due in part to a rare extremely warm period during the LIA. In Northern France in 1788,

after an unusually bad winter, May, June, and July were excessively hot, which caused the grain to shrivel. On July 13, just at harvest time, a severe hailstorm (which typically occurs when there is very cold air aloft) destroyed what little crops were left. From that bad harvest of 1788 came the bread riots of 1789 which led to Marie Antoinette's alleged remark "Let them eat cake," and the storming of the Bastille.

(vii) *Vampires and Violins*

Writers were also influenced by the great change in climate. In 1816, "the year without a summer," many Europeans spent their summers around the fire. ***Mary Shelley*** *(1797–1851) was inspired to write 'Frankenstein', and* ***John Polidori*** *(1795–1821), 'The Vampyre'. Both authors, together with* ***Byron*** *and* ***Percy Shelley****, were in Switzerland, near Lake Geneva where Byron said "We will each write a ghost story." Percy Shelley also referred to a glacier in his poem "Mont Blanc" when he wrote "and wall impregnable of beaming ice. The race of man flies far in dread; his work and dwelling vanish".*

The less intense solar radiation and activity coincided with a sharp decline in temperature, causing a very cold weather in Western Europe. It is clearly seen in tree-ring records from high-elevation forest stands in the European Alps. The long winters and cool summers produced wood that has slow, even growth – desirable properties for producing quality sounding boards.

Antonio Stradivari *of Cremona, Italy, perhaps the most famous of violin makers*[176]*, was born one year before the beginning of the Maunder minimum (1645–1715). He and other violin makers of the area used the only wood available to them from the trees that grew during the Maunder minimum. It was suggested*[177] *that the narrow tree-rings of*

[176] The violin first emerged in Northern Italy in the mid 1500's. Many of the most distinguished violins ever created were produced by famous local families. The most famous makers were: **Andrea Amati** (1520–1578; Italy); **Jacob Stainer** (ca 1617–1683; Austria); **Antonio Stradivari** (1644–1737; Italy); **Francesco Stradivari** (1671–1743; Italy); **Andrea Guarneri** (ca 1626–1698; Italy); **Giuseppe Guarneri** (1666–ca 1739); **Giuseppe del Gesu Guarneri** (1687–1745). **Stradivari**, the most famous of these craftsmen, produced over 1100 violas, guitars, cellos and violins. Around 600 of his instruments are extant today. Narrow tree-rings would not only strengthen the violin but would increase the wood's density. Dense wood with narrow growth-rings may help instill a superior tone and brilliance in violins.

[177] **H. Grissino-Mayer** and **L. Burckle**: *Dendrochronologia*, **21**, 41–45, Lamont-Doherty Earth Observatory, Dec 2003.

these trees played a role in the enhanced sound quality of instruments produced at this time.

1703 CE, Nov. 26 An ocean tempest killed ca 8000 people on the eastern coast of the British isles. Probably the greatest British storm of the last two millennia. Fifteen warships and hundreds of merchant vessels (with about 1500 seamen on board) were lost. Thousands of trees were laid low throughout the country, including ca 4000 large oaks in the New Forest, Cranbourne Chase, and the Forest of Dean. Houses were blown to pieces; church steeples toppled like skittles; floods were widespread, and Bristol's streets ran into water. Eddystone lighthouse, together with its crew, was swept like a heap of rubble into the sea. When it was all over, the Commons presented an address to Queen Anne and she issued a proclamation of general fast.

1704–1709 CE **Abraham Darby** (1678–1717, England). Iron and brass manufacturer who developed a process for smelting iron using *coke*[178]

[178] *Coke* is prepared by carbonizing coal in *coke-ovens*. The old oven consisted of a covered mound of brickwork, in which coal was partly burnt in a limited supply of air, as in charcoal burning. The high temperature produced carbonizes the rest of the coal, and all the volatile products are lost.

The *blast-furnace* consisted of an outer shell of steel plates, lined with refractory bricks. It was a 15–30 m hight, the greatest width being about 8 m at the "boshes". The mouth was closed with a *cap-and-cone* (above) through which the charge of ore, limestone and fuel was fed intermittently, whilst the gases (carbon monoxide and nitrogen) pass away through a pipe to a *dust-catcher*, and are utilized in heating the blast. The furnace below the boshes narrows gradually to a *hearth* at the base, about 3 m in diameter and the same height. This was pierced with holes for the water – jacketed iron blowing-pipes or *tuyeres*, through which air was forced from an annular pipe by means of powerful blowing engines. About 3–5 tons of air were passed through the furnace per ton of iron made, the power for working the blowing-engines being supplied by coke-oven gas obtained in producing the coke for the blast furnace.

The first extensive use of cast-iron was in England (1544). Formerly, charcoal was used as fuel; coal was used by **D. Dudley** (1599–1684) in England (1619).

The chemical reactions in the blast-furnace are as follows:

- Oxygen unites with carbon at a very high temperature in the *hearth* to produce carbon monoxide: $2C + O_2 = 2CO$.

instead of the more expensive charcoal. Darby produced coke to use in *blast-furnaces*. It was made by partly burning coal in a closed chamber. The heating drives out the volatile material from the coal, including most of the sulfur. This produced a high-carbon fuel which was clearer and hotter than coal. Formerly wood-based charcoal was used to smelt iron.

The discovery greatly increased the market for coal and improved iron production. With the invention of the *Newcomen engine*, this breakthrough marked one of the starting points of the Industrial Revolution in England. (Darby employed the cheaper iron to cast thin pots for domestic use, and after his death it was used for the huge cylinders required by the new steam pumping-engines.)

His grandson, Abraham Darby III (1750–1791) constructed the world's first *iron bridge*, over the river Severn at Coalbrookdale, Shropshire.

Darby was born near Dudely, Worcestershire to a Quaker family and trained in engineering, setting up his own business (1698). He visited Holland (1704) and brought back with him some Dutch brass[179] founders, establishing them in Bristol, later moving to Coalbrookdale. They experimented with substituting cast iron for brass in some products and in 1708 Darby took out a patent for a new way of casting iron pots and other ironware in sand only,

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- Above the boshes, at a dull red-heat (500 °C–900 °C) the ferric oxide is reduced by the carbon monoxide to spongy iron: $Fe_2O_3 + 3CO \leftrightharpoons 3Fe + 3CO_2$.

  The reaction is reversible and the escaping gases contain CO and CO_2. Another reaction also occurs which limits the completeness of the reduction: $2Fe + 3CO \leftrightharpoons Fe_2O_3 + 3C$. In this upper zone the limestone is decomposed: $CaCO_3 \leftrightharpoons CaO + CO_2$, and some carbon dioxide is reduced to monoxide: $CO_2 + C \leftrightharpoons 2CO$. The spongy iron absorbs sulphur from the fuel.

- Near the center of the furnace, at bright red heat, finely-divided carbon is deposited by the reaction: $2CO \leftrightharpoons CO_2 + C$. This and the carbon of the charge complete the reduction:

$$Fe_2O_3 + 3C = 2Fe + 3CO.$$

- At the white zone, in the lowest part of the furnace, the spongy iron containing carbon, sulphur, phosphorus and silicon, fuses to molten *cast-iron* which is tapped-off from time to time into sand moulds to form *pig-iron*, or is sent in the fused state to the steel furnaces.

[179] After the restoration of the monarchy England's economy had surged, and the demand for household brass rose rapidly.

without loam or clay. This process cheapened utensils much used by poorer people.

Table 3.6: Iron metallurgy – signposts of progress, 1650–1950

In England, the age of iron marched from triumph to triumph; New techniques were developed for its production and new users were found for it. Vast new smelting houses were built. Such was the demand, that Britain had to import some 50,000 tons a year.

1665 CE Smelting iron with *charcoal* in England.

1709 CE **Darby** (England) developed a process using *coke* in blast furnaces.

1740 CE **Huntsman** (England) rediscovered the crucible process of making *cast steel*. The small ingots produced by this process could not be used yet to build bridges or railways.

1781 CE Darby's grandson constructs in England the world's first *cast-iron bridge* (over the Severn River); the bridge is still used by pedestrians. The 378-ton bridge spans 30 m.

1783–1784 CE The English ironmaster **Henry Cort** (1740–1800) invented a process for purifying iron by *puddling* and a method of producing iron bars by means of grooved rollers. He produced *wrought iron*. It used a '*reverberatory furnace*' where raw coal and low-carbon iron were kept separate to reduce impurities in the finished product (Cort himself was ruined by a prosecution for debt and died poor).

1790 CE *Stainless steel* is produced in England.

1801 CE The engineer **James Finlay** (USA) completed the first modern suspension bridge in Pennsylvania, USA. It used iron chains for support.

1822 CE The engineer **George Stephenson** (1781–1848) built the first *iron railroad bridge* in the world. The first *iron steamship* to cross the Channel was assembled on the Thames from parts fabricated in England.

1845 CE The engineer **William Fairbairn** (1789–1874, Scotland) built the first *steel bridge*.

1851 CE Architect **Joseph Paxton** (1801–1865, England) built the *Crystal Palace* of glass and iron for the London exhibition.

1855 CE **Henry Bessemer** (England) developed the *Bessemer process* of converting *pig iron* into steel: cold air was forced through holes in the base of the furnace and through the molten iron, burning up the carbon. The device produced large amounts of steel cheaply.

1861 CE **William Siemens** (1823–1883, Germany and England) and **Pierre Emile Martin** (1824–1915, France) streamlined steelmaking with their independent invention of the *open-hearth process*: air and hot gas pass over the molten pig iron. The gases from the molten metal are then used to heat the air, to save fuel.

1885 CE The first skyscraper is erected in Chicago, USA, using presaged technique introduced by Paxton (1851).

1902 CE **P. Héroult** (France) began producing steel in a *electric-arc furnace*. This gave very high temperatures, producing much purer steel.

1904 CE **Leon Guillet** (France) developed the first stainless steel that resist corrosion.

1913 CE **H. Brearley** (England) first made stainless steel by adding *chromium* to steel. This prevents rusting.

1947 CE **H. Hartley** (England) added *titanium* to iron to produce much stronger iron.

1948 CE The *basic oxygen process* was introduced in Austria. This is the main method of making steel today: A jet of oxygen is blown on the molten iron, quickly burning up the carbon and producing steel. It is ten times faster than the open-hearth process.

NOMENCLATURE:

Ore A natural deposit of a solid containing an insoluble compound of a metal. Ores contain *minerals* (comparatively pure compounds of the metals of interest) and mixed with relatively large amounts of *gangue* (sand, soil, clay, rock and other materials). *Native ores* is the free state of the less active metals: Au, Ag, Pt, Os, Ir, Ru, Rh, Pd, As, Si, Bi, Cu.

Common classes of ores are:

Oxide Fe_2O_3 (hematite); Fe_3O_4 (magnetite); Al_2O_3 (bauxite); SnO_2 (cassiterite); MgO (periclase); SiO_2 (silica).

Sulfide $CuFeS_2$ (chalcopyrite); Cu_2S (chalcocite); ZnS (sphalerite); PbS (galena); FeS_2 (iron pyrites); HgS (cinnabar).

Chloride NaCl (rock salt); KCl (sylvite); $KCl \cdot MgCl_2$ (carnallite).

Carbonate $CaCO_3$ (limestone); $MgCO_3$ (magnesite); $MgCO_3 \cdot CaCO_3$ (dolomite).

Sulfate $CaSO_4 \cdot 2H_2O$ (gypsum); $MgSO_4 \cdot 7H_2O$ (epsom salt); $BaSO_4$ (barite).

Silicate $Be_3Al_2Si_6O_{18}$ (beryl); $Al_2(Si_2O_8)(OH)_4$ (kalinite); $LiAl(SiO_3)_2$ (spodumence).

Alloy Mixing of a metal with another substance (usually other metals) to modify its properties. Among these are: *bronze* (copper+tin), *brass* (copper+zinc), *pewter* (tin+antimony+copper), *German silver* (copper+nickel+zinc), *yellow gold* (gold+copper), *white gold* (gold+palladium+silver), *sterling silver* (silver+copper), *wrought iron* (iron+small percentage of carbon), *cast iron* (iron+2 percent or more of carbon), *steel* (many different alloys of iron containing carbon and one or more metals such as: manganese, nickel, tungsten, molybdenum, cobalt, vanadium and chromium), *stellite* (cobalt+chromium+tungsten), *carboloy* (tungsten+carbon+cobalt), *woods metal* (bismuth+tin+lead+cadmium).

Metallurgy The overall process by which metals are extracted from ores.

Roasting Heating a compound below its melting point in the presence of air. It removes sulfur, CO_2, moisture and other impurities from the ore. The remaining solid material contains a metallic oxide.

Smelting Chemical reduction of a substance at hight temperature in metallurgy. Basically it is a process of melting the ore in such a way as to remove impurities.

In the case of iron, for example, the ore is placed in a huge, brick-lined furnace called a blast furnace and subjected to high heat by blasting hot air into the bottom half of a furnace, producing temperatures up to 1000 °C. Quantities of coke and limestone are also placed in the furnace. As the heat of the furnace is raised, the coke begins to burn and gives off carbon monoxide. This gas takes oxygen from the iron oxide, helping to purify the metal.

Many of the other impurities of the ore melt and combine with the limestone to form a liquid collection of waste materials (*refuse*), which

is lighter than iron. This refuse rises to the top of the molten metal, and is taken from the furnace as *slag*.

Pig iron The iron as it comes from the blast furnace. It usually contains 95 percent iron, 3 to 4 percent carbon, and smaller amounts of manganese, phosphorus, sulfur, and other elements.

Coke A substance produced from coal by heating it in the absence of air; the heating drives out volatile material from the coal, including most of the sulfur. It also consolidates the carbon into strong lumps, stronger from either coal or charcoal.

Cast iron Reprocessed pig iron: it is remelted in a coke-burning furnace and cooled. It is brittle because it contains much iron carbide Fe_3C, but cheaper to make.

Wrought iron Pig iron is melted and most of the impurities are removed. The molten iron is then poured over a glassy mass of melted sand, or *silica slag*. The iron separates into droplets which quickly start to harden. Gases are trapped inside each droplet. The gases build up pressure and cause the drops to explode. The iron and silicate form spongelike balls of iron. These sponge balls are placed in presses to squeeze out the excess slag and form the wrought iron into blocks. The tiny threads of iron silicate make wrought iron more *malleable* (easier to hammer) and more resistant to corrosion than other kind of iron.

Wrought iron, with no carbon, was stronger but expensive because of the extra work involved in making it. What everyone was looking for was an effective compromise: cheap iron with just a little carbon, what is now called – *wild steel*. But the only kind of steel then available was unsuitable. From about 1000 BCE onwards bars of iron could only be steeled by hand labor with hammer and anvil or by *roasting* with charcoal in a furnace. The secret of melting steel, though practiced in India before the Christian era, was unknown in the West until 1740 CE, when a Yorkshire clock maker devised a process which made Sheffield steel world famous.

Steel An alloy of iron and small definite amounts of other metals. There are many types of steel, containing alloyed metals and other elements in various controlled proportions. Stainless steel show high tensile strength and excellent resistance to corrosion. The most common kind contains 14 to 18% chromium and 7 to 9% nickel.

Pig iron can be converted into steel by burning most of the carbon with O_2 in an oxygen furnace: Oxygen is blown through a heat-resistance tube inserted below the surface of the molten iron. Carbon burns to CO, which subsequently escapes and burns to CO_2.

1704–1711 CE **Luigi Ferdinando Marsigli** (1658–1730, Italy). Naturalist, adventurer, soldier, writer, and student of the sea, whose life was stranger than fiction: A general in the Austrian army, a slave in Turkey, a pensioner of the Queen of Sweden, and a fellow of the French and London Royal Societies. He wrote one of the first textbooks of oceanography, published in Venice in 1711.

While in the Bosporus, Marsigli observed the currents flowing between the Black Sea and the Mediterranean. He found that the surface water flows out of the Black Sea, but the deep water flows in the opposite direction. The local fishermen had been making good use of this fact. To travel from the Black Sea to the Mediterranean, a fisherman merely drifted in the surface current. To proceed in the opposite direction, he lowered his net into the bottom current. The large net acted as a sea anchor, dragging the boat toward the Black Sea against the surface current.

When Marsigli studied the deposits brought up from the sea bottom by fishermen, he became interested in the depth of the sea. He investigated the variation of temperature in the Mediterranean and found that it does not change significantly with depth. He measured the density of seawater with a hydrometer and found that the density increases with depth.

Marsigli was born at Bologna. After a course of scientific studies in his native city he traveled through Turkey, collecting data on the military organization of that empire, as well as on its natural history. On his return he entered the services of the emperor Leopold (1682) and fought with distinction against the Turks, by whom he was wounded and captured in a battle on the River Raab, and sold to a pasha whom he accompanied to the siege of Vienna. His release was purchased in 1684, and he afterwards took part in the war of the Spanish succession. In 1703 he was appointed second in command under Count Arco in the defense of Alt-Breisach. The fortress surrendered to the Duke of Burgundy, and Marsigli was court martialed and forced to give up soldiering. He then devoted the rest of his life to scientific investigations, in the pursuit of which he made many journeys through Europe, spending a considerable time at Marseilles to study the nature of the sea.

1705–1733 CE **Stephen Hales** (1677–1761, England). Clergyman, physiologist, chemist and inventor. Inaugurated the science of plant physiology and is one of the originators of experimental physiology. First to investigate the role of gases in *plant metabolism* and measure *blood pressure*.

The first volume of his book *Vegetable Staticks* (1727) contains an account of numerous experiments on the exchange of gases in plants, flow of fluids in plants and plant respiration, root pressure, leaf growth and the rise of sap under varying plant conditions, weather conditions and time of day. He concluded that plants drew through their *leaves* some part of their nourishment from the air. In the second volume (1733) on *Haemostaticks* he reported methods of determining *blood pressure* in man and animals and also the rate of flow, and the capacity of different vessels. He first reported observing *elasticity in arteries*[180].

Hales was born in Bekesbourne in Kent, grandson of Sir Robert Hales, who was created a baronet by Charles II, in 1670. Studied divinity, anatomy and chemistry at Cambridge and received the degree of doctor of divinity from Oxford (1733). Elected Fellow of the Royal Society (1717) and foreign associate of the French Academy of Sciences (1753).

Hales invented artificial *ventilator* for ships, prisons and hospitals, and devised forceps to aid remove kidney and bladder stones.

1706 CE **William Jones** (1675–1749, England). Mathematician. Introduced the symbol π, adopted by Euler in 1739.

1706 CE **John Machin** (1680–1752, England). Mathematician. Professor of astronomy in London. Calculated π to 100 decimal places[181], using

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180 As the heart *pumps blood* into the arteries during ventricular *systole*, a greater volume of blood enters the arteries from the heart than leaves them to flow into smaller vessels down-stream, because the smaller vessels have a greater resistance to flow. The arteries' elasticity enables them to expand to temporarily hold this excess of volume of ejected blood, storing some of the energy imparted by cardiac contraction, in their *stretched* walls.

When the heart *relaxes* and ceases pumping blood into the arteries, the stretched arterial walls passively *recoil*. This recoil pushes the excess blood contained in the arteries into the vessels down-stream, ensuring continued blood flow to the tissues when the heart is relaxing and not pumping blood into the system.

181 $\tan 4\beta = 4\frac{t(1-t^2)}{1-6t^2+t^4}$; $\tan\left(4\beta - \frac{\pi}{4}\right) = \frac{\tan 4\beta - 1}{\tan 4\beta + 1}$; $t = \tan\beta$.
Taking $\tan\beta = \frac{1}{5}$ we find $\tan 4\beta = \frac{120}{119}$ and $\tan\left(4\beta - \frac{\pi}{4}\right) = \frac{1}{239}$. Consequently:

$$\tan^{-1}\frac{1}{239} = 4\beta - \frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \frac{\pi}{4},$$

which is Machin's formula. Substituting Gregory's series

$$\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots,$$

the arctangent series

$$\frac{\pi}{4} = 4\arctan\left(\frac{1}{5}\right) - \arctan\frac{1}{239}.$$

1706–1716 CE **Roger Cotes** (1682–1716, England). Mathematician. A contemporary of Newton who undertook the publication of the second edition of Newton's *Principia*.

He was the first to develop, in 1714, the important relation

$$i\theta = \log_e(\cos\theta + i\sin\theta),$$

which is usually attributed to Euler. To Cotes we also owe a geometric theorem (1714) which depends on the factorization of a trigonometric function[182], and 'theorem of the *harmonic mean*'.[183]

Cotes was born at Burbage, Leicestershire, the son of a rector. He was educated in Trinity College, Cambridge (1699–1705) and in 1706 was appointed Plumian professor of astronomy and experimental philosophy. He took orders in 1713. After his death, his papers were collected and published under the title *Harmonia Mensurarum* (1722). His meteoric career earned him Newton's exclamation, "*If Cotes had lived, we might have known something*".

1706–1761 CE **Giovanni Battista Morgagni** (1682–1771, Italy). Physician and anatomist. Founder of the science of *pathological anatomy.*

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one finds

$$\frac{\pi}{4} = 4\left[\frac{1}{5} - \frac{1}{3\cdot 5^3} + \frac{1}{5\cdot 5^5} - \cdots\right] - \left[\frac{1}{239} - \frac{1}{3\cdot 239^3} + \frac{1}{5\cdot 239^5} - \cdots\right].$$

182 *The factorization*:

$$x^{2n} + 2x^n y^n \cos n\alpha + y^{2n} = \prod_0^{n-1}\left\{x^2 + 2xy\cos\left(\alpha + \frac{2k\pi}{n}\right) + y^2\right\}$$

is proven by factoring the complex bivariate polynomial $\{x^n + \exp(in\alpha)\times y^n\}$, making use of the n-th root of unity,and taking the modulus assuming x, y to be real.

183 "If, through a fixed point O, a variable line is drawn, cutting an algebraic curve at points $(P_1, P_2, P_3, \ldots, P_n)$, and if a point H is taken on the line such that OH is the harmonic mean of $OP_1, OP_2, \ldots, OP_n$, then the locus of H is a straight line."

His object was to relate the illness to the lesions established at autopsy. His explorations of the female genitals, of the glands of the trachea and of the male urethra, broke new grounds. His work (1761) *De sedibus et Causis Morborum per Anatomen Indagatis* (Caused of Diseases) was grounded on over 600 postmortems. Morgangi was born in Forli. Graduated from the University of Bologna and practiced medicine there. Professor at Padua University 1711–1771.

1706 CE **Francis Hauksbee** (Hawksbee) (c. 1666–1713, England). Physicist. Studied *surface tension* and *capillary* cation in fluids. Invented (1706) an *electrostatic generator*. Constructed a two-cylinder vacuum pump. Experimented with *electroluminescence*. He is called 'the elder' to distinguish his from his nephew of the same name (1688–1763) and the similar scientific interests.

1707–1732 CE **Hermann Boerhaave** (1668–1738, The Netherlands). Physician, chemist and botanist. Founded modern system of clinic instruction. A man with immense academic knowledge who dominated and influenced various branches of science in Europe.

Boerhaave was born in Voorhout, near Leiden. Went to the University of Leiden (1684) where he studied philosophy, botany, languages, chemistry and medicine (graduated 1693). Professor al Leiden (from 1709).

1709–1714 CE **Gabriel Daniel Fahrenheit** (1686–1736, Germany). Physicist. Proposed a temperature scale that bears his name. He also made the thermometer more accurate by using mercury instead of alcohol in the thermometer tube. He determined three fixed temperatures: $0\,^{\circ}$F for the freezing point of ice + salt + water; $32\,^{\circ}$F for the freezing point of pure water and $96\,^{\circ}$F for the normal temperature of the human body. These three temperatures correspond respectively to $-17.77\,^{\circ}$, $0\,^{\circ}$, and $35.55\,^{\circ}$ on the Celsius temperature scale. Later experiments proved the body normal temperature to be $98.6\,^{\circ}$F, or $37\,^{\circ}$C.

Fahrenheit was born in Danzig. For the most part he lived in England and Holland, devoting himself to the study of physics and making a living by the manufacture of meteorological instruments. He also invented an improved form of a hygrometer, of which he published an account in the Phil. Trans. of 1724. He died in Holland.

His temperature scale is still extensively used in the United States and Great Britain.

1710–1744 CE **Giambattista (Giovanni Battista) Vico** (1668–1744, Italy). Historical philosopher and jurist. Criticized radically the aims of science as outlined through the Cartesian system. Claimed that mathematics

does not enable us to promote a knowledge of nature as much as the rationalists thought and therefore tried to discover a 'new science' that was both perfectly knowable and about the real world. Advanced the basic principle that we can know only what we can do or make[184].

Vico's work contains the terms of many developments in the philosophy of the 19^{th} century. His ideas echoed through the *Sturm and Drang* movement in Germany (**Goethe**, **Herder**, 1770) and he extended great influence upon **Karl Marx** (1860), **Benedetto Croce** (1902), **Georges Sorrel** (1908) and **Oswald Spengler** (1918).

In his own time, however, and for fifty years after his death, Vico remained practically unknown. He was born in Naples, son of a small bookseller and lived there or in its environs until his death. Educated by priests, he became, at the age of thirty-one, a minor professor of rhetoric at the university of his native city. This somewhat subordinate position he held until his retirement in 1741. Most of his life he was poor. To keep himself and his family he had to eke out his modest salary by giving private tuition and composing inscriptions, Latin eulogies and laudatory biographies for the nobility. In the last years of his life he was rewarded by being appointed official historiographer to the Austrian Viceroy of Naples.

Vico is known mainly for his *Principi d'una Scienza Nuova* (1725). He founded no school and his philosophy seemed to die with him; his name was soon obscured, especially as the Kantian system dominated the world of thought. His reinstatement was completed by **Michelet** (1827)[185], who translated his books.

According to Vico, mathematics, being an arbitrary construction of the human mind is divorced from nature. It is not as Descartes supposed, a discovery of an objective structure, the eternal and most general characteristic of

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184 His approach was to establish a clear distinction between the world as it really is and the world which we create and cognize through the use of mathematical models and physical experiments. He realized that the understanding one has of something created by oneself is of a different nature to that understanding gleaned from simple observations. This distinction means we can never be free from subjectivism. Vico saw that mathematical models appear intelligible and coherent to our minds because our minds alone have made them. All our inquiry is necessarily anthropocentric because we employ man-made tools and human reasons in its pursuit. Vico believed the 'real' world of nature, which obeyed inaccessible rules, differed in kind from our do-it-yourself model of intelligible but man made laws.

185 **Jules Michelet** (1798–1874, France). Historian. Professor, College de France (1838–1851).

the real world, but rather an *invention*: invention of a symbolic system which men can logically guarantee only because men have made it themselves; but men cannot make the physical world. Nature herself was made by God and therefore only he can fully understand her. As far as man goes, he can learn something about nature by adopting an empirical approach through experiment and observation and not so much through a mathematical procedure[186]. Nature is not completely knowable. His concept of knowledge led Vico to argue further that man can fully know only what he himself invented, created or participated in, i.e., such provinces as participated in, i.e., such provinces as mathematics, mythology, language, symbolism and its own history.

Faced with the choice between a perfect understanding of a philosophical system divorced from reality and an imperfect understanding of the reality of life, Vico chose the latter and developed his concept that, since men could only fully apprehend the reality of their own creations, the task of philosophy should be the study of the universal principles underlying the history of nations. He pleaded that history should be written by philosophers.

Vico was the first thinker who asked, why have we a science of nature, but no science of history? Because our glance can easily be turned outwards and survey the exterior world; but it is far harder to turn the minds eye inwards and contemplate the world of the spirit.

Vico advanced the *cyclical theory of history* which maintains, counter to the Christian concept of time, that humanity advances not in a straight line but along an upward *spiral* staircase, with each spiral bringing man closer to freedom and nearer to God.

He declared that there were three great doors that led into the past: language, myths, and rites (institutional behavior). The task before those who wish to grasp what kind of lives have in the past been led in societies different from their own, is to understand their worlds through each of the above categories. Poetry, for example, is a direct form of self-expression of our remote ancestors, collective and communal. Myths are far-reaching images of past social conflicts out of which many diverse cultures grew.

Vico maintained that the Homeric poems were the sublime expression of a society dominated by the ambition, avarice and cruelty of its ruling class; for only a society of this kind could have produced this vision of life. Later ages may have perfected other aids to existence, but they cannot create the

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[186] Vico failed to see the role it plays in scientific research. At the same time one might allow that there was here a *warning* against unbridled mathematical speculation, which sometimes tries to pass for empirical work. The proper approach lies somewhere between these two extremes.

Iliad, which embodies the modes of thought and expression and emotion of one particular kind of way of life; these men literally saw what we do not see.

The scientific method is adequate for establishing bare facts. However, the task of historians is not merely to establish facts and give causal explanation for them. The knowledge that they need is not knowledge of facts or of logical truths, provided by observation or the science of deductive reasoning. They must possess imaginative power of a higher degree. Without this power of entering into minds and situations, the past will remain a dead collection of objects in a museum for us. Without some ability to get into the skin of others, the human condition, history cannot be understood. This use of informed imagination about, and insight into, systems of value, conceptions of life of entire societies, is not required in mathematics or physics.

The ideas of Vico provided a natural prologue to the more critical analysis which were to be developed by **David Hume** and **Immanuel Kant**.

1711 CE Austria and Germany devastated by plague. About 500,000 died.

1712 CE **Thomas Newcomen** (1663–1729, England). One of the inventors of the early steam engine. His 'fire engine' (1712) was used for pumping water from mines until **James Watt** invented one with a separate condenser. Newcomen's engine was the first to use a piston and a cylinder. Earlier models of the machine were constructed by him during 1705–1712 with the aid of the military engineer **Thomas Savery** (1650–1715). The whole situation is confused by a patent granted to Savery and in later years Newcomen paid royalties to Savery. It is also known that Newcomen corresponded with **Robert Hooke** about the previous investigations of **Denis Papin**.

Newcomen was born in Dartmouth, Devon, and set up a blacksmith's shop there, assisted by a plumber called **John Calley**. The Newcomen's engine consumed an enormous amount of coal, because fresh hot steam had to be raised for each piston stroke. The early engines were very expensive, because the cylinder was made by brass; later, iron cylinders were produced but they were thick-walled and consequently even less efficient in terms of coal consumed. However, they were mostly used in coal mines. It was with the Newcomen's engine that the age of steam began.

As late as the French revolution (1793 CE), it has been estimated, Europe drew energy from about 14 million horses and 24 million oxen. All these societies exploited energy sources that where *renewable*: nature could eventually replenished the forests they cut, the wind that filled their sails, the rivers that turned their puddle wheels. Even animals and people were replenished "energy slaves".

A revolutionary shift began after Newcomen's engine. Societies, by contrast, drew their energy from coal, gas and oil – from *irreplaceable* fossil funds.

It meant that for the first time a civilization was eating into natures capital rather than merely living off the interest it provided.

This dipping into the earth's energy reserves provided a hidden subsidy for industrial civilization – a vastly accelerated economic growth. And from that day to this, nations built towering technologies and economic structures on the assumption that cheap fossil fuels would be endlessly available.

1712–1715 CE **Brook Taylor** (1685–1731, England). Mathematician. Discovered the polynomial approximation of analytic functions near a given point[187]. Also, contributed to the general development of the calculus. In

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[187] Taylor expansion of a function $f(x)$ $[f^{(n)}(x)$ continuous] about a point $x = a$ is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + R(x),$$

with a *remainder*

$$R(x) = \frac{1}{(n-1)!}\int_a^x (x-s)^{n-1} f(s)ds = \frac{f^{(n)}(\xi)}{n!}(x-a)^n, \qquad a < \xi < x.$$

Newton's method of finding an approximate local solution to an equation of the form $f(x) = 0$ follows from Taylor's expansion in the following way: Suppose c denotes the solution to the above equation and $f''(x)$ exists on an interval containing both c and the initial value x_0. Expanding $f(x)$ in Taylor series about x_0 we have

$$0 = f(c) = f(x_0) + (c - x_0)f'(x_0) + \frac{1}{2}f''(\xi)(c - x_0)^2,$$

or

$$c - x_0 + \frac{f(x_0)}{f'(x_0)} = -\frac{1}{2}\frac{f''(\xi)}{f'(x_0)}(c - x_0)^2,$$

where $f'(x_0) \neq 0$. Denoting $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ (Newton's first approximation), Taylor's expansion implies that

$$c - x_1 = -\frac{1}{2}\frac{f''(\xi)}{f'(x_0)}(c - x_0)^2.$$

If a bound M is known for the second derivative of f on an interval about c and x_0 is within the interval, then $|c - x_1| \leq \frac{M}{|2f'(x_0)|}|c - x_0|^2$. This inequality implies that Newton's method has the tendency to approximately double the number of digits of accuracy with each successful approximation.
Succeeding approximations are generated by applying the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

1715 he introduced the idea of "*integration by parts*"[188]. During 1715–1717, Taylor invented the concept of *finite difference*, thus initiating the calculus of finite differences. Devised the basis principle of perspective in his *Linear Perspective* (1715).

Taylor was born in Edmonton, Middlesex. He was elected a Fellow of the Royal Society (1712) and was appointed in that year to the committee for adjudicating the claims of Newton and Leibniz to have invented the calculus.

'Taylor's Expansion' on a Sumerian Cuneiform Tablet?

The ancient Sumerians in Mesopotamia gave some interesting approximations to the square root of nonsquare numbers, like $\frac{17}{12}$ *for* $\sqrt{2}$ *and* $\frac{17}{24}$ *for* $\frac{1}{\sqrt{2}}$. *A remarkable approximation for* $\sqrt{2}$ *is*

$$1 + \frac{24}{60} + \frac{51}{(60)^2} + \frac{10}{(60)^3} = \mathbf{1.414215}5$$

Choosing $f(x) = x^n - k$ and taking x_0 to be an approximation to $\sqrt[n]{k}$, we find $x_1 = \frac{n-1}{n}x_0 + \frac{k}{nx_0^{n-1}}$.

A useful generalization of the Taylor expansion in which one function is expanded in terms of *another given function*, was discovered by **Heinrich Bürmann** (1799):

$$f(x) = f(a) + \sum_{k=1}^{n-1} \frac{\alpha_k(a)}{k!}\big[g(x) - g(a)\big]^k + R(x),$$

where $\alpha_k(x) = \frac{\alpha'_{k-1}(x)}{g'(x)}$, $k = 1, 2, \ldots$, $\alpha_0(x) = f(x)$, $R(x) = \frac{\alpha_n(\xi)}{n!}\big[g(x) - g(a)\big]^n$, $a < \xi < x$, $f^{(n)}(x)$ and $g^{(n)}(x)$ continuous; $g'(x) \neq 0$. The function $\alpha_k(x)$ is given explicitly by the expression

$$\alpha_k(x) = \Big[\frac{1}{g'(x)}\frac{d}{dx}\Big]^k f(x).$$

[188] The name "*integration by parts*" first appeared in 1797 in a book by **Sylvestre Francois Lacroix** (1765–1843, France).

(correct to 5 decimals), found on the Yale tablet YBC 7289 *dated about* 1600 *BCE.*

When searching for $\sqrt{x}$*, the Sumerian would start with some approximation* a *and then generate a sequence of increasingly better approximations. In modern notation they calculated*

$$a_1 = \frac{1}{2}\left(a + \frac{x}{a}\right), \quad a_2 = \frac{1}{2}\left(a_1 + \frac{x}{a_1}\right), \quad \cdots, \quad a_{n+1} = \frac{1}{2}\left(a_n + \frac{x}{a_n}\right).$$

This iterative algorithm by successive approximation was known to the Greeks, as is evident from the writings of ***Heron*** *(ca 50* CE*).*

Let us apply their technique to evaluate $\sqrt{2}$ *and take* $a = 1$ *as our initial guess. Then*

$$\begin{aligned} a_1 &= \frac{1}{2}\left(1 + \frac{2}{1}\right) = \frac{3}{2} \equiv 1 + \frac{30}{60}, \\ a_2 &= \frac{1}{2}\left(\frac{3}{2} + \frac{2}{3/2}\right) = \frac{17}{12} \equiv 1 + \frac{25}{60}, \\ a_3 &= \frac{1}{2}\left(\frac{17}{12} + \frac{2}{17/12}\right) = \frac{577}{408} \equiv 1 + \frac{24}{60} + \frac{51}{(60)^2} + \frac{10}{(60)^3}, \end{aligned}$$

which leads to the result inscribed on the Yale tablet!

Al-Khowarizmi *(ca 825* CE*) spoke of rational numbers as* audible *and surds as* inaudible*, and it is the latter that gave rise to the word* surd *(deaf, mute in Arabic). The European use of this word begins with* ***Gerhardo of Cremona*** *(ca 1150* CE*).*

The Arab mathematicians (e.g. ***Al-Karkhi****, 1020* CE*) and medieval writers used the approximation*

$$a + \frac{h}{2a+1} < \sqrt{a^2 + h} < a + \frac{h}{2a}, \qquad 0 < h \le a.$$

Now, on the r.h.s. we recognize the old Sumerian first approximation. Indeed, take $x = a^2 + h$ *and then*

$$a_1 = \frac{1}{2}\left(a + \frac{a^2 + h}{a}\right) = a + \frac{h}{2a}.$$

The European mathematicians before Newton generalized the surd approximation to the case of cube roots. Thus, ***Joannes Buteo*** *(1492–1572,* France*) derived* $\sqrt[3]{a^3 + h} \approx a + \frac{h}{3a(a+1)}$ *(1559) and* ***Stevin*** *followed suit (1634) with* $\sqrt[3]{a^3 + h} \approx a + \frac{h}{3a(a+1)+1}$*. With the development of the Newtonian calculus*

and the expansions of Taylor and Maclaurin that followed in its wake, it was recognized that the approximation

$$\sqrt[n]{a^n + h} \approx a + \frac{h}{na^{n-1}} \qquad (h < a)$$

corresponds to the first approximation used by Heron for $n = 2$ *(the relevant Sumerian analogs were discovered only in 1943).*

Moreover, Taylor's polynomial expansion shed some light on the entire Sumerian method of approximating square roots. for, if we apply their technique to $\sqrt{1+x}$, *starting with the approximate value* $a = 1$, *we find*

$$\begin{aligned} a_1 &= \frac{1}{2}\left[1 + \frac{1+x}{1}\right] = 1 + \frac{x}{2} \\ a_2 &= \frac{1}{2}\left[1 + \frac{x}{2} + \frac{1+x}{1+x/2}\right]. \end{aligned}$$

But for $|x| < 1$, $\frac{1}{1+x/2} = 1 - \frac{x}{2} + \frac{x^2}{4} - \cdots$ *renders*

$$a_2 = 1 + \frac{x}{2} - \frac{x^2}{8} + \cdots .$$

Continuing this process will lead to the Taylor polynomial of higher and higher degrees. The Sumerian approximation, however, had the clear advantage of being valid for all values of x *and not just* $|x| < 1$.

1715–1750 CE **Giulio Carlo Fagnano dei Toschi** (1682–1766, Italy). Mathematician. Discovered the formula $\pi = 2i \log_e \frac{1-i}{1+i}$, in which he anticipated **L. Euler** in the use of imaginary exponents and logarithms. His studies on the rectification of the ellipse, the hyperbola and the lemniscate are the starting-points of the *theory of elliptic functions*. Suggested new methods in solving equations of degree 3 and 4. He gave expert advice to Pope Benedict XIV regarding the safety of the cupola of St. Peter's at Rome. In return the Pope promised to publish his mathematical investigations. For some reason, the promise was not fulfilled and they were not published until 1750.

1716–1720 CE **Jacob Hermann** (1678–1733, Switzerland). Mathematician. Worked in mechanics and first to study the '*inverse problem*', where one has to determine the orbit from the knowledge of the law of force. One

of the pioneers of '*theoretical mechanics*'. Hermann was a pupil of **Jakob Bernoulli** and was a professor of mathematics in the University of Padua (1707–1713), at Frankfurt a.d.O. (1713–1724), at St. Petersburg (1724–1731) and at Basel.

The Evolution of Trigonometry (280 BCE–1720)

Trigonometry, in its essential form of showing how to deduce the values of the angles and sides of a triangle when other angles and sides are given, is an invention of the Greeks, although the basic trigonometry of the right-angled triangle was known to the Babylonians and the Egyptians.

Thus, the history of trigonometry stretches over a period of some 2000 *years from* ***Aristarchos of Samos*** *to* ***Euler****.*

Trigonometry found its origin in the computations demanded for the reduction of astronomical observations and in other problems connected with astronomical science: After the 3^{rd} *century BCE, mathematical research shifted increasingly away from the pure forms of constructive geometry toward areas related to applied disciplines, in particular to astronomy. Also, in the* 2^{nd} *century BCE, the Greeks first came into contact with the fully developed Mesopotamian astronomical systems and took from them many of their observations and parameters.*

While retaining their own commitment to geometric models rather than adopting the arithmetic schemes of the Mesopotamians, the Greeks nevertheless followed the Mesopotamians' lead in seeking a predictive astronomy based on a combination of mathematical theory and observational parameters. They thus made it their goal not merely to describe but to calculate *the angular positions of the planets on the basis of the numerical and geometric content of the theory. This major restructuring of Greek astronomy, in both its theoretical and practical aspects was primarily due to* ***Hipparchos*** *(ca 150 BCE), whose work was consolidated further by* ***Ptolemy****.*

To facilitate their astronomical researches, the Greeks developed techniques for numerical measurements of angles, a precursor of trigonometry, and produced tables for practical computations. Early efforts to measure numerical ratios in triangles were made by ***Archimedes*** *and* ***Aristarchos****. Their*

results were soon extended, and comprehensive treatises on the measurement of chords (effectively tables of values of the sine function) were produced by ***Hipparchos*** *and by* ***Menelaos of Alexandria*** *(ca 98* CE*). These works are now lost, but the essential theorems and tables are preserved in Ptolemy's Almagest. For computing with angles the Greeks adopted the Mesopotamian sexagesimal method in arithmetic, whence it survived in the standard units for angles and time employed to this day.*

It so happened that spherical trigonometry was developed before the simpler plane trigonometry.

In place of sine, cosine and tangent, the Greek astronomers Hipparchos and Ptolemy (150 CE) always used chords of arcs of circles. In fact it makes little difference whether one operates with cords or with sines, since what we now call the sine of an angle is the quotient by the radius of one half of the chord of twice the intercepted arc i.e.

$$\sin\alpha = \frac{1}{2R} chord(2\alpha)$$

[it is only since around 1800 CE that we divide by the radius and regard the l.h.s. as more fundamental than chord (2α)*].*

As early as the 5^{th} *century CE, the Hindu astronomers changed from the chords to the sines.*

The Hindus, who were much more adept calculators than the Greek, availed themselves of the Greek geometry which came from Alexandria, and made it the basis of trigonometrical calculations. The principal improvement which they introduced consists in the formation of tables of half-cords (or sines) instead of chords.

Although the Hindus could calculate sines and cosines of one degree [$\sin 1^\circ = \frac{10}{573}$, $\cos 1^\circ = \frac{6568}{6569}$] *with greater accuracy than Ptolemy, they did not apply their trigonometrical knowledge to the solution of triangles. For astronomical purposes they solved right-angled plane and spherical triangles by geometry.*

The Arabs were acquainted with Ptolemy's almagest and they probably learned from the Hindus the use of the sine. The Arab astronomer ***Albategnius*** *(850–929) employed the sine regularly, and was fully aware of the advantage of the sine over the chord. He was also acquainted with the formula*

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

for a spherical triangle ABC.

Abu al-Wafa of Baghdad *(940–998) was the first to introduce the tangent as an independent function. This improvement was forgotten, however, and the tangent was reinvented in the* 15^{th} *century.*

Ibn Yunus of Cairo *(d. 1008), Alhazen's contemporary and countryman (they both lived in Egypt), introduced the formula*

$$2\cos x \cos y = \cos(x+y) + \cos(x-y).$$

[This formula, and 3 similar ones, were used in 16^{th} *century Europe to convert products to sums before the invention of the logarithm!] He showed even more skill then Albategnius in the solution of problems in spherical trigonometry, and gave improved approximate formulae for the calculation of sines.*

The Western Muslim astronomer, ***Jabir Ibn Aflah of Seville*** *(frequently called* ***Jabir*** *or* ***Geber****), who flourished ca* 1130*, discovered the relation* $\cos B = \cos b \sin A$*, valid for a spherical triangle* ABC *with a right angle at* C*. This formula escaped the notice of* ***Ptolemy****. Strangely enough, Jabir made no progress in plane trigonometry.*

George Peurbach *(***Purbach***) (1423–1461), professor of mathematics at Vienna, wrote a work entitled* Tractatus super propositiones Ptolemaei de sinubus et chordis *(Nuremberg, 1541). This treatise consists of a development of a method of interpolation for the calculation of tables of sines, and was published posthumously by Regiomontanus at the end of his works.*

Johannes Müller *(1436–1476), known as Regiomontanus, was a pupil of Purbach and taught astronomy at Padua; he wrote an exposition of the* Almagest*, and a more important work,* De triangulis planis et sphericis cum tabulis sinuum*, which was published in* 1533*, a later edition appearing in* 1561*. He reinvented the tangent and calculated a table of tangents for each degree, but did not make any practical applications of this table, and did not use formulae involving the tangent. His work was the first complete European treatise on trigonometry, and contains a number of interesting problems; but his methods were in some respects behind those of the Arabs.*

Copernicus *(1473–1543) gave the first simple demonstration of the fundamental formulae of spherical trigonometry; the* Trigonometria Copernici *was published by* ***George Joachim*** *(1514–1576), known as Rheticus (1542). He wrote* Opus palatinum de triangulis*, which contains tables of sines, tangents and secants of angles at intervals of* $10''$ *from* $0°$ *to* $90°$*. His method of calculation depends upon the formulae which give* $\sin(n\alpha)$ *and* $\cos(n\alpha)$ *in terms of the sines and cosines of* $(n-1)\alpha$ *and* $(n-2)\alpha$*; thus these formulae may be regarded as due to him. Rheticus found the formulae for the sines of the half and third of an angle in terms of the sine of the whole angle.*

In 1595 there appeared an important work by ***Bartholomaeus Pitiscus*** *(1561–1613), entitled Trigonometriae seu De dimensione triangulorum, in which the word 'trigonometry' was first coined; this contained several important theorems on the trigonometrical functions of two angles, some of which had been given before by* ***Thomas Fincke***, ***Landsberg*** *(or* Lansberghe de *Meuleblecke) and* ***Adrian van Roomen***.

Francois Viète *or* ***Vieta*** *(1540–1603) employed the equation*

$$\left(2\cos\frac{1}{3}\phi\right)^3 - 3\left(2\cos\frac{1}{3}\phi\right) = 2\cos\phi$$

to solve the cubic $x^3 - 3a^2x = a^2b$ $\left(a > \frac{1}{2}b\right)$; *he obtained, however, only one root of the cubic. In* 1593 *Van Roomen proposed, as a problem for all mathematicians, to solve the equation*

$$45y - 3795y^3 + 95634y^5 - \cdots + 945y^{41} - 45y^{43} + y^{45} = C.$$

Vieta gave $y = 2\sin\frac{1}{45}\phi$, *where* $C = 2\sin\phi$, *as a solution, and also twenty-two of the other solutions, but he failed to obtain the negative roots. In his work Ad angulares sectiones* Vieta *gave formulae for the chords of multiples of a given angle in terms of the chord of the angle itself.*

A new stage in the development of the science was commenced after ***John Napier****'s invention of logarithms in* 1614. *Napier also simplified the solution of spherical triangles by his rules for the solution of right-angled triangles. The first tables of logarithmic sines and tangents were constructed by* ***Edmund Gunter*** *(1581–1626), professor of astronomy at Gresham College, London; he was also the first to employ the expressions cosine, cotangent and cosecant for the sine, tangent and secant of the complement of an angle.*

A treatise by ***Albert Girard*** *(1590–1634), published at the Hague in* 1629, *contains the theorems which give areas of spherical triangles and polygons, and applications of the properties of the supplementary triangles to the reduction of the number of different cases in the solution of spherical triangles. He used the notation sin, tan, sec for the sine, tangent and secant of an angle.*

In the second half of the 17^{th} *century the theory of infinite series was developed by* ***John Wallis***, ***Gregory***, ***Mercator***, *and afterwards by* ***Newton*** and ***Leibniz***. *In the* Analysis per aequationes numero terminorum infinitas, *which was written before* 1669, *Newton gave the series for the angle in powers of its sine; from this he obtained the series for the sine and cosine in powers of the angle; but these series were given in such a form that the law of the formation of the coefficients was hidden.*

James Gregory discovered in 1670 *the series for the angle in powers of the tangent and for the tangent and secant in powers of the angle. The*

first of these series was also discovered independently by Leibniz in 1673, *and published without proof in the Acta eruditorum* for 1682. *The series for the sine in powers of the angle he published in* 1693*; this he obtained by differentiation of a series with undetermined coefficients.*

In the 18^{th} *century the science began to take a more analytical form; evidence of this is given in the works of* ***Jakub Kresa*** *(1648–1715) in* 1720 *and* ***Mayer*** *in* 1727. ***Friedrich Wilhelm von Oppel****'s* Analysis triangulorum *(1746) was the first complete work on analytical trigonometry. None of these mathematicians used the notation sin, cos, tan, which is the more surprising in the case of Oppel, since* ***Leonhard Euler*** *had, in* 1744, *employed it in a memoir in the* Acta eruditorum. ***Johann Bernoulli*** *was the first to obtain real results by the use of the symbol* $\sqrt{-1}$*; he published in* 1712 *the general formula for* $\tan(n\phi)$ *in terms of* $\tan\phi$, *which he obtained by means of transformation of the angle into imaginary logarithms.*

Further advance was made by ***Euler****, who brought the science in all essential respects into the state in which it is at present. He introduced the present notation into general use; until his time the trigonometrical functions had been, except by Girard, indicated by special letters, and had been regarded as certain straight lines, the absolute lengths of which depended on the radius of the circle in which they were drawn.*

Euler's great improvement consisted in his regarding the sine, cosine, &c., as functions of the angle only, thereby giving to equations connecting these functions a purely analytical interpretation, instead of a geometrical one as before. The exponential values of the sine and cosine, ***de Moivre****'s theorem, and a great number of other analytical properties of the trigonometric functions, are due to Euler, most of whose writings are to be found in the* Memoirs of the St. Petersburg Academy.

1722 CE **Johann Sebastian Bach** (1685–1750, Germany). Composed *The Well-Tempered Clavier*, written in the tempered scale with 12 notes per octave having a *fixed frequency ratio* of $2^{1/12} = 1.0595$. Although there is controversy as to whether Bach ever played on an instrument tuned according to equal temperament, his *Well-Tempered Clavier* had considerable influence on the use of the system.

This scale may have originated in China long before the time of **Pythagoras** (ca 540 BCE). **Michael Stifel** (1544) introduced the scale to Europe. **Mersenne** (1636), however, was the first to give the correct frequency ratios

for equal temperament. One should note that Bach was able to employ the tempered scale only because **Napier** had invented the logarithms before him, shortly after 1600.

1724 CE **Jacopo Francesco Riccati** (1676–1754, Italy). An Italian savant who wrote on mathematics, physics and philosophy. He was chiefly responsible for introducing the ideas of Newton to Italy. At one point he was offered the presidency of the St. Petersburg Academy of Sciences, but preferred the leisure and comfort of his aristocratic life in Italy. Though widely known in scientific circles of his time, he did very little original work and his name now survives only through the differential equation $y' = p(x) + q(x)y + r(x)y^2$, bearing his name. Even this was an accident of history, for Riccati only discussed special cases of this equation without offering any solutions, and even most of these were treated by various members of the Bernoulli family. The term '*Riccati's equation*' was given by **d'Alembert** (1763).

1725–1741 CE **Vitus Bering** (1680–1741, Denmark). Navigator and explorer in Russian service. Was dispatched by Peter the Great to explore the waters off north-eastern Siberia. The Bering Island, sea and strait take their name from him. In a series of voyages he discovered the Bering strait, crossed to Kamchatka and explored the Aleutian Islands.

1728–1748 CE **James Bradley** (1693–1762, England). Astronomer. Discovered the phenomenon of *starlight aberration* (1728): because of the earth's orbital motion, if starlight is to pass through the length of the telescope[189] the telescope must be slightly tilted forward in the direction of the earth's motion relative to the *actual* line of sight to the star's position; i.e. the apparent direction of the star is displaced slightly from its geometrical direction, and the displacement is in the direction of the earth's orbital motion.

Since the speed of light is about 10,000 times that of the earth in its orbit, the angle through which a telescope must be tilted forward can be as large as ten-thousandth of a radian, or about $20.5''$. The effect is greatest when the earth is moving at right angles to the star's direction.

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[189] *Analogy*: A man stands still, holding a straight drain pipe in a vertical upright position. If it is raining, and if raindrops fall vertically (no wind), they will fall through the length of the pipe. But if the man walks forward with a fixed speed, v, he must tilt the pipe forward so that drops entering the top will fall out at the bottom without being swept up by the approaching inside of the wall of the pipe. If the raindrops fall with speed V in the earth's frame, the pipe must be tilted at an angle α to the vertical such that $\tan\alpha = \frac{v}{V}$.

Bradley found that if a telescope is pointed in a certain direction to observe a particular star on one night, then 6 month later the telescope must be pointed in a slightly different direction to observe the same star. Let a star be located on a line from the sun that is perpendicular to the earth's orbital plane. Because of the earth's motion with orbital velocity V, the tilt angle α of the star's rays is given by $\tan\alpha = V/c$, where c is the speed of light in vacuum. Bradley measured the difference in sighting angles 6 months apart, obtaining $2\alpha = 40.4''$ (arc seconds). Combined with $V = 30$ km/sec (known independently from celestial mechanics), this value for 2α radians gives

$$c = \frac{V}{\tan\alpha} \approx \frac{V}{\alpha} = \frac{3.0\times 10^4\,\frac{m}{sec}}{20.2\times\pi/(3600\times 180)} = 3.06\times 10^8\ m/sec.$$

The *relativistic* expression is

$$\tan\alpha = \beta(1-\beta^2)^{-1/2},$$

$\beta = V/c$. When $\beta \ll 1$, this expression reduces, as in the present case, to $\tan\alpha \approx \beta$.

A star that is on the ecliptic plane appears to shift back and forth in a straight line during the year. A star in a direction perpendicular to the earth's orbit appears to describe a small circle in the sky. [In 1862, **J. Foucault** reversed the logic of the above calculation by using his measurement of the speed of light to calculate the earth's speed, and hence verify its distance to the sun.][190]

In 1737 Bradley discovered the *nutation* of the earth's axis, which is a motion caused by the temporal irregularity of the forces that cause the precession: for instance when the sun or moon is in the plane of the earth's bulge, no tidal torque is applied by the respective body. The sun crosses the celestial equator twice a year and the moon crosses it twice a month, and at these moments of crossing there will be no torque effecting precession due to one or the other. In addition, there are variations in the orientation of the moon's orbit with respect to the ecliptic. All these factors affect precession by causing slight fluctuations in its rate and in the tilt angle of the earth's axis to the ecliptic[191]. These are the nutations (Latin for "nodding"), which

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[190] The distance is approximately equal to $\frac{VT}{2\pi}$ where V is the earth's orbital speed and T is its orbital period (1 year). Knowing this, the mass of the sun can then be estimated from Kepler's third law. This idealized calculation assumes a circular orbit.

[191] However, even a simple spinning top on a flat table, generally undergoes nutation, since precession competes with the tendency of the tilted top to fall.

result in an overall periodic motion of the earth's pole relative to the "fixed stars" much faster than its precession (period: 18.6 years).

Bradley was born in Sherborne, Gloucestershire. He graduated from Oxford University in 1717 and was trained in astronomical observations by his uncle, a skilled astronomer. He became a professor of astronomy at Oxford in 1721 and served until 1742. He then became the director of Greenwich Observatory, succeeding **Edmund Halley** as astronomer royal.

1728–1749 CE **Pierre Bouguer** (1698–1758, France). Mathematician and physicist. Invented the *photometer* (1748) and the *heliophotometer*. Considered as the father of *photometry*. Devised a method to relate gravity anomalies to deficiency of mass in the earth's crust. Participated in the French astro-geodetic expedition[192] to Peru (1735–1744), and made his measurements in the high Andes.

Bouguer was appointed in 1723 to succeed his father as professor of hydrography. In 1730 he was made professor of hydrography at Havre, and succeeded Maupertuis as member of the Académie des Sciences.

1728–1755 CE **Daniel Bernoulli** (1700–1782, Switzerland). A distinguished mathematician of the 18^{th} century. Son of Johann Bernoulli. Studied medicine like his father, and like him gave it up to become a professor of mathematics, at St. Petersburg. In 1733 he returned to Basel and was successively a professor of botany, anatomy and physics. He won 10 prizes from the French Academy, for one of which his father was among the competitors. In a fit of jealous rage Johann threw his son out of the house for winning the prize that he coveted for himself.

His famous book '*Hydrodynamica*' (1738) includes the earliest treatment of the *kinetic theory of gases* and the famous *Bernoulli principle*[193] [obtained

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[192] This expedition, and another to Lapland (1736–1737), established that the earth is flattened at the poles. The flattening, predicted theoretically by **Newton** (1687), was confirmed by the expeditions' measurements of 110,600 m for the length of a degree of latitude in Peru and 111,900 m for the corresponding length in Lapland.

[193] *The Bernoulli principle*: If an incompressible frictionless homogeneous fluid with density ρ moves without friction, the sum of pressure p and kinetic energy per unit volume remains constant (along a streamline): $p + \frac{1}{2}\rho v^2 = \text{constant}$. Thus an increase in the flow velocity v is accompanied by a decrease in the pressure exerted by the fluid on the walls of the container.

Many aerodynamic effects are consequences of Bernoulli's principle. For example, subsonic aircraft obtain most of their lift from the pressure difference between underside and top of the wing, whose profile makes the air flow faster

before the discovery of the Euler equation, by considerations similar to the modern principle of conservation of energy]. He also used the Fourier series expansion long before Fourier (1828).

He is considered by many to have been the first genuine mathematical physicist.

Apart from Jakob, Johann and Daniel, the Bernoulli family produced another 6 mathematicians of distinction:

- **Nicolas** (**Nicolaus**, 1687–1759). Professor of mathematics at Padua. Contributed to probability theory and the theory of infinite series. Nephew of Jakob and Johann.
- **Nicolas II** (1695–1726). Professor of mathematics at St. Petersburg. Empress Catherine ordered him a state funeral upon his premature death.
- **Johann II** (1710–1790). Professor of mathematics at Basel. Contributed to the theory of heat diffusion and light propagation.
- **Johann III** (1744–1807). Astronomer at the Academy of Berlin. Son of Johann II.
- **Jakob II** (1759–1789). Professor of mathematics at St. Petersburg. Tragically drowned while bathing in the Neva, a few months after his marriage to the granddaughter of Leonhard Euler. Son of Johann II.
- **Daniel II** (1751–1834). Professor of mathematics at Basel. Son of Johann II.

The Bernoulli family, with all its mathematical talent, also had more than its share of arrogance and jealousy, which turned brother against brother and father against son. In three successive generations, fathers tried to steer their sons into nonmathematical careers, only to see them gravitate back to mathematics. The fiercest conflict occurred between James, John, and Daniel.

During his teens Daniel was tutored by his older brother Nicholas II; his father wanted him to go into business, but when that career failed Daniel was permitted to study medicine. During his years at St. Petersburg Academy (1725–1733) he conceived his ideas on modes of vibrations and produced the

over the top than along the underside. Similarly, the "curve ball" familiar to baseball aficionados is the Bernoulli effect on a moving sphere whose spin will cause a difference in air flow velocity and thus a pressure difference on opposite sides of the sphere.

first draft of his *Hydrodynamica*. Although he missed the basic partial differential equations of hydrodynamics, his book used systematically the principle of conservation of energy. Unfortunately, publication of *Hydrodynamica* was delayed until 1738. His father, John then published a book on hydrodynamics in 1743, dating his book to 1732! - one of the most blatant priority theft in the history of mathematics. Daniel complained to Leonhard Euler (1743) with the result that John's reputation was so tarnished by the episode that he did not even receive credit for parts of his work that *were* original.

1728 CE **Pierre Fauchard** (1678–1761, France). Dentist. Founder of modern dentistry. He practiced in Paris from 1715 and was influential in raising dentistry from a trade into a profession. He advocated the sharing of dental knowledge and wrote the two volume '*La chirurgien Dentiste ou traité des dents*' (1728). It includes detailed discussions on the treatment of caries, the making and using of removable dentures, and a variety of dental instruments. After removal of the carious material, with pain reduced through application of oil of cinnamon, the cavity is filled with small pieces of thin foil of tin, or gold.

After the publication of Fauchard's work the practice of dentistry became more specialized and distinctly separated from medical practice, the best exponents of the art being trained as apprentices by practitioners of ability, who had acquired their training in the same way from their predecessors.

Fauchard suggested porcelain as an improvement upon bone and ivory for the manufacture of artificial teeth.

1729–1753 CE **Jean Astruc** (1684–1766, France). Physician, medical researcher and historian, and a pioneer biblical critique and exegetic. One of the most prolific medical authors of the 18^{th} century. Physician to August II, King of Poland; consultant to Louis XV of France. Descendant from a Jewish Marrano family. Wrote extensively on venereal and skin diseases. Considered as the progenitor of the modern scholarly and textual investigation of sources of Pentateuch (1753).

1730–1746 CE Identification, study and industrial working of the metallic element *zinc*[194] by **Isaac Lawson** (1730, England), **John Champion** (1743,

[194] **Plato** (ca. 400 BCE) refers to brass, an alloy of zinc and copper. An alloy containing 23 percent of zinc and 10 percent of tin was found at *Gezer* (ancient Israel), already in 1500 BCE. The name zinc derived from *tusku* (mentioned in Assyrian tablets of 650 BCE), probably zinc carbonate, $ZnCO_3$, used also by the alchemists. Deposits of *calamine* (native zinc carbonate) occur in the old Greek silver mines of Laurion. The extraction of zinc from its ores was in

England) and **Andreas Sigismund Marggraf** (1746, Germany). Marggraf made the earliest complete study of zinc.

1730 CE **James Stirling** (1692–1770, Scotland). Mathematician. Presented the approximation of the factorial function for large argument $n! \approx \sqrt{2\pi n}(n/e)^n$, or more generally[195]: $\Gamma(x) \sim \sqrt{\frac{2\pi}{x}}(x/e)^x$. He also contributed to the calculus of finite differences [*Stirling's interpolation formula*, *Stirling numbers*[196], and *Stirling factorial series*[197]].

Stirling, a descendant of a noble Scottish family, was educated at Glasgow and Oxford. He was expelled from Oxford for supporting the Jacobite cause, and lived in Venice during 1716–1724. His return to Britain is supposed to have been hastened because he had learned some secrets of the glass industry, and may have feared for his life. Newton, whose friendship he enjoyed, helped him secure fellowship in the Royal Society. Stirling's book *Methodus differentialis*, which appeared in 1730, included most of his mathematical discoveries.

In 1735 he was asked to reorganize the work of the Scottish Mining Company in the lead mines at Leadhills, Lenarkshire. Stirling was a successful administrator and spent most of his time after 1735 in that remote village. In 1748, he was elected to the Berlin Academy of Sciences, even though his mathematical activities had ceased. Stirling's own political principles prevented him from succeeding to the Edinburgh chair left vacant at **Maclaurin**'s death.

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operation on an extensive scale in Bristol (1743), the roasted ore ZnO being distilled with carbon at high temperatures in a crucible.

195 The asymptotic expansion

$$\Gamma(z) = e^{-z} z^{z-1/2}\sqrt{2\pi}\left[1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} - \frac{571}{2488320z^4} + O(z^{-5})\right]$$

is known today as *Stirling's expansion*. This is a misnomer, however, since it was discovered earlier by **de Moivre**.

196 *Stirling numbers* of the first kind $S_k^{(n)}$ are the coefficients of x^{n-k} in the factorial polynomial of degree n, i.e.

$$x(x-1)\cdots(x-n+1) = S_0^{(n)}x^n + S_1^{(n)}x^{n-1} + \cdots + S_{n-1}^{(n)}x.$$

197 The *Stirling series* $\sum_{s=0}^{\infty} a_s\left\{\frac{s!}{x(x+1)\cdots(x+s)}\right\}$ are of importance in the theory of linear difference equations, where they play a part analogous to that of power series in the theory of differential equations.

1730–1735 CE Advances in navigational instruments: **John Hadley** (1682–1744, England). A country gentleman (b. Hertfordshire) of independent means and instrument-maker of East Barnet, London, and independently **Thomas Godfrey** (1704–1749), a poor glazier in Philadelphia, invented in 1730 the *reflecting sextant*[198] to measure the angle between a star and the horizon. The frame of the sextant supports the graduated arc of a sixth part of a circle, a moveable arm which represents the radius of the circle, two mirrors and a small telescope. One of the mirrors is fixed (known as the *horizon glass*). The second mirror is screwed to the moveable arm, and is called the *index glass*, both mirrors being perpendicular to the plane of the sextant.

Light from a star is reflected from the index glass to the silvered half of the horizon glass and thence through the telescope to the observer's eye. If the moveable arm has been moved so as to make the *image* of the star coincide with that of the horizon, it is seen that the altitude of the star is equal twice the angle which the moveable arm reads on the graduated arc.

In 1731 John Hadley invented the *bubble-sextant*, or *artificial horizon* sextant (it made long-range air navigation possible 200 years later).

In 1735 **John Harrison** made the first accurate chronometer in England.

[198] It was introduced by **Tycho Brahe** to measure angular distances between any two points on the celestial sphere. Originally it was equipped with two sights: one on a fixed radius, the other on a moveable radius, which the observer pointed to the two objects of which the angular distance was to be measured.

Geodesy[199] *and the Gyrostatic Equilibrium of Liquid-like Bodies*

Man has been concerned about the earth on which he lives for many centuries. During very early times this concern was naturally limited to his foraging range or to the immediate vicinity of his dwelling place; later it expanded to encompass a village, a region of land or sea, a country; and finally, with the development of advanced means of transportation man became interested in his whole world. Much of this early "world interest" was evidenced by speculation concerning the size, shape, and composition of the earth.

The early Greeks, in their speculation and theorizing, ranged from the flat disc advocated by ***Homer*** *to Pythagoras' spherical figure — an idea supported later by* ***Aristotle*** *(ca 350 BCE).* ***Pythagoras*** *(ca 530 BCE) was a mathematician, and to him the most perfect figure was a sphere. He reasoned that the gods would create a perfect figure and therefore the earth was created to be spherical in shape.* ***Anaximenes*** *(ca 540 BCE) held that the earth was rectangular in shape.*

The early astronomers, however, had no doubts: reasoning from the uniform level appearance of the horizon, the variations in latitude of the circumpolar stars as one travels toward the north or south, the disappearance of a ship sailing out to sea, and perhaps other phenomena — they came to regard the earth as a sphere.

Since the spherical shape was the most widely supported during the Greek era, efforts to determine its size followed. ***Plato*** *(ca 380 BCE) determined the circumference of the earth to be* 64,000 *km, while* ***Archimedes*** *(ca 250 BCE)*

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199 *Geodesy* determines by observation and measurement the exact position of points and the figures and areas of large portions of the earth's surface, the shape and size of the earth, and the variation of terrestrial gravity.

Geophysics deals with the physical phenomena and properties of the whole earth (or of its more extensive regions). One of the branches of geophysics is *gravimetry*, which is the science of the earth's gravity field [from the Latin *gravis* (= heavy) and the Greek *meterin* (= to measure)]. The gravimetrician measures gravity, studies the figure and dimensions of the earth's body and relates these to the internal structure and composition of the earth's interior.

The determination of the *figure of the earth* is a problem of the highest importance in astronomy, inasmuch as the diameter of the earth is the unit to which all celestial distances must be referred.

estimated it to be 50, 000 *km. Plato's figure was a guess and Archimedes' a more conservative approximation. Meanwhile, in Egypt,* ***Eratosthenes*** *(ca 250 BCE) set out to make more explicit measurements.*

He had observed that on the day of the summer solstice, the midday sun shone to the bottom of a well in the town of Syene (Aswan). At the same time, he observed that the sun was not directly overhead at Alexandria; instead, a vertical pole cast a shadow with subtended angle equal to $(1/50)^{th}$ *of a circle* ($7^\circ 12'$)*. To these observations, Eratosthenes added certain "known" facts: (1) that on the day of the summer solstice, the midday sun was directly over the line of the summer Tropic Zone (Tropic of Cancer) — Syene was therefore concluded to be on this line; (2) the linear distance between Alexandria and Syene was (in today's units)* 804.5 *km; (3) Alexandria and Syene lay on a direct north-south line.*

From these observations and "known" facts, Eratosthenes concluded that, since the angular deviation of the sun from the vertical at Alexandria was also the angle of the subtended arc, the linear distance between Alexandria and Syene on the earth's surface was $\frac{1}{50}$ *of the circumference of the earth; the latter thus came out* $50 \times 804.5 = 40,225$ *km. [A currently accepted value for the earth's circumference at the Equator is* 40, 065 *km, based upon the equatorial radius of the World Geodetic System.] The actual unit of measure used by Eratosthenes was called the "stadia". No one knows for certain what the stadium that he used is in today's units. The measurements given above in km were derived assuming one stadia to be* 160 *meters.*

It is remarkable that such accuracy was achieved in view of the fact that most of the "known" facts, and his observations too, were incorrect: (1) although it is true that the sun at noon is directly overhead at the Tropic of Cancer on the day of the summer solstice, it was erroneously concluded that Syene lay on that line. Actually, Syene is 60 *km to the north; (2) the true distance between Alexandria and Syene is* 729 *km and not* 804.5 *km; (3) Syene lies* $3^\circ 30'$ *east of the meridian of Alexandria; (4) the difference of latitude between Alexandria and Syene is* $7^\circ 5'$ *rather than* $7^\circ 12'$ *as Eratosthenes had concluded.*

Nevertheless, Eratosthenes appears to have seen the first who entertained an accurate idea of the principles on which determination of the figure of the earth really depends, and attempted to reduce them to practice. His method, the comparison of a line measured on earth with the corresponding arc of the heavens, is still valid.

Another ancient measurement of the size of the earth was made by the Greek ***Poseidonios*** *(ca 100 BCE). He noted that a certain star was hidden from view in most parts of Greece but that it just grazed the horizon at Rhodes. Poseidonios measured the elevation of the same star at Alexandria*

and determined that the angle was $\frac{1}{48}^{th}$ *of circle. Assuming the distance from Alexandria to Rhodes to be* 800 *km, he computed the circumference of the earth as* 38,600 *km. While both his measurements were approximations, when combined, one error compensated for another and he achieved a fairly accurate result.*

Revising the figures of Poseidonios, another Greek philosopher determined 29,000 *km as the earth's circumference. This last figure was promulgated by* ***Ptolemy*** *(ca 150* CE*) through his world maps. The maps of Ptolemy strongly influenced the cartographers of the Middle Ages. It is probable that Columbus, using such maps, was led to believe that Asia was* 5000–6500 *km west of Europe. It was not until the* 16^{th} *century that this concept of the earth's size was revised.*

No improvement on the Greek methods was forthcoming until 1528*, when* ***Jean Francois Fernel*** *repeated the Eratosthenes procedure with greater accuracy.*

G. Mercator *(1568* CE*) made successive reductions in the size of the Mediterranean Sea and all of Europe which had the effect of modifying the size of the earth. The telescope, logarithmic tables, and the method of triangulation were contributed to the science of geodesy during the* 17^{th} *century. Indeed, during* 1617–1669*, measurements of this type (employing a spherical earth model) were redone by* ***Snell*** *(1617),* ***Richard Norwood*** *(1637), and* ***Jean Picard*** *(1669) who was the first to apply the telescope to angular measurements. He performed an arc measurement that is modern in some respects; he measured a base line with the aid of wooden rods, used a telescope in his angle measurements, and computed with logarithms. Earth models departing from spherical symmetry date from* 1672*, when* ***Jean Richer*** *discovered that the magnitude of the force of gravity depends on latitude.*

The first, rather inaccurate measurements of the acceleration of gravity were made by ***Galilei*** *(1564–1642). In ca* 1590 *he discovered that the distance traversed by a falling body in the first second is equal to half the value of the acceleration of gravity at the point of observation. The possibility of determining the shape of the earth from measurements of gravity on its surface, occurred to both* ***Newton*** *(1642–1727) and* ***Huygens*** *(1629–1695) who became interested in the observation of* ***Richer*** *(1630–1696) that the period of a pendulum depends on the latitude of its location via the dependence of its period on* g*. Newton, assuming the earth to be a gravitating rotating ellipsoid of revolution of uniform fluid in hydrostatic equilibrium, demonstrated that a slowly rotating liquid body must necessarily be flattened at the poles. He found its ellipticity to be* $\epsilon = \frac{a-c}{a} = \frac{g_p - g_e}{g_e} \approx \frac{1}{231}$*, where* $\{g_p, g_e\}$ *are the respective values of gravity at the pole and the equator.*

Huygens, on the other hand, assumed a uniform earth with its total mass concentrated at its center. Using the condition of the Geoid[200] *as an equipotential surface, he obtained* $\epsilon = \frac{1}{578}$.

The disagreements between the theoretical considerations of Newton and Huygens were explained by ***Clairaut*** *(1713–1765), who also showed how the flattening of the earth could be computed from gravimetric observations.*

The prediction of Newton (1687) that the earth was oblate at the poles was contrary to the best astronomical evidence available at the time, and for years after Newton's death, the Parisian school of ***Cassini*** *(1625–1712) vigorously supported the view that the earth was actually prolate*[201]*. To settle the controversy once and for all, the French Academy of Sciences sent a geodetic expedition to Peru in* 1735 *to measure the length of a meridian degree close to the equator, and another to Lapland to make a similar measurement near the Arctic Circle.*

When the leader of the Arctic party, ***Maupertuis****, returned to Paris, after suffering hunger and shipwreck, with proof that the earth is oblate (as Newton had forecast), Voltaire congratulated him on having "flattened the poles and Cassini".*

We now know that the actual ellipticity of the earth is $\sim \frac{1}{294}$, *substantially smaller than Newton's predicted value of* $\sim \frac{1}{230}$. *This discrepancy is interpreted in terms of the inhomogeneity of the earth.*

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200 The *Geoid* is a theoretical smooth surface whose normal at each point is in the direction of gravity at that point, i.e., a surface of constant gravity potential. The shape of the Geoid is that which the surface of water would take, were it to cover the whole surface of the earth. [Sea-level, undisturbed by winds or tides, is an equipotential surface of the earth's gravitation.] The *niveau spheroid* is a mathematical approximation of the Geoid, where all local irregularities caused by lateral density variations were removed. This spheroid is very close to an *ellipsoid of revolution*.

201 **G.D. Cassini** continued Picard's arc northward to Dunkirk and southward to the Spanish boundary, dividing the measured arc into two parts, one northward from Paris, another southward. When he computed the length of a degree from both chains, he found that the length of one degree in the northern part of the chain was shorter than that in the southern part. This unexpected result could have been caused only by an egg-shaped earth or by observational errors.
The results started an intense controversy between French and English scientists. The English claimed that the earth must be flattened, as Newton and Huygens had shown theoretically, while the Frenchmen defended their own measurement and were inclined to keep the earth egg-shaped.

Newton's model of the earth as a rotating homogeneous incompressible fluid body was valid only for small rotation speeds. However, in 1742 ***Maclaurin*** *(1698–1746) generalized Newton's result to the case where the ellipticity caused by the rotation cannot be considered small. He found a class of exact solutions for the equilibrium of a rotating body. In these solutions, known as* Maclaurin spheroids *(the fluid surface is an oblate ellipsoid of revolution) the eccentricity is a function of the angular velocity. Moreover,* ***Simpson*** *and* ***d'Alembert*** *(1717–1783) have shown (1743) that Maclaurin's solution implies: (1) for slow rotation there are two possibilities, one nearly spherical and the other very much flattened; (2) above a critical rotation rate* no *spheroid is a figure of equilibrium.*

In 1834 ***Jacobi*** *proved that when the rate of rotation is not too great there is an ellipsoid of* 3 unequal *axes which is a figure of equilibrium. For a certain rate of rotation it coincides with the more nearly spherical shape of the Maclaurin spheroids. These figures are known as* Jacobi ellipsoids*. In* 1860, ***Riemann*** *went one step further than Jacobi by showing that even the Jacobi ellipsoids are only special members of a much larger family of ellipsoidal equilibrium configurations, the* Riemann ellipsoids.

In 1885, ***Poincaré*** *showed that the Jacobi ellipsoids are actually the preferred configurations of rapidly rotating fluid bodies because they have lower energy for fixed angular momentum and mass.*

A more detailed account of the development of these ideas is as follows: ***Maclaurin*** *showed that for every volume V and for each angular velocity $\omega \le \omega_L = 1.188\sqrt{\rho G}$ (G = Newton's gravitational constant, ρ = density of liquid), there exist two different rotating oblate spheroids that are in gyrostatic equilibrium. As ω approaches ω_L, the shapes of both spheroids approach that of the same rotating spheroid, rotating with the angular velocity ω_L. As ω approaches a value of zero (that is, as the rotation slows down to zero), one branch of the Maclaurin spheroids will increasingly resemble a ball of volume V, the well-known equilibrium configuration at absolute rest ($\omega = 0$), whereas the other branch will grow into a disc of "infinite diameter".*

For nearly a century it was believed that Maclaurin's spheroids were the only shapes possible for uniformly rotating bodies of homogeneous fluids in gyrostatic equilibrium. ***Lagrange*** *claimed that there could not be any other equilibrium configurations; yet this was not true. In 1834* ***Jacobi*** *discovered that, for every volume V and every value ω of the angular velocity which is neither zero nor too large, there exists an equilibrium configuration in the shape of an asymmetric ellipsoid ($a > b > c$) that rotates about the axis of the smallest principal radius c. Jacobi showed that ω should stay below $\omega_J = 1.084\sqrt{\rho G} < \omega_L$.*

If ω approaches the value of ω_J, the Jacobi ellipsoid will eventually resemble one of the Maclaurin spheroids (which rotate with the angular velocity ω_J) and, if ω approaches a value of zero, the Jacobi ellipsoid will come to resemble a needle of infinite length. [What would life be like on a planet that was very thin and very long, and that rotated very, very slowly?].

***Poincaré** found that a new branch of pear-shaped equilibrium configurations bifurcates from the family of Jacobi ellipsoids, much as the Jacobi ellipsoids branch off one class of the Maclaurin spheroids. Poincaré conjectures "that the bifurcation of the pear-shaped body leads onward stably and continuously to a planet attended by a satellite". He furthermore proclaimed that along the Jacobi sequence there must be other points of bifurcation that give rise to other stable branches that would eventually develop into planets with two, three, or more satellites. In this way Poincaré envisioned a grand scheme that could explain the birth of our solar system by an evolutionary process rather than by sudden catastrophes.*

If one follows the cosmogonic hypotheses of Kant and Laplace, our solar system was at first a huge and slowly rotating gas ball of very low density. Self-gravitation would then lead to a contraction of the gas, thereby increasing density and angular velocity, with the matter then changing from a gaseous into a liquid state. As density and speed increased, the originally sphere-shaped matter would become a more and more oblate Maclaurin spheroid, until the bifurcation point at which the Jacobi ellipsoids became stable configurations. The liquid body would change into a Jacobi ellipsoid and then, with even stronger contraction, into a pear-shaped body, which eventually would fission into a main body and a satellite.

*Poincaré never made the detailed calculations necessary to substantiate such a scenario. Such calculations were instead carried out by **George Darwin** (1898) who claimed that he had proved the stability of the pear forms. Unfortunately, **Lyapunov** (1903) was able to refute Darwin's calculations, and other scientists reached the same conclusion. Thus Poincaré's wonderful model collapsed. Nevertheless, the theory of equilibrium configurations developed by Poincaré and Lyapunov was the beginning of bifurcation theory in nonlinear dynamics. This important theory is a principal tool in such diverse areas as fluid mechanics, mathematical biology, and elasticity theory.*

What happens for large values of ω? it was known that a figure cannot be in gyrostatic equilibrium if its angular velocity ω is too large. There is, in fact, no possible gyrostatic equilibrium if ω^2 is greater than $2\pi G\rho$. The only way out of this dilemma is to consider rotating liquid bodies in which the liquid is in internal motion, but whose shape does not alter. This is a much weaker type of equilibrium, but very likely a more realistic one. It was studied by

Dirichlet and ***Riemann*** *(1858–1860) and completed by* ***Chandrasekhar*** *in the* 1960*s.*

The importance of the Jacobi ellipsoids for galactic dynamics is that their very existence suggests that a rapidly rotating galaxy may not remain axisymmetric.

The equilibrium shape of rotating liquid masses is a special case of a more general problem: the rotation of a homogeneous fluid [either simply connected (like a ball) or multiply connected (like a handle, ring etc.)] under forces generated in the fluid itself such as self-gravitation, surface tension, electrostatic Coulomb attraction and centrifugal forces. In gyrostatic equilibrium, the above four forces balance each other: the contractive forces of surface tension and self gravitation counterbalance the dispersive electrostatic (for charged fluid) and centrifugal forces. The problem is then to find the possible shapes of liquid bodies in stable gyrostatic equilibrium.

According to ***Johann Bernoulli****'s principle of virtual work (1717), the equilibria are stationary states of the potential energy, and the stable equilibrium correspond to the minima of potential energy. The total potential energy of a liquid body is the sum of four terms: total energy = surface energy (proportional to the surface area) + electrostatic energy + gravitational energy + rotational energy (potential energy of the centrifugal forces).*

We have seen that the earliest example was that of rotating bodies of liquids, which served as models of the planets and, later on, of the stars and galaxies. Here the forces of self-attraction caused by gravitation are so large that the influence of surface tension can be neglected. If charged celestial bodies are excluded, the potential energy reduces to the sum of gravitational and rotational energies.

Another special case of interest in physics is where surface tension is the dominant force, whereas self-attraction is virtually nill. This situation was realized by ***Plateau*** *(1873) who derived an apparatus for rotating small uncharged drops of oil, immersed in another liquid of the same density; with increasing angular velocity, the drop decomposes first into a drop + ring, then into a drop with droplet satellites of different sizes. This circumstance brings to mind a solar system with a large central body circled by small satellites. The ring resembles the rings of Saturn or Jupiter*[202].

[202] Recently, Plateau's experiments were repeated and improved on by scientists at the Jet Propulsion Laboratory in Pasadena, CA. Besides the axisymmetric and ring-shaped figures of Plateau, they discovered two-, three- and four-lobed equilibrium shapes. With increasing angular speed, all figures were seen to decay into a one-lobed shape. No satisfactory method of explaining all this is currently available, because, in fact, the friction between the host liquid and the

The more complicated case of a charged drop found application in the nuclear drop model of ***G. Gamow*** *(1929). Here, theory predicts various hourglass figures corresponding to different values of the physical parameters. If energy is introduced into the nucleus — say, by bombarding it with a neutron — the nucleus can be excited into a state of free oscillations; if these vibrations take the nucleus above a certain energy barrier, it will then be split into two parts (fission).*

Can this physical model be applied to unicellular organisms, which are drops of protoplasm, a very viscous fluid, suspended in water? It is believed that tension forces at the surface of a cell can only partly explain its shape, and that internal structures are to a large extent responsible for cell shape.

1731–1743 CE **Alexis Claude Clairaut** (1713–1765, France). Mathematician. Born in Paris and spent most of his life in his native city.

Under his father's tutelage he made such rapid progress that at age 13 he read before the French Academy an account of four curves which he had then discovered. He was first to give analytic expressions to non-planar space curves and study their differential geometry (1729). This procured him admission into the Paris Academy of Sciences in 1731 although he was then below the legal age. He became the youngest person ever elected to the Academy.

During 1736–1738 he participated in the Lapland expedition of Maupertuis. Their geodetic measurements of length of meridian arcs at different latitudes, afforded data which showed conclusively the flattening of the earth at the poles.

Later in 1743, he deduced a theoretical relation between the variation of gravity from equator to poles and the ellipticity of a spheroidal earth model[203]

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oil drop cannot be neglected. This friction causes internal flows, which become rather significant as soon as lobes form.

[203] $g(\varphi) = g(0)[1 + \beta \sin^2 \varphi]$, where $g(\varphi)$, $g(0)$ are the respective values of gravity at latitude φ and the equator. The ellipticity is then given by $\epsilon = \frac{5}{2}\frac{\omega^2 a}{g(0)} - \beta$, where β is known from observations. For further reading, see:

- Webster, A.G., *Dynamics – Lectures on Mathematical Physics*, Hafner Publishing Co., 1949, 588 pp.
- Jeffreys, H., *The Earth*, Cambridge University Press, 1976, 574 pp.

of mean radius a [his development is valid only to terms of the first order in the flattening]. His results showed for the first time how the oblateness of the earth could be computed from gravimetric observations.

In 1750 Clairaut gained the prize of the St. Petersburg Academy for his essay *Théorie de la lune*, and in 1759 he calculated the perihelion of Halley's comet. The first-order ODE, $y = xy' + f(y')$, bears his name. He studied the 3-body problem and wrote several important memoirs on the calculus.

1733 CE The *Industrial Revolution* is launched in England with the invention of the *Flying Shuttle* by **John Kay**.

1733–1740 CE **Charles Francois de Cisternai Du Fay** (1698–1739, France). Physicist. Demonstrated that there are two different kinds of electric charges, one of which was to be found in rubbed amber and one in rubbed glass. In 1750 they were named '*negative*' and '*positive*' electrical fluids, respectively, by **Benjamin Franklin** (1706–1790, U.S.A.). He noticed that if an object carrying one kind of electricity touched an object carrying an equal quantity of the other kind, the two kinds neutralized each other, leaving both objects electrically 'uncharged'. He then coined the names, 'positive charge' and 'negative charge'.

1733 CE **Girolimo Saccheri** (1667–1733, Italy). Mathematician. Composed '*Euclides ab omni naevo vindicatus*' (Euclid vindicated from all fault), where he inadvertently laid the foundation of non-Euclidean geometry.

1733–1773 CE **(Francois Marie Arouet de) Voltaire** (1694–1778, France). Writer. A universal man of the 18^{th} century. *In his writings* he was at once dramatist, poet, philosopher, scientist, novelist, moralist, satirist, polemicist, letter-writer, and historian. *In his life* he was imprisoned in the Bastille, spent years abroad in England and Prussia, was courtier at Versailles and a wealthy landowner at Ferney.

His long life and his voluminous writings, which show a strong sense of engagement in the world around him, made him a man who bestrided the Age of Enlightenment, and in many ways is the epitome of it. Helped spread Newtonian science through the European intellectual community (*Lettres Philosophiques sur les Anglais*, 1773; *Elements de la Philosophie de Newton*, 1738), and presented new ideas in the field of optics, most particularly regarding the psychology of perception. During 1751–1772 Voltaire participated in the composition of the *French Encyclopedia of the Sciences, Arts and Trades*. In 1772, he encouraged the serious study of *probability theory* in his *Essay on Probabilities Applied to the Law*.

Voltaire (pseud, 1718) was born at Paris, son of a notary who belonged to a class of yeoman-tradesman. He was educated by the Jesuits, and soon began to appear in Paris society, particularly in free-thinking and neo-Epicurean circles. His satiric writings incurred him nearly a year in the Bastille (1717–1718). A quarrel with an influential aristocrat led to exile in England (1726), where he encountered a society living in a state of relative justice and freedom.

The publication of his *Lettres Philosophiques* (1773) occasioned another scandal and forced him to retreat at Cirey; he remained there for a decade, apart from brief visits to Paris, Prussia and the Low Countries [an inheritance from his father in 1721 made him economically independent]. There he continued writing, conducting experiments in physics in his own well-equipped laboratory.

In 1744 he was recalled to Versailles and given official positions at Court. During 1750–1753 he visited the court of Frederick II in Berlin. Eventually he settled (1755) at Les Delices on the outskirts of Geneva, and later moved to Ferney (purchased in 1758), a few miles away inside France. He remained there until 1778, when he went back to Paris and died there.

Voltaire was, in general, an ardent defender of victims of religious persecution with, however, one exception:

In his entry '*Juifs*' [*Dictionnaire Philosophique*, 1764], Voltaire writes:

"We find in the *Jews* only an ignorant and barbarous people, who have long united the most sordid avarice with the most detestable superstition and the most invincible hatred for every people by whom they are tolerated and enriched".

Thus, Voltaire echoes the familiar litany of insults drawn from classical pagan antisemitism, which he, no doubt, owed to his Jesuit upbringing. Not only did he repeat the pagan canard that Jews were the '*enemies of the mankind*', but he even justified the long history of persecution and massacres to which they had been subjected. These diatribes, shared by other prominent thinkers of the French Enlightenment like **Diderot**, **Baron d'Holbach**, and **Rousseau**, should be seen as philosophical expression of a crisis of a religious belief, in which war was conducted against the very roots of the Christian faith led logically to an assault on its Jewish origins.

The achromatic lens story (1733–1758)

Chester Moor Hall *(1703–1717, England) and* ***John Dollond*** *(1706–1761, England). Opticians. Independently invented the* achromatic telescope*, using an objective lens, composed of two kinds of glass so that the* chromatic aberration *in one kind of glass is compensated for by the other kind of glass.*

Hall, an amateur scientist from Essex, made achromatic lenses for his own use (1733) to little notice. Dollond developed the achromatic telescope (1758) with his son Peter (1738–1820). Earlier (1754) Dollond, a London optician of Huguenot descent, invented the heliometer*, a telescope that produces two images that can be manipulated to determine angular distances accurately, for finding the diameter of the sun or distances between stars. Hall brought an action against the Dollonds on the ground of the priority of his earlier work, but the action was dismissed by the courts. In 1761 Dollond was appointed optician to King George III, only a few month before his death.*

Isaac Newton *(1704) gave up trying to remove chromatic aberration from refracting telescope lenses, erroneously concluding that it could not be done: he falsely maintained that two lenses of different refraction indices require an infinite focal distance. Consequently he turned to the design of reflectors.*

Euler *proposed that the undesirable color effect seen in a lens were absent in the eye (which is an erroneous assumption) because the different media present negated dispersion. He suggested that achromatic lenses might be constructed in a similar way. Enthused by this work,* ***Samuel Klingenstjerna*** *(1698–1765, Sweden), professor at Uppsala, reperformed Newton's experiments on achromatism and determined them to be in error. Klingenstjerna was in communication with John Dollond, who was observing similar results. Dollond finally (1758), combined two elements, one of crown glass and the other of flint glass, to form a single achromatic lens. This was an accomplishment of great importance. The full mathematical theory of combinations of thin lenses and its application for correction of* spherical *and* chromatic aberration *was given in 1840 by Hungarian mathematician* ***Joseph Max Petzval*** *(1807–1891).*

1734 CE **George Berkeley** (1685–1753, England). Philosopher, economist, mathematician, physicist and bishop. Argued that 'absolute space'

does not exist by itself, since it is not a fundamental thing but an attribute, like color or harmony. Hence motion is relative and must be measured against some fiducial. He claimed that "Esse is percipi" — the existence of a thing is our perceiving it and since absolute space cannot be perceived, it cannot do as a reference frame. This led him to the notion that inertia is not intrinsic to a body but produced by motion relative to the fixed stars, i.e. all motion, including acceleration and rotation, should be regarded relative to the fixed stars, not space itself.

In his book "*The Analyst*: *A Discourse Addressed to an Infidel Mathematician*" (1734), he ridiculed infinitesimals as "the ghosts of departed quantities". His criticism was a much needed breath of fresh air; in the early days of calculus, it was practiced by a handful of fanatics, and so it had to be, for the theories of *fluxions* and *fluents*[204] were virtually devoid of rigor. Berkeley's attack forced mathematicians to re-examine the foundations of analysis. There followed 200 years of intense efforts by the best minds in Europe. The result was the rigorous calculus we know it today.

He was some 150 years ahead of his time and his arguments were lost in Newton's shadow.

[204] Berkeley's criticism was meant indirectly against Newton. Newton, however, knew exactly what he was doing; he just could not find a precise way to express it.

Do the infinitesimals really exist?

The concept of the infinitesimal, a number that is infinitely small yet greater than zero, has roots stretching back into antiquity. In spite of its importance as a tool in mechanics and geometry since the golden age of Greece, a never ending war between the finite and the infinite has been going on for the past 24 centuries.

In the 19^{th} century infinitesimal were driven out of mathematics once and for all, or so it seemed. To meet the demands of logic, the infinitesimal calculus of ***Isaac Newton*** *and* ***Gottfried Whilhelm von Leibniz*** *was reformulated by* ***Karl Weierstrass*** *(1872) without infinitesimals. Let us briefly survey the evolution of this idea:*

In ***Euclid's*** *geometry, both the infinite and the infinitesimal are deliberately excluded; we read in Euclid that a point is that which has a position but no magnitude. Certainly this meaningless definition is just a pledge not to use infinitesimal arguments. This was a rejection of earlier concepts in Greek thought: the atomism of* ***Democritos*** *had been meant to refer not only to matter but also to time and space. But then the arguments of* ***Zeno*** *had made untenable the motion of time as a row of successive instants, or the line as a row of successive "indivisibles".* ***Aristotle****, the father of systematic logic, banished the infinitely large or small from geometry.*

One of the first thinkers who came forth in defense of infinitesimals was ***Nicolas of Cusa*** *(ca. 1450). It behooved him to do just that because, as a cardinal of the Church, he believed that the infinite was the "source and means, and at the same time the unattainable goal, of all knowledge". Nicolas was followed in his mysticism by* ***Johannes Kepler*** *who in 1612 used infinitesimals to find the best proportions for a wine cask! He was not troubled by the self contradictions in his method; he relied on divine inspiration, and he wrote that "nature teaches geometry by instinct alone even without ratiocination". Moreover, his formulas for the volumes of wine casks are correct.*

The most famous mathematical mystic was no doubt ***Blaise Pascal****. In answering those of his contemporaries who objected to his reasoning with infinitely small quantities, Pascal said that "the heart intervenes to make the work clear" (1656). Pascal looked on the infinitely large and infinitely small as mysteries proposed by nature to man for him to admire but not to understand.*

The full flower of infinitesimal reasoning came with ***Newton*** *(1664),* ***Leibniz*** *(1673),* ***Jakob Bernoulli*** *(ca. 1700),* ***Johann Bernoulli*** *(ca. 1700) and* ***Leonhard Euler*** *(ca. 1740). The first textbook on the calculus was written*

in 1696 by ***G. F. A. de L'Hospital****, a pupil of Leibniz and Johann Bernoulli. In it are found two axioms that Aristotle outlawed 2000 years earlier:*

- *Two quantities differing by an infinitesimal can be considered equal (i.e. equal and unequal at the same time).*
- *A curve is the totality of an infinite number of straight segments.*

Curiously enough Newton and Leibniz did not endorse these views: Leibniz did not claim that infinitesimal really existed[205]*, only that one could reason without error as if they did exist. Newton, on the other hand, tried to avoid the infinitesimal: in his Principia Mathematica (as in Archimedes' 'On the Quadrature of the Parabola'), results originally found by infinitesimal methods are presented in a purely finite Euclidean fashion.*

The first critique of the infinitesimal method appeared by ***George Berkeley*** *in his book The Analyst (1734). It was addressed to "an infidel mathematician" [He meant Edmund Halley who financed the publication of the* Principia *and helped prepare it for the press]. In this book he accused both Newton and Leibniz of false reasoning, calling infinitesimals "the ghosts of departed quantities" and naming Newton's fluxions*[206] *"obscure, repugnant and precarious".*

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[205] This syndrome repeats itself at every revolutionary stage in the history of science: e.g. **Einstein** did not believe in the inherent probabilistic interpretation of quantum mechanics and **Schrödinger** did not believe in the reality of his own wave mechanics!

[206] Newton called "*fluent*" what we call today the instantaneous position function (of time) of a moving body. By "*fluxion*" he meant the instantaneous velocity of the same body. In the case of a falling stone, the fluent is given by the formula $s(t) = 16t^2$ with distances measured in feet and time in seconds. To evaluate the velocity of the falling stone at $t = 1$, we let dt stand for the infinitesimal increment of time. The corresponding increment in position is $ds = 16(1 + dt)^2 - 16 = 32dt + 16dt^2$. The ratio $\frac{ds}{dt}$ which yields the average velocity over ds at time dt is equal to $32 + 16dt$. To compute instantaneous velocity one must drop the infinitesimal $16dt$, i.e. assume that $32 + 16dt$ is the same as 32. That is precisely what Bishop Berkeley would not let them do. He said: "dt is either equal to zero or not equal to zero. If dt is *not* zero then $32 + 16dt$ is not the same is 32. If dt *is* zero, then the increment in distance ds is also zero, and the fraction $\frac{ds}{dt}$ is not $32 + 16dt$ but a meaningless expression $\frac{0}{0}$".

Weierstrass, 138 years later, gave the following answer (1872): to find an instantaneous velocity we abandon any attempt to compute the speed as a *ratio*. Instead we define the speed as a *limit*, which is approximated by ratios of *finite*

Although Berkeley's logic could not be refuted at the time, mathematicians went on using infinitesimals for another century, and with great success. Indeed, physicists and engineers have never stopped using them. The 18th century, the great age of the infinitesimals, was the time when no barrier between mathematics and physics was recognized. The leading physicists and the leading mathematicians were the same people. When pure mathematics reappeared as a separate discipline, mathematicians again made sure that the foundations of their work contained no obvious contradictions. "Just go on, and faith will soon return", said ***Jean Le Rond d'Alembert*** *to hesitate mathematical friend, who lacked experience and intuition in handling infinitesimals.*

By the beginning of the 19th century a clear distinction had been established between analysis (the study of infinite processes) and algebra (the study of operations of discrete entities such as natural numbers). A major objective of much of 19th-century mathematical effort was to unify (or at any rate to bridge) these two branches of mathematics. This endeavor was termed 'the arithmetization of analysis'. It was realized that the prime task was to construct a sound logical foundation for the real number system. Although the basic concepts of analysis - function, continuity, limit, convergence, infinity, were progressively clarified and refined during the first half of the 19th century, notably by ***Cauchy****, much remained to be done. This task was left to* ***Weierstrass****,* ***Dedekind*** *and* ***Cantor****, who eventually restored Greek standard of rigor: Modern analysis secured its foundation by doing what the Greeks had done: outlawing infinitesimals.*

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increments. Let Δt be a variable finite time-increment and Δs be the corresponding variable space-increment. Then $\frac{\Delta s}{\Delta t}$ is the variable quantity $32+16\Delta t$. By choosing Δt sufficiently small we can make $\frac{\Delta s}{\Delta t}$ take on values as close as we like to the value 32, and so, by definition, the speed at $t = 1$ is exactly 32.

This approach succeeds in removing any reference to numbers that are not finite. It also avoids any attempt to set directly $\Delta t = 0$ in the fraction $\frac{\Delta s}{\Delta t}$. There is, however, a price to pay: the intuitively clear and physically measurable quantity, the instantaneous velocity, becomes subject to the surprisingly subtle notion of '*limit*'. In the mathematical terminology it means that:
"The velocity is v if, for any positive number ϵ, $\left|\frac{\Delta s}{\Delta t} - v\right| < \epsilon$ for all values of $|\Delta t| < \delta$ for some $\delta = \delta(\epsilon, t) > 0$".
We have defined v by means of a relation between two new quantities, ϵ and δ, which are irrelevant to v itself. Ignorance of ϵ and δ never prevented Euler or Bernoulli from finding a velocity. The truth is that in a real sense we already knew what instantaneous velocity was before we learned this definition; for the sake of logical consistency we accept a definition that is much harder to understand than the concept being defined.

The reconstruction of the calculus on the basis of the limit concept and it's 'epsilon-delta' definitions amounted to a reduction of the calculus to the arithmetic of real numbers. The momentum gathered by these foundational clarifications led naturally to an assault on the logical foundation of the real-number system. This was a return, after 2500 years, to the problem of irrational numbers, which the Greeks had abandoned as hopeless after Pythagoras (One of the tools of these efforts was the newly developed field of symbolic logic). For rigorous certainty one had to resort to the cumbersome Archimedean method of exhaustion in it's modern version: the Weierstrass epsilon-delta method.

When we say that infinitesimals do exist, we do mean this in the sense it would be understood by Euclid and Berkeley. Until 100 years ago it was tacitly assumed by all philosophers and mathematicians that the subject matter of mathematics was objectively real in the sense close to the sense in which the subject matter of physics is real. Whether infinitesimals did or did not exist was a question of fact not too different from the question of whether material atoms do or do not exist.

Today, most mathematicians have no such conviction of the objective existence of the objects and structures they study. What mathematicians want from infinitesimals is not material existence *but rather the right to use them in proofs. This, of course, is of no concern to applied mathematicians and physicists since it is quite true that whatever can be done with infinitesimals can in principal be done without them.*

1734 CE **Emanuel Swedenborg** (1688–1772, Sweden). Scientist, philosopher and mystic[207]. Believed that the world evolved from a material point-source and the solar system originated from a sudden explosion of material from the sun.

1734–1742 CE **Colin Maclaurin** (1698–1746, Scotland). One of the ablest mathematicians of the 18^{th} century. He is best known today for his

[207] He wrote a treatise *A new system of reckoning which turns* 8 *instead of the usual turn at* 10 (1718) in which he defended the number 8 as a base. In his own words: "*Should the practice of the use and the use of the practice give its approval, I suppose that the learned world will gain incredible benefits from this octonary reckoning*". Modern computers have long been using a base-8 arithmetic. [It has been recently discovered that crows are capable of counting up to 7.]

exact solution for the spheroidal figure of equilibrium of a uniformly rotating homogeneous fluid mass (1742).

He did not discover the 'Maclaurin expansion' $f(x) = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!}$, since it is a special case of the '*Taylor expansion*' (1715) and was also given by **James Stirling** a quarter of a century before Maclaurin used it (as acknowledged by Maclaurin himself). He did, however, devise in 1729 a means of finding solutions to systems of linear equations, long before Cramer published it in 1750.

Maclaurin did notable work in geometry, particularly in the study of higher plane curves, and he showed great prowess in applying classical geometry to physical problems. Among his many papers in applied mathematics is a prize-winning memoir on the mathematical theory of tides (1740). In his '*Treatise on Fluxions*' (1742) he undertook the defense of the calculus techniques of Newton, which came under attack in 1734 by a nonmathematician, Bishop **George Berkeley**. In his book Maclaurin treated calculus on the basis of Greek geometry and thus answered all objections to its method as being founded on false reasoning and full of mystery. It is in this work that he expounded his discovery of the '*Maclaurin spheroids*'.

Maclaurin was a mathematical prodigy. At age 11 he entered the University of Glasgow. In 1717 he was elected professor of mathematics in Marischal College, Aberdeen. Two years later he was admitted as Fellow of the Royal Society and made acquaintance with Isaac Newton.

In 1722 Maclaurin traveled as tutor and companion to the eldest son of Lord Polward. In 1725 he was elected professor of mathematics in the University of Edinburgh on the recommendation of Newton. There was some difficulty in obtaining a grant to cover his salary, and Newton offered to bear the cost personally.

In 1745, when the rebels of Charles the pretender were marching on Edinburgh, Maclaurin took part in preparing trenches and barricades for its defense. The hardships to which he was thus exposed, caused a disease to which he later succumbed. He died at Edinburgh.

1735 CE **George Hadley** (1685–1768, England). One of the first contributors to the classical model of the general circulation in the atmosphere. In his paper "*Concerning the cause of the general trade winds*"[208], he revised

[208] *Trade winds*: steady winds, with speed between 5–$7\frac{1}{2}$ m/sec, occupy belts between latitudes 25° and 5° on either side of the equator. North of the equator they blow from the northeast; in the Southern Hemisphere their direction is generally from the southeast. Along the equator, the atmospheric pressure tends to be low and the winds weak. This is the region of the *doldrums*, where sailing vessels can make but very little headway.

and improved Edmund Halley's earlier explanation of the trade winds (1686).

Hadley was well aware of the fact that solar energy drives the winds. He proposed that the large temperature contrast between the poles and the equator would create a thermal circulation very similar to that of sea breeze. As long as the earth's surface is heated unequally, air will move in an attempt to balance the inequalities.

Hadley suggested that on nonrotating earth the air movement would take the form of one large *convection cell* in each hemisphere. The more intensely heated equatorial air would rise and move poleward. Eventually, this upper-level flow would reach the poles where it would sink and spread out at the surface and return to the equator. As the cold polar air approached the equator, it would be reheated and rise again. If the earth were not rotating, this would produce winds blowing from the poles to the equator along the earth's surface. Because of the rotation of the earth, the air moving towards the equator is deflected to blow from east to west (easterly wind), while the flow aloft will be deflected from west to east (westerly wind).

Hadley's paper remained unnoticed for many years. His ideas were based on a *single* thermally direct cell and required high pressure over the poles and low pressure over the equator, with uniform pressure gradient between them. In the 19^{th} century, new observations of surface pressures contradicted this, for belts of high pressure were observed in the subtropics as well as at the poles, with low pressure in middle latitudes as well as at the equator; such distribution required the existence of *three cells* (**Ferrel**, 1856), not one[209]. In the 1920's, the three-cell model (in each Hemisphere) was definitely accepted as correct (and sufficient to accomplish the task of maintaining the earth's heat balance).

In the zone between the equator and roughly 30 degrees latitude, the circulation closely resembles the convective model used by Hadley for the whole earth; hence, the name *Hadley cell* is generally applied to it.

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[209] Hadley's model does not take into account the thermal conditions in the upper atmosphere. Clearly, heating and cooling are not restricted to the earth's surface. However, there is yet another reason why the single-cell model is not acceptable: because of gravity, the atmosphere must rotate with the earth. In the Hadley model the surface winds blow toward the west and would, because of *friction*, oppose the earth's rotation, which is toward the east. Since the atmosphere is attached to the earth, however, its average motion relative to the solid earth's surface must vanish. Thus, easterly flow at one latitude must be balanced by westerly flow at another.

It is of historical interest to note that Hadley realized the effect of the earth's rotation on winds 100 years ahead of **Coriolis**(!), basing his theory upon the law of conservation of angular momentum.

1735–1743 CE **Charles Marie de la Condamine** (1701–1774, France). Naturalist and mathematician. Member of an expedition to Peru (with **Bouguer**) to measure the length of a degree of a meridian arc at the equator (1735). Made first scientific exploration and account of the Amazon river in a 4-month raft journey (1743). It was published in 1751.

1736–1740 CE **Claudius Aymand** (1660–1740, France). Surgeon. Performed the first recorded successful appendectomy (1736).

1736–1760 CE **Israel ben Eliezer, Ba'al Shem Tov** (Master of The Name; Acr. BESHT, ca. 1700–1760, Poland). Religious leader and philosopher. Founder (1736) of the Jewish *Hasidic* ("The Pious") movement in Eastern Europe that influenced the course of Jewish life for over two centuries. The new cult was directly in the line of the traditional Jewish mystics of the *Kabbalah*, but the BESHT endeared it to the masses with a poetic earthiness and love of life and people which the old ascetic Kabbalah lacked.

Upon the rapid decline of Jewish life in Slavic countries following the great devastation and massacres by the Cossack hordes (1648–9), religious worship had become even more formalistic and the great majority of the Jews sank into the most abject poverty and ignorance. In his teachings, the BESHT revived the cult of the *Tzadik*[210], the righteous saintly person who mediates between the Upper and the Lower worlds. He invented a revolutionary form of popular prayer, through which man breaks down the barriers of his natural existence and reaches into the divine world. Thus Hasidism emphasized *joyful* worship of God in prayer and in *all* of one's actions.

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210 The Talmudists placed Ba'al Shem and his followers under the ban, but to no avail. Since eventually *Hasidism* became orthodoxy it could not be excommunicated. One of the dedicated enemy of Hasidism was the great Talmudic scholar **Eliahu ben Shlomo Zalman** (Known as HAGA'ON MI-VILNA; 1720–1797, Lithuania). He was a man of awesome secular and religious knowledge, probably one of the greatest Hebrew scholars ever. He purchased a small house outside Vilna and concentrated entirely on study. He never slept more than two hours a day, and not more than a half an hour at a time. To eliminate distractions he kept his shutters closed even in daytime and studied by candlelight. To stop himself from falling asleep, he cut off the heating and put his feet in a bowl of cold water. He expressed interest in *secular science* as an aid to understanding the Torah. "all knowledge is necessary for understanding the Holy Torah and is included in it".

Nothing of what the BESHT taught was new to Judaism; he merely gave added emphasis to ideas which had been current for millennia [e.g. *Psalms* **47**, 2; **100**, 2]. His techniques, however, offered the Jewish masses an escape from their troubled life and the oppressive authority of the rabbis. Consequently his movement spread with remarkable speed trough Southeastern Europe.

The Ba'al Shem Tov was born in Okop in backward Podolia to very old parents. Orphaned at an early age, his early manhood was spent in the wilderness, in utter poverty, performing miracles, faith-healing, and exorcizing evil spirits (unlike Jesus, he was twice married).

In his fortieth year, (1740), Ba'al Shem threw off his cloak of boorishness and revealed himself in the splendor of a messenger of God. He established himself in Medzibezh, Podolia, where he remained until his death. Henceforth, he led a life of saintliness and piety. The BESHT left no writings. His homilies were put down by his disciples.

1736–1761 CE **Thomas Bayes** (1702–1761, England). Mathematician. Introduced the '*Principle of inverse probability*'[211]: "*The posterior probabilities of the hypotheses are proportional to the products of the prior probabilities and the likelihoods*". In other words: if there is no ground to believe one of a set of alternative hypotheses rather than another, their prior probabilities are equal. When, in addition, posterior evidence is available, then in retrospect the most probable hypothesis is the one that would have been most likely to lead to that evidence. Thus, if the data were equally likely to occur on any of the hypotheses, the former tell us nothing new with respect to the credibility of the latter, and we shall retain our previous opinion, whatever it was. This principle provides a formal rule, in general accordance with common sense, that enables a decision between hypotheses on the basis of available evidence.

Bayes initiated the "Bayesian" school of 'inductive probability' (probability of causes) with his extension of the definition of conditional probability. It has been promoted by **Harold Jeffreys** (1891–1989, England) and applied with considerable success to diagnosis of medical conditions and many other applications of statistical inference and fuzzy logic.

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211 The essence of Bayes' theorem is found already in the Talmud (*Yevamoth*, **4**). The Jewish rabbis solved the problem very much earlier (ca. 100 CE), but expressed the argument in words, not numbers. They also thought of the analysis as a way to solve moral and legal problems, not as an end in itself. The Talmud, though, lacks the clarity of the Bayes' analysis and is content with *relative*, not absolute probabilities. Yet, the rabbis understood the logic underlying this analysis, and their relative probabilities are the same as obtained by the Bayes' formula.

Bayes was born in London. He was privately educated and his mature life were spent as an Anglican minister at the chapel of Turnbridge Wells.

In 1761 he attempted to use the theory of probability to prove the existence of God. To this end he started from a statement expressing the relationship between the conditional probability and its inverse ("The other way around"). The conditional probability of event B, given A is

$$prob(B|A) = prob(A|B)\frac{prob(B)}{prob(A)}$$

Now, let a set of mutually exclusive events $B_1, B_2, \dots, B_n$, be given, and let us assume that the occurrence of one or another of them is a necessary condition for the occurrence of an event A. Since

$$prob(A) \equiv P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + \cdots P(B_n)P(A|B_n),$$

there follows *Bayes' theorem*

$$P(B_i|A) = \frac{P(B_i)P(A|B_i)}{P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + \cdots + P(B_n)P(A|B_n)}$$

Two years later (1763) Bayes died and his "*Essay towards solving a problem in the doctrine of chance*" was discovered and published. It caused a great stir in the mathematical community, establishing an entire new field of science, now called *Bayesian Statistics*, and having far reaching implications about scientific inference.[212]

1737, Oct 11 CE A cyclonic storm in the Bay of Bengal killed about 300,000 people in Calcutta, India.

1737 CE **Georg Brandt** (1694–1768, Sweden). Chemist. Discovered and isolated the element *cobalt*. The coper-miners of the Harz Mountains frequently obtained ores looking like copper-ore; these gave an unpleasant smell of garlic or roasting and furnished no copper. The miners attributed their occurrence to the pranks of a mischievous spirit, *Kobold* (from the Greek *kobalos*), and the material was called "false-ore", or *cobalt*. The use of cobalt as a constituent of some blue glazes and blue glass, made in imitation of *lapis*

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[212] Using Bayes' theorem in cases where the prior probabilities of B_i cannot be measured directly, may lead to controversial results: misrepresentation of data. if one is not careful enough.
In the 20^{th} century, a leading proponent of Bayesian probability theory was Bruno de Finetti (1906–1985, Italy), who expounded the mathematical relationships between independence and *exchangeability*.

lazuli, has been established for ancient Egyptian (1735 BCE) and Babylonian (1450 BCE) specimens by analysis.

Cobalt chloride solutions were introduced as *invisible ink* in 1705. The pale pink, almost colorless complex $2[Co(H_2O)_6]Cl_2$ in dilute aqueous solution is all but invisible when used to write on white paper. Gentle warming of the paper shifts the equilibrium to $Co(CoCl_4)+12H_2O$ by driving out the complexed water molecules, forming the easily identified blue cobalt chloride complex. However, if the paper is allowed to sit about at room temperature for a while, all is soon invisible again as the anhydrous complex picks up moisture from the atmosphere.

With the advent of nuclear physics in the 20^{th} century, a great variety of radioactive isotopes for medical and industrial use were produced by neutron irradiation of various elements. The most widely used of these is cobalt-60, which is prepared from the normal metal cobalt-59. A moveable cobalt-60 "gun" produces a beam of gamma rays for the irradiation of a selected spot on the patients body.

Since cobalt-60 has a half life of 5 years, the "gun" does not have to be loaded very often. Cobalt-60 is also an exceptionally effective material for making a *dirty bomb*. It can be wrapped around a large hydrogen bomb in almost unlimited amounts to absorb the superfluous neutrons and produce fallout enormously more potent than that from an ordinary atomic bomb. Cobalt bombs have been mentioned as possibly the main ingredient of a *doomsday machine*.

1737–1753 CE **Bernard Forest de Belidor** (1698–1761, France). Civil engineer. First to apply the Newtonian calculus to practical architectural problem. This he expounded in his influential 4-volume treatise *Architecture hydraulique* (1737–1753). He wrote numerous books dealing with hydraulic, civil and military engineering.

Bedidor was a professor at the Ecole de Artilleric.

1737–1776 CE **David Hume** (1711–1776, Scotland). Agnostic philosopher, historian and political economist. A major figure of the *enlightenment* (1715–1789). Pioneered in the sciences of political and cultural history, economics, comparative history of religions, sociology and psychology. Scandalized Britain with his anti-religious ideas.

After his graduation from Edinburgh University and after fruitless attempts to make a place for himself in law and in business (1734), David

Hume went to France and there wrote *Treatise on Human Nature*[213] (1735–1740). Settling in his family estate of Ninewells (1740) he wrote an *Enquiry Concerning Human Understanding* (1748). His reputation made by this essay, was solidified in England and on the continent with *An Enquiry Concerning the Principles of Morals* (1752).

Hume's reputation as a skeptic led to his failure to obtain the Chair of Ethics and Philosophy at Edinburgh University (1744). Later on he was a soldier (1745–1746), librarian (1751–1757), diplomat (1763–1765) and Under-Secretary of State (1767–1768). Thereafter he settled down in Edinburgh living among his friends with Epicurean ease, and dedicating his time to the writing of a history of England and his posthumously published *Dialogues Concerning Natural Religion* (1779).

In his *Dialogues*, Hume mounted a skeptical attack on the logical structure of many naive features of the Newtonian clockwork-universe ("God wound up the Universe and set it going") and indeed also upon the rational basis of any form of scientific inquiry. Hume calls the *Design Argument* 'the religious hypothesis', and proceeds to attack its foundation from a variety of directions. Hume's approach was entirely negative; whereas most of his contemporaries accepted the rationality and ordered structure of the world without question, Hume did not. A commonsense view of the world, along with the metaphysical trimmings that had been added to the Newtonian world model, Hume rejected.

His objections are threefold: Firstly, the *Design Argument* is unscientific; there can be no causal explanation for the order of nature because the uniqueness of the world removes all grounds for comparative reference. Secondly, analogical reasoning is so weak and subjective that it could not even provide us with a reasonable conjecture, let along a definite proof. And finally: all negative evidence has been conveniently neglected.

Hume maintains that a dispassionate approach could argue as well for a disorderly cause if it were to concentrate upon the disorderly aspects of the world structure.

Humean tirade against the simple design arguments of the English physicists fell upon deaf ears, and must have seemed rather naive when held up against the staggering quantitative achievements of the Newtonian system. He became an isolated and ignored figure in literary circles even during his lifetime, and appeared a 'crank' to the Newtonians.

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213 In Germany, the philosopher, statistician and scientist **Johannes Nicolaus Tetens** (1736–1807) expounded a similar theory in his treatise (1777): "*Philosophische Versuche über die Menschliche Natur und ihre Entwicklungen.*" He is therefore known as the 'German Hume'.

Hume's theory of knowledge can be summarized as follows:

- *Science* and *religion* are mutually exclusive. Religion is not a form of *knowledge*; it is not even a form of *knowing*; it is rather a complex kind of *feeling*, without recourse to science and reasons. Religion simply postulated unknown causes: reason is limited to the realm of human experience, and therefore it cannot decide ultimate questions such as the origin of the cosmos.

- Except for abstract reasoning concerning quantity and numbers (mathematics) reasoning involves belief rather than knowledge and is referable to human feelings, instincts and emotions (passions).

- The problem of *truth* in questions of fact and existence is referable to psychology rather than to logic. Hume stressed that empirical facts must be given due weight against the testimony of men. He gives a number of examples showing that testimony of otherwise reputable men cannot be trusted. *Mass delusion*[214] can occur: given enough time for people to talk to each other, the delusion can develop consistency. Delusions are especially likely in cases where people are trying to interpret an extraordinary event, e.g. the natural law (confirmed billions of times by all humans everywhere) that the dead do not rise must outweigh testimony to the contrary.

- The idea that bases the existence of God on the majestic and wondrous design of the Universe (i.e., natural theology) is rejected. The world could have existed throughout eternity, requiring no first cause.

- There is no observable 'soul' behind the process of thought; what we call 'mind' is only an abstract name for perceptions, memories and feelings. (By dissipating the concept of soul, Hume destroyed orthodox religion.)

- Man cannot know ultimate reality or achieve any knowledge beyond a mere awareness of phenomenal sensory images. The only knowledge we can possess consists of a mere sequence of ideas (perceptions, or assumptions) none of which can be proved to be true; all knowledge is therefore restricted to mental states or experiences; of those only we can be certain. (He thus challenged all alleged truths except those of mathematics and the immediate intuitive awareness of our sense experience.) This made him reject the idea of *scientific law* as objective reality, i.e., science must

[214] We have now vastly more experience with mass delusions than Hume had, experience primarily obtained in the process of investigating UFOs. The modern world does not believe in messengers from God (angels), but it does believe in extraterrestrial intelligent life: imaginary super-beings from other stars that play the same psychological role in modern society as angels did in the past.

limit itself strictly to mathematics and direct experiment; it cannot trust to unverified deduction from "laws".
There is no such thing as 'natural law' or 'necessity' in the sequence of effect upon cause; we never perceive causes, or laws: we perceive events and sequences, and *infer* causation and necessity. 'Law' is an observed *custom* in the sequence of events; but there is no 'necessity' in custom.

- An event C and a subsequent event E are related as *cause* and *effect*, if the occurrence of C (or a situation *similar* to it[215]) is always followed by E, and if E never occurs unless C has occurred previously. But the fact that we have become aware of a particular cause sequence (like C to E) even a very large number of times, is no proof that C will be followed by E on all future occasions. He concluded that our belief in causality is no more than a *habit* which is not an adequate basis for belief.

Causality according to this definition[216] cannot be gained from material given by the senses. To connect one occurrence with some other by the notion of cause and effect is not the result of rational knowledge but of a habit of expecting the perception of the second after having perceived the first; because that sequence has previously taken place in innumerable cases. This *habit* is founded upon a belief which can be explained psychologically but cannot be derived by abstraction from the ideas of the two events (objects) or the impressions of the senses. Hume did not deny that causality works. He only denied that reason is capable of understanding it.

There can be no causal explanation for the order of nature: perhaps the development of the world is *random* but has had an infinite amount of time available to it so all possible configurations arise until eventually a stable self-perpetuating form is found or — matter may posses some intrinsic *self-ordering property* (1748).

Hume has influenced the development of the best philosophers who came after him (**Kant**, **James**, **Russel**, **Santayana**) and gave speculative philosophy a new direction[217].

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[215] Hume included it in his definition because he wanted to make causality *experimentally verifiable*.

[216] Modern physics, whose causal laws are elaborated inferences from the observed course of nature, have supported Hume's challenge to the traditional causal connection.

[217] **Immanuel Kant** read Hume's Dialogues in 1780 and subsequently acknowledged his debt to him for awakening him 'from his dogmatic slumbers'.
Hume's idea that the world might have been gradually *evolved* from very small beginning, increasing by the activity of its inherent principles rather than by a sudden decree of God was taken up by the zoologist **Erasmus Darwin**

(1731–1802), who was Charles Darwin grandfather. Erasmus was starting to take up early steps (1794) toward an evolutionary theory of animal biology, maintaining that the components of an animal or plant were not designed for the use to which they are currently applied, but rather, have grown to fit that use by a process of gradual improvement.

Hume had also influenced young **Albert Einstein**, who said of him: "*One is amazed that many, sometimes highly esteemed, philosophers after him have been able to write so much obscure stuff and even find grateful readers for it*".

Worldview XII: David Hume

* * *

"Reason is limited to the realm of human experience, and therefore it cannot decide ultimate questions such as the origin of the cosmos".

* * *

"While Newton seemed to draw off the veil from some of the mysteries of nature, he showed at the same time the imperfections of the mechanical philosophy; and thereby restored her ultimate secrets to that obscurity in which they ever did and will remain".

* * *

"If we take in our hand any volume, of divinity or school metaphysics, for instance; let us ask 'Does it contain any abstract reasoning concerning quantity or number?' No. 'Does it contain any experimental reasoning concerning matter of fact and existence?' No. Commit it then to the flames: for it can contain nothing but sophistry and illusion".

* * *

"Look around this world: Contemplate the whole and every part of it. You will find it to be nothing but one great machine, subdivided into an infinite number of lesser machines... All these various machines and even their most minute parts, are adjusted to each other with an accuracy, which ravishes into admiration all men who have ever contemplated them. The curious adapting of means to ends, throughout all nature, resembles exactly, though it much exceeds, the productions of human contrivance; of human design, thought, wisdom and intelligence...".

* * *

"... We are guilty of the grossest, and most narrow partiality, and make ourselves the model of the Universe... What peculiar privilege has this little agitation of brain which we call thought, that we must thus make it the model of the whole Universe".

* * *

"A very small part of this great system, during a very short time is very imperfectly discovered to us: And do we thence pronounce decisively concerning the origin of the whole?... Let us remember the story of the Indian philosopher and his elephant. It was never more applicable than to the present subject. If the material world rests upon a similar ideal world, this ideal world must rest upon some other; and so on, without end. It were better, therefore never to look beyond the present material world".

* * *

"But were this world ever so perfect a production, it must still remain uncertain whether all the excellences of the work can justly be ascribed to the workman... Many worlds might have been botched and bungled, throughout an eternity, ere this system was struck out; much labour lost, many fruitless trials made; and a slow but continued improvement carried on during infinite ages of world-making".

* * *

1737–1783 CE **Leonhard Euler**[218] (1707–1783, Switzerland). The greatest mathematician of the 18^{th} century. Freed the analytical calculus from all geometric bounds, and thus established analysis as an independent science, which from his time on has maintained an unchallenged leadership in the field of mathematics.

Euler was born in Basel, the son of a preacher. The father's wish was that his son should follow in his footsteps, but on entering the University of Basel in 1720, Euler met **Johann Bernoulli** and learned much mathematics from him. Nevertheless he emerged with a master of philosophy degree and joined the department of theology, with ample knowledge in Greek and Hebrew and a strong religious conviction that stayed with him all his life.

At 19 he won a prize from the French Academy of Sciences for a paper on the masting of ships, and was consequently invited to join the Academy of Sciences in St. Petersburg. In 1740 he lost sight in one eye, and at the request of Frederick the Great joined the scientific community in Berlin, where he stayed for the next 25 years. In 1766 he returned to St. Petersburg, and by 1771 he had become totally blind.

Euler's output, range and energy were phenomenal: he published hundreds of papers in almost every field in the pure and applied mathematics of his day, plus several books on a wide range of topics. While in Berlin he supervised the observatory, the botanic gardens and the publication of maps and calendars. He also advised on financial matters, including lotteries and pensions. In addition he was required to work on canal improvements and to translate a military book into German. He continued to work in his blindness for twelve years, producing an 800-page book on lunar motion and 50 research papers totaling 1000 pages. He fathered 13 children, and enjoyed playing with them whilst simultaneously contemplating mathematics.

Euler contributed new essential ideas in number theory, algebra, calculus, calculus of variations, functions of complex variables, differential geometry, difference and differential equations, special functions, acoustics, optics[219], mechanics, fluid dynamics, astronomy, artillery, navigation, statistics, finance and philosophy of science. In addition, he was also a prodigious calculator.

In his memoir '*De Fractionibus Continius*' he laid the foundations for the modern theory of continued fractions which play an important role in present day mathematics. They constitute a most important tool for new

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[218] For further reading, see:

- Wittle, T., *Leonhard Euler 1707–1783*: *Beiträge zu Leben und Werke*, Gedenkband des Kantons Basel-Stadt, Birkhäuser: Basel, 1983.

[219] E.g., suggested (1766) a design for *achromatic lenses*.

discoveries in number theory and in the field of Diophantine approximations. In numerical analysis, continued fractions are used to give rapid numerical approximation.

Euler (1728) introduced the notation e for the base of the natural logarithms[220]. In 1739 he adopted the symbol π (**Jones**, 1706). Later (1750) he introduced the functional notation $f(x)$, the summation symbol $\sum$; and in 1777, the symbol $i = \sqrt{-1}$. In 1755 he discovered the differential equations of motion of non-viscous fluids. During 1758–1765, he proved that the instantaneous displacement of a rigid body can be expressed as a sum of an axial rotation and a translation. To describe the rotation he introduced the '*Euler angles*', thus establishing the basis for the algebra of finite rotations. He then derived the equations of motion of a rigid body about a point, thus laying the mathematical foundation for the analysis of gyroscopic behavior. His formulation emphasizes the crucial role of the components of the *inertia tensor*, the first tensor entity to enter physics.

In 1765 he suggested that the earth might undergo a free *nutation* with a period of $A/(C-A) = 305$ sidereal days. Euler started the systematic investigation of variational problems, a subset of which is known as *isoperimetric problems*. These maximum-minimum problems attracted the interest of the best minds — such as **Newton**, **Leibniz**, **Jakob** and **Johann Bernoulli** — from the very start of the infinitesimal calculus. Euler found a differential equation which gave the implicit solution of an extended class of such problems.

In 1768, Euler published his three-volume treatise '*Institutiones calculi integralis*' in which he presented exhaustive methods for evaluating definite and indefinite integrals in terms of elementary functions. He also developed the theory of ordinary and partial differential equations[221].

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[220] Seeking a function $f(x)$ whose derivative is equal to itself, **Newton** (1665) had shown that $f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$. From this he deduced that $f(1) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots$.
Euler (1728) proved that $f(1) = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$ and gave this number the symbol e.
Lambert (1776) proved that e is irrational and **Hermite** (1873) showed that e is also transcendental.
Euler discovered the beautiful relationship $e^{\pi i} = -1$ and, more generally that e is related to the trigonometric functions by $e^{i\theta} = \cos\theta + i\sin\theta$.

[221] *Example*: Euler derived a closed-form solution to the general linear ODE of the second order in the form of a *continued fraction*. Starting from $y(x) = P_1(x)\frac{d^2y}{dx^2} + Q_0(x)\frac{dy}{dx}$ this equation is differentiated and becomes $y' = Q_1 y'' + P_2 y'''$, where $Q_1 = \frac{Q_0 + P_1'}{1 - Q_0'}$, $P_2 = \frac{P_1}{1 - Q_0'}$. This procedure is repeated indefinitely, and a set of relations $y^{(n)} = Q_n y^{(n+1)} + P_{n+1} y^{(n+2)}$ is obtained, where

The legacy of Leonhard Euler in unsurpassed in the long history of mathematics. In body, quantity and quality his achievements are overwhelming. Euler's collected works fill over 70 volumes, a testament to the genius of this unassuming man who changed the face of mathematics so profoundly.

Throughout his career, Euler was blessed with a phenomenal memory. His number-theoretic investigations were aided by the fact that he had memorized not only the first 100 prime numbers, but all their first six powers as well. While others were digging through tables or pulling out pencil and paper, Euler could simply recite from memory such quantities as 241^4 or 337^6. He was able to do mental calculations requiring him to retain in his head up to 50 significant figures, and that without apparent effort, "*just as men breath, as eagles sustain themselves in the air*" — in the words of Francois Arago. Yet this extraordinary mind still had room for the entire text of Virgil's *Aeneid*, which Euler had committed to memory as a boy, and still could recite flawlessly half a century later. No writer of fiction would dare provide a character with a memory of this caliber[222].

$n = 1, 2, 3, \ldots,$ and $Q_n = \frac{Q_{n-1}+P_n'}{1-Q_{n-1}'}$, $P_{n+1} = \frac{P_n}{1-Q_{n-1}'}$. Then

$$\frac{y}{y'} = Q_0 + \frac{P_1}{(y'/y'')} = Q_0 + \frac{P_1}{Q_1 + \frac{P_2}{(y''/y''')}}.$$

Let

$$\lambda(x) = Q_0 + \cfrac{P_1}{Q_1 + \cfrac{P_2}{Q_2 + \cfrac{P_3}{Q_3 + \cfrac{}{\ddots + \frac{P_n}{Q_n+R_n}}}}}$$

where $R_n = P_{n+1}\frac{y^{(n+2)}}{y^{(n+1)}}$.

If the fraction terminates, $y(x) = \mathrm{e}^{\int \frac{dx}{\lambda(x)}}$; if it does not terminate, the problem of its convergence arises. To this end the following fundamental theorem is available: $\{\lambda(x)\}^{-1}$ converges and has the value y'/y if $y \neq 0$ and (1) $P_n \to P$, $Q_n \to Q$ as $n \to \infty$, (2) the roots ρ_1 and ρ_2 of the equation $\rho^2 = Q\rho + P$ are of unequal modulus; if further $|\rho_2| < |\rho_1|$ then $\lim |y^{(n)}|^{1/n} < |\rho_2|^{-1}$ provided that $|\rho_2| \neq 0$. When $|\rho_2| = 0$ the last condition is replaced by the condition that the limit is finite.

222 However, even the great Euler was not always right. His conjecture (1769) that $x^n + y^n + z^n = c^n$ has no solution if $n \geq 4$ was proven wrong:

Noam Elkies found (1988) the counterexample

$$2,682,440^4 + 15,365,639^4 + 18,756,760^4 = 20,615,673^4$$

Part of Euler's well-deserved reputation rests upon the textbook he authored. In all his texts, Euler's exposition was quite lucid and his mathematical notation was chosen so as to clarify, not obscure, the underlying ideas.

Euler's *Opera Omnia*, consists of 73 volumes. It contains 886 books and articles — written variously in Latin, French and German. His output was so huge and the pace of its production so rapid — even in the darkness of his later life — that a publication backlog is reported to have lasted 47 years after his death. It has been estimated that if one were to collect *all* publications in the mathematical sciences produced during 1725–1800, roughly 1/3 of these were from the pen of Leonhard Euler.

Virtually every branch of mathematics has theorems of major significance that are attributed to Euler.

One can get a feel for Euler's profound insight through the following examples:

To prove that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ Euler began with the key equation

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots = \left[1 - \frac{x^2}{\pi^2}\right]\left[1 - \frac{x^2}{4\pi^2}\right]\left[1 - \frac{x^2}{9\pi^2}\right]\cdots.$$

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and **Roger Frye** followed with the simpler result

$$95,800^4 + 217,519^4 + 414,560^4 = 422,481^4.$$

Moreover, Elkies showed that there are *infinitely many solutions* of

$$x^4 + y^4 + z^4 = c^4$$

in coprime natural numbers x, y, z, and c. He also provided a second solution in four astronomical numbers, each width 70 digits:

$$\begin{aligned}
x &= 1439965710\ 6489544922\ 6850677183\ 3175267850\ 2014266153\ 0044221829\ 2336336633,\\
y &= 4417264698\ 9945384969\ 4359748975\ 4952845854\ 6722971790\ 4789886412\ 4209346920,\\
z &= 9033964577\ 4825324980\ 5948245939\ 8457291004\ 9479250057\ 4302814746\ 5732645880,\\
c &= 9161781830\ 0354368478\ 3245239826\ 7266038227\ 0029622572\ 4366207037\ 0888722169.
\end{aligned}$$

Euler further conjectured (1769) that the general Diophantine equation

$$x_1^n + x_2^n + \cdots + x_{n-1}^n = x_n^n \quad (n \geq 4)$$

has no solutions in positive integers. Yet, **L. J. Lander** and **T. R. Parkin** were able to furnish the first counterexample (1966) for $n = 5$

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5.$$

Performing the infinite multiplication on the right hand side, Euler obtained

$$1-\left[\frac{1}{\pi^2}+\frac{1}{4\pi^2}+\frac{1}{9\pi^2}+\frac{1}{16\pi^2}+\cdots\right]x^2+(\cdots)x^4-\cdots.$$

Equating the coefficient of x^2 on both sides, the required result follows. Thus, Euler found the answer that had escaped mathematicians for decades[223].

From the above key equation Euler deduced, for $x=\frac{\pi}{2}$, the known Wallis' infinite-product representation

$$\frac{2}{\pi}=\big(1-\frac{1}{4}\big)\big(1-\frac{1}{16}\big)\big(1-\frac{1}{36}\big)\cdots$$

or

$$\frac{\pi}{2}=\frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdots}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdots}.$$

Moreover, by equating the coefficients of x^4 on both sides of the above key equation, he could announce that $\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}$.

Another of Euler's beautiful results is the relation (1737) known as '*Euler's product formula*' $\sum_{n=0}^{\infty}\frac{1}{n^s}=\prod_p\big(1-\frac{1}{p^s}\big)^{-1}$, where s is a real number greater than 1 and the expression on the right denotes an infinite product in which p runs over all primes.

In the field of Diophantine equations, Euler found that $p=a^3-9ab^2$; $q=3a^2b-b^3$; $r=a^2+3b^2$ solve the equation $p^2+3q^2=r^3$

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[223] Historically,

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots$$

was the first series that mathematicians were unable to sum using elementary algebraic methods. After the Bernoulli family had tried and failed, Euler finally cracked the problem (1734) by means of a brilliant unorthodox argument. Today, such results can be derived in a systematic way using *residues*:
Consider the function $g(z)=\frac{cot(\pi z)}{z^2}$. With N a positive integer, let S be the origin – centered square with vertices $(N+\frac{1}{2})(\pm 1\pm i)$. Adding up the residue inside S,

$$\frac{1}{2\pi i}\oint_s g(z)=Res\,[g(z),0]+\sum_{n=-N}^{-1}Res[g(z),n]+\sum_{n=1}^{N}Res\,[g(z),n]=$$

$$=-\frac{\pi}{3}+\frac{2}{\pi}\sum_{n=1}^{N}\frac{1}{n^2}$$

As N tends to infinity, the integral on the LHS tends to zero and from this fact we immediately deduce Euler's result.

It is incredible how Euler could have discovered the identity:

$$x^4 + y^4 = z^4 + t^4$$

where

$$\begin{aligned} x &= a^7 + a^5b^2 - 2a^3b^4 + 3a^2b^5 + ab^6, \\ y &= a^6b - 3a^5b^2 - 2a^4b^3 + a^2b^5 + b^7, \\ z &= a^7 + a^5b^2 - 2a^3b^4 - 3a^2b^5 + ab^6, \\ t &= a^6b + 3a^5b^2 - 2a^4b^3 + a^2b^5 + b^7. \end{aligned}$$

One of Euler's achievements (1748) was the discovery of the four-square analogue of $(a^2 + b^2)(c^2 + d^2) = (ac \pm bd)^2 + (ad \mp bc)^2$, namely:

$$\begin{aligned} (x_1^2 + x_2^2 + x_3^2 + x_4^2)(y_1^2 + y_2^2 + y_3^2 + y_4^2) = \quad & (x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4)^2 \\ & +(x_1y_2 - x_2y_1 - x_3y_4 + x_4y_3)^2 \\ & +(x_1y_3 + x_2y_4 - x_3y_1 - x_4y_2)^2 \\ & +(x_1y_4 - x_2y_3 + x_3y_2 - x_4y_1)^2. \end{aligned}$$

Euler also showed (1737) that

$$\tanh(1) = \frac{e^2 - 1}{e^2 + 1} = \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{5 + \cfrac{1}{7 + \cfrac{1}{\ddots}}}}} = 0.76159415\ldots$$

where e is the basis of the natural logarithms.

The Saga of $i = \sqrt{-1}$

During the 16^{th} century, mathematicians encountered square roots of negative numbers through the general solutions of quadratic and cubic equations.

Since the square of every real number is either positive or zero, the equation $x^2+1=0$ *cannot be solved in the field of real numbers.*

A book on algebra by ***Rafael Bombelli****, which dates from* 1572*, contains a consistent theory of roots of negative numbers. These numbers were used by mathematicians since the middle of the* 17^{th} *century and were since known as imaginary numbers. Later, the theory of numbers of the form* $a+b\sqrt{-1}$ *(complex numbers) was advanced by* ***Johann Bernoulli****. However, the symbol* $\sqrt{-1}$*, was not satisfactory as it lead to paradoxes such as:* $-1=(\sqrt{-1})^2=\sqrt{-1}\sqrt{-1}=\sqrt{(-1)(-1)}=\sqrt{1}=1$*. To avoid this,* ***Leonhard Euler***[224] *introduced in* 1777 *the notation* i *with the basic property* $i^2=-1$*. The two roots of the equation* $x^2=-1$ *are now* $\pm i$*. The symbol* i *is called the imaginary unit. The choice of the word imaginary is unfortunate, but it indicates the distrust with which complex numbers were viewed. These suspicions slowly vanished at the end of the* 18^{th} *century, when* ***Caspar Wessel*** *in* 1797*,* ***Carl Friedrich Gauss*** *in 1799 (doctoral thesis) and* ***Jean Robert Argand*** *in* 1806*, gave simple geometric representation to complex numbers* $a+ib$ *as vectors (points) in the Cartesian plane.*

This simple interpretation of complex numbers made mathematicians feel much more comfortable with them, and their existence was slowly accepted.

With Euler began the study of functions and power series in a complex variable. He observed that the formal substitution of x *by* ix *in the exponential function*

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$

leads to

$$e^{ix} = \left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)+i\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right),$$

that is, $e^{ix}=\cos x+i\sin x$*. However, all these formal results were lacking in mathematical rigor and often led to paradoxes*[225]*.*

It was not until the 19^{th} *century that this naive approach to complex analysis was replaced by a rigorous treatment. In* 1833*,* ***William Rowan Hamilton*** *presented a paper before the Royal Irish Academy in which he*

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[224] For further reading, see:

- Nahin, P.J., *The Imaginary Tale*, Princeton University Press, 1998, 257 pp.

[225] For example, we know that the real-valued function $y=\tan x$ for $-\pi/2<x<\pi/2$ takes on all real values. Suppose that this function could be generalized so as to take on all complex values while retaining the ordinary law of the tangent of sums. There should then exist a complex number x_0 such that $\tan x_0=-i$. Thus for any complex number x with $\tan x\neq\pm i$, we should have $\tan(x+x_0)=\frac{\tan x+\tan x_0}{1-\tan x\tan x_0}=\frac{\tan x-i}{1+i\tan x}=-i$, which is absurd.

introduced a formal algebra of ordered pairs of real numbers, the rules of combination being precisely those given today for the system of complex numbers.

The founders of the theory of functions of complex variable (and of all analysis), were **Augustin Louis Cauchy**, *professor at the École Polytechnique in Paris (1848),* **Karl Weierstrass**, *professor at the University of Berlin (1864), and* **Bernhard Riemann**, *professor at Göttingen (1859). Cauchy introduced the concept of the complex line integral in* 1814 *and published his basic theorems on functions of complex variable in* 1825. *During the second half of the* 19^{th} *century, Riemann developed the theory of complex functions from a physico-geometrical standpoint, and Weierstrass developed it from a logically rigorous standpoint.*

The invention of set theory by **Georg Cantor** *at the end of the* 19^{th} *century helped enormously in the development of the foundations of complex analysis.*

Euler versus Bernoulli versus Leibniz

Nothing sheds more light on the state of knowledge in any given era than the issues debated among the scientists of that time. One of the investigations continued from the 17^{th} *century was the solution of polynomial equations. In this context, the question was raised whether an arbitrary polynomial with real coefficients can be decomposed into a product of linear factors (or a product of linear and quadratic factors with real coefficients, to avoid the use of complex numbers).* **Leibniz** *(1702) thought this was not possible and gave the example*

$$x^4 + a^4 = (x^2 - ia^2)(x^2 + ia^2) = (x + a\sqrt{i})(x - a\sqrt{i})(x + a\sqrt{-i})(x - a\sqrt{-i}),$$

claiming that no two of these four factors render a quadratic factor with real coefficients upon multiplication. Had he been able to express the square root

of i and $-i$ as ordinary complex numbers, he would have seen his error. Indeed, ***Nicholas Bernoulli*** *(1687–1759), a nephew of James and John, pointed out in* 1719 *that*

$$x^4 + a^4 = (a^2 + x^2)^2 - 2a^2x^2 = (a^2 + x^2 + ax\sqrt{2})(a^2 + x^2 - ax\sqrt{2})$$

[which meant that the function $(x^4 + a^4)^{-1}$ could be integrated in terms of trigonometric and logarithmic functions!]

Notwithstanding this result, Nicholas did not believe that his decomposition can be effected for every *polynomial with real coefficients. Euler, however, took the correct stand: In a letter to Nicholas of October* 1, 1742*, Euler affirmed (without proof) that a polynomial of* arbitrary *degree with real coefficients could be decomposed into linear and quadratic factors with real coefficients.*

Nicholas replied on December 15, 1742 *with an example of his own, which he said, contradicts Euler's assertion:*

$$f(x) = x^4 - 4x^3 + 2x^2 + 4x + 4 = (x - z_1)(x - z_2)(x - z_3)(x - z_4)$$

where

$$z_1 = 1 + \sqrt{2 + i\sqrt{3}}; \qquad z_2 = 1 + \sqrt{2 - i\sqrt{3}};$$
$$z_3 = 1 - \sqrt{2 + i\sqrt{3}}; \qquad z_4 = 1 - \sqrt{2 - i\sqrt{3}}.$$

Euler then showed that since $z_1 = \overline{z}_2$, $z_3 = \overline{z_4}$ (conjugate pairs), the factorization yields

$$f(x) = \left[x^2 - 2(1+p)x + (1+p)^2 + q^2\right]\left[x^2 - 2(1-p)x + (1-p)^2 + q^2\right],$$

where p, q are real and

$$z_1 = 1 + p + iq; \qquad z_2 = 1 + p - iq;$$
$$z_3 = (1-p) - iq; \qquad z_4 = (1-p) + iq.$$

Euler thus proved Nicholas wrong on this count, but he still lacked a general proof. The kernel of the problem of factoring a real polynomial into linear and quadratic factors with real coefficients was to prove that every such polynomial had at least one real or complex root. The proof of this fact, called the fundamental theorem of algebra*, became a major goal. Proofs afforded by* **d'Alembert** *and* **Euler** *were incomplete. The first substantial proof was given by* ***Gauss****, in his doctoral thesis (1799).*

Worldview XIII: Leonhard Euler

* * *

"If a nonnegative quantity was so small that it is smaller than any given one, then it certainly could not be anything but zero. To those who ask what the infinitely small quantity in mathematics is, we answer that it is actually zero. Hence there are not so many mysteries hidden in this concept as they are usually believed to be. These supposed mysteries have rendered the calculus of the infinitely small quite suspect to many people. Those doubts that remain we shall thoroughly remove in the following pages, where we shall explain this calculus".

* * *

"Mathematicians have tried in vain to this day to discover some order in the sequence of prime numbers, and we have reason to believe that it is a mystery into which the human mind will never penetrate".

* * *

[Upon losing the use of his right eye] "Now I will have less distraction".

* * *

On Euler

* * *

The whole form of modern mathematical thinking was created by Euler. It is only with the greatest difficulty that one is able to follow the writings of any author preceding Euler, because it was not yet known how to let the formulae speak for themselves. This art Euler was the first to teach.

F. Rudio

* * *

The Earth's Rotation (350 BCE–1765)

*The revolution of the earth about the sun and its rotation about its own axis were known to **Heracleides** (ca 355 BCE). During the next two centuries, the Greeks accumulated sufficient astronomical data which enabled them to discover the forced precession of the earth's axis of rotation. This astronomical lore was put in 'deep freeze' for some* 1700 *years and resurged in Europe after the Copernican revolution (1543).*

*By the time of **Euler** (1758) the following facts about the earth's rotation were known:*

(1) The inertial behavior of the earth is compatible with a figure that closely resembles an ellipsoid of revolution[226] *(spheroid), slightly flattened at the poles with ellipticity (flattening) of* $\frac{1}{297}$*, i.e. possessing an equatorial diameter greater by about* 43 km *than its polar diameter.*

The figure has its principal moment of inertia C, in the direction of the symmetry axis. The points at which the axis of symmetry pierces the surface of the earth is the geometric North Pole.

(2) The earth is not rotating at present about a principal axis. Therefore the axis of rotation is not fixed relative to itself. The points at which the angular velocity vector cuts through the earth's surface are the celestial poles of rotation. Vertically above these points stars would have no diurnal motion. (The star Polaris is near the celestial North Pole). This axis of spin[227] *is called the polar axis. The polar axis is inclined to the plane of ecliptic by* $23°27'$.

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[226] This is known as the 'reference ellipsoid'. It is generated by the rotation of the ellipse $R_1 = b(1 - \epsilon^2 \cos^2 \varphi_1)^{-1/2}$, where b is the semi-minor axis, R_1 is the geocentric radius, φ_1 is the geocentric latitude and $\epsilon^2 = (a^2 - b^2)/a^2$, where a is the semi-major axis $[a = 6378.388 \text{ km},\ b = 6356.912 \text{ km}]$ and the flattening is $(a - b)/a$.

[227] To an earthbound observer, the principal axes of inertia of the earth are fixed. The angular velocity vector $\boldsymbol{\omega}$ is fixed in magnitude and rotates about the major axis of inertia $\mathbf{e}_3$ (*free Eulerian precession*). In the *forced precession*, the inertia axis $\mathbf{e}_3$ rotates about the normal to the ecliptic, as seen to an observer outside the earth.

Spin is the component of the angular velocity vector in the direction of the greatest moment of inertia. Its value is $\omega_3 = 7.29211 \times 10^{-5}$ rad/sec.

(3) *According to the Eulerian theory of free precession, the tip of the angular velocity vector (relative to the body axes) describes a circle (the 'polohods') about the axis of symmetry. In the case of the earth, the shape of this curve is somewhat irregular, but its diameter never exceeds* 15 *meters, with a mean radius of* 4 *m. The period of revolution about the figure axis is about* 305 *days.*

(4) *Because of the earth's oblateness ('equatorial bulge') and the inclination of the polar axis just mentioned, the resultant attraction of the sun or moon does not pass through the earth's center of mass. The resultant attraction is therefore equivalent to a force acting through the center of mass and a couple which tends to bring the earth's equator into coincidence with the ecliptic. The earth, being an enormous gyroscope, reacts to this couple by a precession of its spin axis (roughly directed along its axis of symmetry) around a normal to the ecliptic (Newton, 1687). The precession due to the moon is more than twice as great as that due to the sun. This precession manifests itself through two equivalent motions:*

(a) *a slow conical motion of the earth's polar axis in space (about a normal to the ecliptic);*

(b) *a continual revolution of the line of nodes (line of equinoxes in astronomical usage, i.e. the intersection of the plane of the earth's equator with the plane of the ecliptic) in the plane of the ecliptic.*

Both motions have a period of about $25,800$ *years and amount to a rotation of* $50.4''$ *per year*[228]. *The diameter of the apparent circular motion at the earth's poles is about* 52 *cm which is about* 10 *times smaller than the corresponding amplitude of the Eulerian free precession.*

(5) *Superposed on the precessional motion of the principal inertial axis, there is an additional periodic irregularity, called forced nutation [from Latin nutare, to nod]. This appears as a small elliptical motion*[229] *about the mean pole, depending on the moon's inclination, with a period of about* 18.6 *years and amplitude of about* $9''$ *(**Bradley**, 1748).*

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228 The average precession rate per year due to the combined action of the sun and the moon is $-\frac{6\pi^2}{\Omega}\left(\frac{C-A}{C}\right)\cos\theta\left[1+\frac{m_L}{m+m_L}\frac{1}{\tau_1^2}\right]$ where Ω = angular velocity of the earth's rotation relative to the inertial frame of the fixed stars $= 2\pi \times 365\frac{1}{4}$ radians per year; $(C-A)/C = 0.0032$; $\theta = 23°27'$; $\frac{m_L}{m+m_L} = \frac{1}{82.5}$ (m = mass of earth; m_L = mass of moon); τ_1 = period of revolution of the moon around the earth (27.32 days) in units of solar years $= \frac{27.32}{365\frac{1}{4}}$. The total result is $-15.9'' - 34.5'' = -50.4''$ per year.

229 The major axis of this ellipse points towards the Pole of the ecliptic and is only $18''$ long, and the minor axis is $14''$ long. These are about the angular dimensions of a lemon seen from a distance of a kilometer.

The Enlightenment (Age of Reason 1687–1789)

Central Europe emerged from the calamity of the Thirty Years' War greatly diminished in population, badly disorganized economically and more than ever broken up into tiny political states. The largest of this states was Austria, whose king was usually also emperor of the so-called Holy Roman Empire*, though his powers as emperor were purely nominal. The next in size was Prussia, whose ruler (1700) changed his title of elector to that of King. Then there where hundreds of almost independent principalities, cities, bishoprics. The head of each of these, or its group of ruling patricians, exercised authority with few limitations, the common people having no say whatever in the government. Few of the rulers, moreover, had any idea how to improve the economic situation of their people. Their interests were confined to taxing, conducting war and living in as grand a style as possible.*

The burgers, on the other hand, were bound to the old ways of doing things and feared the slightest change in the methods of commerce and industry. Their guilds regulated every thing to a degree that it was impossible for trade to make any progress. England and Holland had broken with these old-fashioned methods and were rising rapidly in wealth and power, while the rest of Europe was standing still. Alone among the princes of the Continent, Frederick William, elector of Prussia (reigned 1640 – 1688), saw the need for reforming the economic conditions of his country by breaking the stranglehold of the guilds, thus starting Prussia on the road to military and political power. This power derived from the changes he forced in the economic order of his lands.

The one hundred years that preceded the French revolution witnessed the rise of kings to unmatched power and influence in European affairs. These years also encompassed the birth, maturation, and waning of the Enlightenment*. In the latter half of the period, during the rule of* "philosopher kings" *(the enlightened absolutists), the monarchical tradition and the new intellectual development was reflected. This period was brought to an abrupt end by the movement toward representative government and the stirrings of political and social revolts.*

The overwhelming success of the Newtonian physics and world-view induced the thinkers of the 18^{th} *century to apply the methods and principles*

of 17^{th} *century mathematics and physics to heal the economic, social, political and ecclesiastical elements of society. The Paris-centered* enlightenment *movement dates from the visit of* ***Voltaire*** *(1694–1778) to England (1726–1729), and the subsequent dissemination of the ideas of* ***Newton*** *and* ***Locke*** *(1632–1704) across the channel. Other leaders of the enlightenment were* ***Montesquieu*** *(1689–1755, France) [who tried to apply methods of the natural sciences to the study of governmental forms], and* ***Denis Diderot*** *(1713–1784, France).*

The period produced many important advances in the fields of astronomy, chemistry, mathematics and physics. Philosophers of the Age of Reason organized knowledge in encyclopedias *and founded scientific institutes. They explored issues in education, law, philosophy and politics, and attacked tyranny, social injustice, superstition and ignorance. Many of their ideas contributed directly to the outbreak of the American and French revolutions*[230].

The principal publication of the Age of Reason is the 'Encyclopédie'[231], *edited by* ***Diderot*** *and* ***d'Alembert*** *in* 17 *volumes of text and* 11 *volumes of illustrations during* 1751–1772. *This monumental endeavor became important in the democratization of scientific knowledge. Technology was given equal importance to that of pure science or philosophy. For the first time in the history of science a group of scholars and savants addressed their ideas in writing to a broad public.*

While science, literature and philosophy flourished in France, there was little evidence for such cultural activity in Germany; here, the setback caused by the Thirty Year's War (1618–1648) lasted until the second half of the 18^{th} *century. The like was true of Jewish culture: the ghetto imprisonment, the impoverishment and the terrors of war had not only destroyed schools, but also crushed the independence of spirit necessary for cultural progress*[232].

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[230] In the Middle Ages, opposite forces were held together by the pressure of the Church. As this pressure has diminished, the opposite forces rebelled against each other, leading to revolutions.

[231] It was by no means the first endeavor of its kind. The Chinese encyclopedia 'Yung lo ta tien' was written in the 15^{th} century.

[232] As odd as it may seen, the Enlightenment in France marks the beginning of modern secular anti-judaism. The torchbearers of this new age, namely **Voltaire**, **Diderot** and to a lesser degree **Jean-Jacques Rousseau**, carried over into the mainstream of Enlightenment thinking the medieval Christian stereotypes of the Jew.

Instead of disappearing with Enlightenment, antisemitism simply found new guise, one which no longer blamed the Jews for crucifixion of Christ but held them responsible for all the crimes and perversities committed in the name of

Other major figures of the enlightenment are: ***Swift*** *(1667–1745),* ***Berkeley*** *(1685–1753),* ***Buffon*** *(1707–1788),* ***Frederick the Great*** *(1712–1786),* ***Sterne*** *(1718–1768),* ***Helvetius*** *(1715–1771),* ***Winckelmann*** *(1717–1768),* ***Adam Smith*** *(1723–1790),* ***Lessing*** *(1729–1781),* ***Moses Mendelssohn*** *(1729–1786),* ***Burke*** *(1729–1797),* ***Priestley*** *(1733–1804),* ***Wieland*** *(1733–1813),* ***Coulomb*** *(1736–1806),* ***Gibbon*** *(1737–1794),* ***Galvani*** *(1737–1798),* ***Lavoisier*** *(1743–1794).*

The late enlightenment created [toward the end of the 18^{th} *century] a reaction to its materialism*[233] *and rationalism. It is generally called 'Romanticism' in France and England. In art, music and literature this reaction emphasized the great elemental motions and denied the supremacy of reason. Romanticism did not come to fruition in most countries until the first half of the* 19^{th} *century.*

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the monotheistic religions: The Jews were judged to be inherently perverse, and their 'fossilized' religion to be *an obstacle to human progress*!

Interestingly enough it was *Jews*, and non other, that helped Germany recover from the destructive effects of the Thirty Years' War by strengthening its shaky economy and catalyzing the diffusion of the spirit of the age of reason into Germany. Jews were being called out of the ghetto-prisons to help rebuild the lands ravaged by war. However the *Christian* population continued in its accustomed hostility, despite the new ideas which were slowly spreading among the new cultured classes.

As one of the means to achieve his ends, Frederick William of Prussia made use of the Jews. To overcome the opposition to the burgers (who, on the slightest suspicion of competition by Jews bestirred themselves to force the Jews out of the occupation involved) He granted the Jews certain trading privileges, and then used these privileges to bargain with the burgers, either to gain greater authority for himself or to make some change in general economic situation, which enhanced interstate and international commerce for the benefit of all concerned. Thus there emerged the figure of the Court-Jews who served the various princelings as financial agents and as civilian quartermasters for their armies. The two most noted of these were Samuel Oppenheimer of Vienna (1630–1703) and his distant relative Joseph Süss Oppenheimer ("*Jud Süss*") (1692–1738, Feb 04).

233 A philosophical doctrine which examined both nature and social life from a mechanistic point of view. Basing themselves on *mechanics*, which in those days was the height of science, materialist philosophers imagined that the same mechanical laws can be applied automatically to life and nature. Moreover, since these laws are immutable, society changes very little except for repeating itself mechanically via wars, hunger, government etc. Consequently, mankind can do nothing to change things.

The romantic era marked an interlude between the more disciplined, rationalist 'Weltanschauung' of the 18^{th} *century, which reflected the order and regularity of the Newtonian universe, and the science-oriented outlook that was to triumph in the second half of the* 19^{th} *century.*

At any rate, what remained of the romantic mood was shattered by the mid-century revolutions of 1848. *The abortive uprisings of that year seemed to prove that ideals were not enough, that in the last analysis physical force, material resources, and power were what counted in human relations*

1738 CE, Feb 04 **Joseph Süss Oppenheimer** (1692–1738). The Jewish Financial Minister of Karl Alexander, the Duke of Württemberg. He has been envied and hated for his role in planning and implementing radical economics reforms. Arrested after the Duke's death and placed in an iron cage suspended from a high beam over the Württemberg city square, for all to see and mock. When the mob tired of the spectacle, Oppenheimer was strangled. In 1939, the German Nazis made an anti-semitic movie named *Jüde Süss*, which was very successful all over Germany. The public murder of Oppenheimer took place in the middle of European *Enlightenment* (Age of Reason) in the days of Voltaire, Montesquieu, Diderot, Rousseau and Lessing.

1739 CE, Oct **Jose Antonio da Silva** (1705–1739, Brazil and Portugal). A Converso writer, was garroted and burnt at a Lisbon auto-da-fe (on charges of Sabbath observance) by the Inquisition – *Enlightenment* Portuguese style. His wife, who witnessed his death, did not long survive him. Da Silva's tragic story has inspired several modern writers, including the Portuguese Camilo Castelo Branco (author of the novel *O Judeu*), who was himself of Converso origin.

1740 CE **Benjamin Huntsman** (1704–1776, England) inventor and steel-manufacturer. Produced a satisfactory *cast steel*, purer and harder than any steel then in use[234]. Born to German parents in Lincolnshire. He started business as a clock, lock and tool maker at Doncaster, and attended a considerable local reputation for scientific knowledge and skilled workmanship.

[234] Steel had been made in small quantities even before Christian era. However, in 1722 **René Antoine de Réaumur**, a French physicist, learned how to make larger quantities by placing malleable iron in a bath of cast iron.

Finding that the bad quality of the steel then available for his products seriously hampered him, he began to experiment in steel manufacture, first at Doncaster and subsequently in Handsworth, near Sheffield, to where he moved in 1740 to secure cheaper fuel for his furnaces.

After several year's trials he at last produced satisfactory cast steel[235]. The Sheffield cutlery manufactures, however, refused to buy it, on the ground that it was too hard, and for a long time, Huntsman exported his whole output to France. The growing competition of imported French cutlery made from Huntsman's cast-steel at length alarmed the Sheffield cutlers (who vainly endeavored to get the exportation of steel prohibited by the British government) and compelled them in self-defense to use it.

Huntsman had not patented his process and its secret was discovered by a Sheffield ironfounder. Huntsman's business was subsequently greatly developed by his son **William Huntsman** (1733–1809).

1740–1747 CE **Moshe Hayyim Luzzatto**[236] (1707–1747, Italy Amsterdam and Israel; acronym RaMHaL). Philosopher, mystic moralist, accomplished linguist, poet and the progenitor of a Hebrew revival. His philosophy

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[235] *Steel* is a purified alloy of iron, carbon, and other elements that is manufactured in the liquid sate. Most steels are almost freed from phosphorus, sulfur, and silicon, and contain between 0.15 to 1.5 percent of carbon. High-carbon steels (0.70–1.5 percent) are used for making razors, surgical instruments, drills and other tools.

The *Crucible* is the oldest method of making steel, it is a small pot made of clay and graphite. A number of crucibles are placed on the hearth of a furnace, which is heated by gas. Carefully selected scrap is melted in these crucibles. Huntsman melted together pieces of iron and charcoal (Ca 20 kg) in a covered crucible for a few hours. The resulting steel, with a relatively high but evenly distributed carbon content, was exceptionally hard. Because he cast it in molds, Huntsman called it *cast steel*. However, small ingots of this size could not yet be used to build bridges and railways.

[236] Luzzatto (Luzzatti) is the name of Italian scholars that is derived from the province of *Lausitz* in Eastern Germany (Lat. *Lussatia*). According to tradition the family emigrated into Italy in ca 1450, settling in the Venetian territories. The earliest member of the family of whom there is a record is Abraham Luzzatto (1586); one of his sons settled in Safed, Israel.

During 1500–1900 CE, the Luzzatto family has provided an uninterrupted lineage of some 14 generations of creative scholars in many fields of human intellectual endeavor: philosophers, scientists, physicians, historians, statesmen, authors and religious leaders. Among them: (i) **Shmuel David Luzzatto** (acronym *SHaDaL*; 1800–1865), philosopher, philologist, translator, Bible commentator; (ii) **Luigi Luzzatti** (1841–1927); Prime Minister of Italy 1910–1911.

inspired millions of people in Europe and continues to be a living tradition in Judaism.

His treatise *The Path of the Upright* (*Mesillat Yesharim*, 1740) stands on it's owns as one of the most influential and inspirational ethical works of Judaism [alongside with the *Bible book of Genesis* (ca. 750 BCE); *Book of Job* (ca. 600 BCE); *Book of Ecclesiastes* (ca. 250 BCE); *Book of the Khazar* (Yehuda Halevi, 1139) and *The Guide for the perplexed* (Maimonides, 1190)][237]. Luzzatto was born in Padua, son of a wealthy merchant. Since his childhood prodigy, he became thoroughly knowledgeable in Judaic literature, classical and modern languages, contemporary Italian culture and the secular sciences. In his early poetry and dramas (1724–1727) he created a new school of Hebrew literature. But through 1727–1734 he began to lean toward Kabbalistic mysticism, becoming leader of a group of religious thinkers. This brought him to a direct conflict with the Venetian Rabbinate who, fearing a new messianic pretender, put Luzzatto under the ban (1734) for "*practicing sorcery and pronouncing incantations*".

Subjected thus to persecution and excommunication, Luzatto went to Amsterdam (1736) where he could freely teach and write on diverse topics, such as ethics, philosophy, poetry and Kabbalah. Like Spinoza before him, he earned

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[237] The spiritual giants of the Jewish people can be divided into two groups: One is called *"Ma'atikei Shmu'a"* – those who faithfully record what has been passed down to them for posterity. The other is composed of those who have tried to rewrite the tradition that passed down to them.

The members of the first group earned admiration, trust, and love during their lifetimes. The members of the second group were not trusted and were considered to be controversial; after their death, however, they where given unlimited admiration, to the point of becoming legends. It is to this second mysterious group that Maimonides and Rabbi Moshe Hayyim Luzzatto belong. An intellectual and tragic common denominator connects them: it was after their death that both earned total admiration, which has not diminished since. Based upon the following famous Talmudic passage Luzatto wrote *Mesillat Yesharim* in order to blaze a trail that man must follow to attain ethical perfection: Rabbi Pinchas ben-Yair says: (Mishna; *Sota*, **9**)

> "Watchfullness leads to alertness,
> Alertness leads to cleanliness,
> Cleanliness leads to abstinence,
> Abstinence leads to purity,
> Purity leads to saintliness,
> Saintliness leads to humility,
> Humility leads to fear of sin,
> Fear of sin leads to holiness."

his living by grinding optical lenses, but unlike him he remained ardently devoted to the cause of Judaism. However, he did not turn his thoughts from the mysticism that not only incited his loftiest aspirations but also inspired him to the conception of high ethical principle.

Indeed, it is in Amsterdam that he wrote his important philosophical treatise *Mesillat Yesharim* (The path of the Upright) on the path man must follow to attain ethical perfection[238]. This ethical work, written in Hebrew, became one of the most influential books read by Eastern European Jewry in the late 18^{th} and 19^{th} centuries.

In other ethical and theosophical works Luzzatto studied some basic theological questions: the ways of divine justice, the ways to overcome evil desires, prayer, the Commandments, relationship between the just and the sinner, original sin, the aim of creation, the next world and the world of redemption. All his works in this field were widely read and accepted, and contributed to his metamorphosis to sage and saint.

Luzatto visited London, but finally he was determined to escape from the prohibition to teach Kabbalah. Filled with longing for the Holy Land, and after many hardships he moved with his wife (m. 1731) and son to Safed, the Kabbalistic center in Israel at that time. He died of the plague on May 06, 1747 in Kfar Yassif near Acre and was buried at Tiberias beside Rabbi Akiva

Luzatto, though persecuted when alive, was accepted by the three main 19^{th} century Jewish movements, which were fighting bitterly among themselves: the *Hassidim* saw him as a saintly mystic and used some of his Kabbalistic ideas. Their opponents, the *Mitnaggedim* regarded his ethical works as the clearest pointers toward a Jewish ethical way of life; and the enlightenment (*Haskalah*) writers saw Luzzatto as a progenitor of their own movement, and his works as the beginning of Hebrew aesthetic writings. Every facet of

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238 This treatise has been compared to John Bunyan's *The Pilgrim's Progress* (1675), though it was not influenced by the latter. Though written in the 18^{th} century, *Messilat Yeshrim* is essentially a medieval book, for Jewish medievalism outlasted European medievalism by almost 400 years. The work was printed many times, and translated into many languages.

A common groundless accusation against Judaism, which is repeated *ad nauseam*, is that Judaism was nothing but a formal system of practices which exacted outward conformity regardless of inner meaning of mind and heart. This misleading disinformation, which is all too opt to be accepted uncritically, is shuttered in the face of the vast ethical literature which the Jews have produced. *Messilat Yesharim* cultivates the inwardness of the laws and duties which the Jew has to live up.

Luzatto's work, therefore, remained alive and creative in the divided and confused Jewish culture of 19 th century.

1740–1744 CE **Pierre Louis Moreau de Maupertuis** (1698–1759, France). Mathematician and astronomer.

Was born in St. Malo. At 20 he entered the army, becoming lieutenant in a regiment of cavalry and spending his leisure on mathematical studies. In 1723 he quit the army and was admitted as a member of the Academy of Sciences. In 1728 he visited London and was elected a fellow of the Royal Society. In 1736 he was the head of an expedition sent by Louis XV into Lapland to measure the length of a degree of the meridian for the sake of determining the oblateness of the earth. On the basis of these measurements he found that the earth is flattened at the poles and oblate at the equator, as predicted by Newton. His findings corrected earlier results of Cassini.

In 1740 Maupertuis went to Berlin on the invitation of the King of Prussia, and took part in the battle of Mollwitz where he was taken prisoner by the Austrians. Returning to Berlin in 1744 at the request of Frederick II, he was chosen president of the Royal Academy of Sciences.

In 1744, he stated his "*principle of least action*" and applied it to optics and mechanics. He believed that it is a mathematical principle through which nature acts in the grand scheme of the universe to secure greatest economy.

Maupertuis was a man of considerable ability, but his restless, gloomy disposition involved him in constant quarrels, of which his controversy with Voltaire during the latter part of his life furnishes an example.

1741–1765 CE **Johann Peter Süssmilch** (1707–1767, Germany). Prussian regimental pastor and a pioneer in the field of population statistics. In his book (1761): "*Die göttliche Ordnung in den Veränderungen des menschlichen Geschlechts aus der Geburt, dem Tode, und der Fortpflanzung desselben erweisen*" he made a systematic attempt to make use of a class of facts which up to that time had been regarded as belonging to "political arithmetic" (today — "vital statistics").

In his book, Süssmilch investigated whether war and plague were part of God's plan[239] for the decimation of human surplus on earth. To this end he

[239] In this he was influenced by a paper (1710) of **John Arbuthnot** (1667–1735) called "*An argument for Divine Providence, taken from the constant regularity observed in the birth of both sexes*". In this note Arbuthnot (Physician to Queen Anne during 1709–1714, satirical writer and collaborator of Jonathan Swift) claimed to demonstrate that divine providence, not chance, governed sex-ratio of birth.

estimated that the earth had then a billion people (an overestimate), and calculated that the population could grow for centuries before it reached the maximum number of people the earth could support. This number he estimated at 13.9 billion. Süsmilch then piously concluded that war and pestilence were *not* part of the divine plan for reducing human population.

Süssmilch had arrived at a perception of what has been later termed the "*laws of large numbers*". He endeavored to form a general theory of society, based upon quantitative aggregate observation. Although he did not enter his investigation with an open mind, his work was nevertheless a most valuable one since it *pointed out the road* which other unbiased researchers were not slow to follow. Thus, Süssmilch's success was the origin of a mathematical school of statisticians[240].

How Many People Have Ever Lived on Earth?

Leeuwenhoek *(1769) published the first quantitative estimate of 13.4 billion people. Since then, 65 different estimates have been made, ranging from 1 billion to 1000 billion, depending upon initial assumptions made.* ***Süssmilch*** *(1761) gave an estimate of 13.9 billion.*

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240 The word *statistics* is derived from the Latin *status*, which in the Middle Ages came to mean a *state* in the political sense, denoting inquiries into the condition of a polity.

As human societies became more and more organized, a considerable body of official statistics came into existence, and intended to aid administration. The Romans were careful to obtain accurate information regarding the resources of the state, and they appear to have taken the census with a regularity which has hardly been surpassed in modern times. The material for statistics therefore existed at a very early period, but it was not until within the last four centuries that systematic use of the information available began to be made for purposes other than mere administration.

Statistics in the modern sense of the word, did not really come into existence until Süssmilch's publication.

The total number of people who have ever lived on earth can be estimated in a reasonable way as follows:

One begins by determining the mean population size for a birth-death stochastic process (i.e., the average behavior of a population whose size varies stochastically, growing over time due to random occurrence of births and deaths). One then assumes a starting population of two persons 1.5 millions years ago and divides the total time span into a number of smaller subintervals by using times for which estimates of world population have been made (e.g., N *(8000 BCE)* $= 5 \times 10^6$*;* N *(0 BCE)* $= 250 \times 10^6$*;* N *(1750 CE)* $= 800 \times 10^6$*;* N *(1825 CE)* $= 10^9$*;* N *(1930)* $= 2 \times 10^9$*;* N *(1960)* $= 3 \times 10^9$*;* N *(1980)* $= 4.4 \times 10^9$*). The total number of people who ever lived since 1.5 million years before present is then found to range from 50 to 100 billion* (10^{11})*.*

One can also show that the number of people living today (2008 CE) is almost equal to the number of offsprings from a single pair of parents (The primordial Adam and Eve of, say, the Homo Sapiens branch) 140,000 years ago.

1742 CE **Christian Goldbach** (1690–1764, Germany and Russia). Mathematician. Made notable contributions to the theory of infinite series and the integration of differential equations, but is mainly known on account of the *Goldbach conjecture*[241]: in a letter to **L. Euler** (1742) he claimed: (1) Each *even* positive integer $n > 2$ is expressible as a sum of two primes. (2) Each positive integer greater than 2 can be represented as a sum of three primes.

Goldbach was born in Königsberg, Prussia. He became a professor of mathematics at St. Petersburg (1725). In 1728 he became a tutor to Tzar Peter II, and from 1742 on served as a staff member of the Russian Ministry of Foreign Affairs.

[241] The first conjecture was found valid up to $n = 100,000$, but no definitive proof has been found. In 1937, **Ivan Matveyevich Vinogradov** (1891–1983, Russia) gave a partial proof of the second conjecture, restricting n to be *a sufficiently large odd number* $(\geq 3^{3^{15}})$.

1742 CE **Benjamin Robins** (1707–1751, England). Military engineer. Laid the groundwork for modern ballistic theory. Invented the *ballistic pendulum*[242], first described in his *New Principles of Gunnery* (1742).

Artillery men of 18^{th} century endeavored to improve their cannon-firing with little success. Robins noted that one of the causes of imperfection was the deflection of the bullet's path due to *friction against the bore of the gun.* He suggested remedying this by *scoring the bore longitudinally.*

Euler rejected both Robin's *observations* and his solution. It was more than a century before they were recognized as fully justified[243].

1742 CE **Anders Celsius** (1701–1744, Sweden). Astronomer. Described the centigrade thermometer in a paper read before the Swedish Academy of Sciences. Born in Uppsala. Occupied the chair of astronomy in the university of his native town (1730–1744), but traveled during 1732 and some subsequent years in Germany, Italy and France. In Paris he advocated the measurement

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242 *Ballistic pendulum*: A device used to measure the velocity of such projectiles as bullets, arrows, and darts by applying the *momentum principle.* It consists of a rather massive block of wood that is suspended by parallel cords and is initially hanging at rest. A test projectile (e.g., a bullet) is fired horizontally into the block, which is thick enough to bring the bullet to rest, embedded inside it (inelastic collision). The block and embedded bullet swing up to a maximum deviation h. From the known masses and h, the final velocity of the bullet is $v = \frac{m+M}{m}\sqrt{2gh}$. Because h is generally small and difficult to measure, this result is expressed in terms of x_m (the maximum horizontal displacement) and L (the length of the pendulum chord). Thus, for $x_m^2 >> h^2$ we find $h \cong \frac{x_m^2}{2L}$, $v = \frac{m+M}{m}x_m\sqrt{\frac{g}{L}}$. Note that during the stopping time of the bullet, momentum is conserved, but kinetic energy is *not*, whereas later, as the pendulum begins to swing, energy is conserved, but the momentum of the pendulum of the block changes due to the unbalanced forces that then begin to act.

If $m = 10$ g, $M = 3$ kg, $x_m = 25$ cm, $L = 1$ m, calculations yield $v = 235$ m/sec.

Momentum conservation allows us to obtain a result for the velocity of the bullet even though the *force* exerted on the bullet by the block during the stopping time is extremely complicated (even unknowable). To obtain the result from the time-developed equations of motion, using Newton's second law, would be exceedingly difficult (even impossible).

243 **Euler**, like most of his contemporaries, adopted the wrong philosophy of explaining science rather then observing it. He thus made an impressive blunder which halted the progress of ballistics for a hundred years. There are innumerable examples of this type throughout the history of science.

of an arc of the meridian in Lapland, and took part, in 1736, in the expedition organized for that purpose by the French Academy.

1742–1747 CE **Jean Le Rond d'Alembert** (1717–1783, France). Mathematician, physicist and man of letters. Pioneer in the study of partial differential equations and their application in physics. Studied the equilibrium and motion of fluids, hydrodynamics, mechanics of rigid bodies, the 3-body problem in astronomy and atmospheric circulation. In science he is remembered for his four contributions:

(1) *d'Alembert's principle* (1742): by introducing the concept of 'force of inertia', which is created by the body's own motion, it is possible to reduce problems of motion to problems of equilibrium in the body's co-moving frame of reference. d'Alembert went further to generalize the *principle of virtual work* to all mechanical systems, thus furnishing a bridge between the Newtonian formulation of the laws of mechanics and the later Lagrangian formulation.

The first veiled formulation of the *principle of virtual work* is contained in the *Physics* of **Aristotle** (384–322 BCE). However, he used *virtual velocities* rather than *virtual displacements*, and this is the form in which the principle was used up to the 19^{th} century. Aristotle derived the law of the lever from his principle and **Stevinus** (1548–1620) used it to deduce the equilibrium of pulleys. **Galileo** (1564–1642) improved the formulation of Aristotle's principle by recognizing that it is not the velocity, but rather the velocity in the direction of the force which counts.

His method amounts to the recognition of the "*work*" as the "product of the force and the displacement in the direction of the force". He applied the principle of virtual work to the equilibrium of a body on an inclined plane, and showed how his principle gives the same result that Stevinus found on the basis of the energy principle. **Johann Bernoulli** (1667–1748) was first to formulate the principle of virtual work as a general principle of statics with which problems of equilibrium could be solved (1717).

The principle states: "If a system of n material points $A_1, \dots, A_n$ is without friction, then the necessary and sufficient condition for the equilibrium of the acting forces $\mathbf{F}_1, \dots, \mathbf{F}_n$, is that to every virtual displacement $\delta\mathbf{r}_1, \dots, \delta\mathbf{r}_n$ the inequality $\sum_{j=1}^{n} \mathbf{F}_j \cdot \delta\mathbf{r}_j \le 0$ holds for the virtual work of the acting forces" [virtual displacements = infinitesimal displacements *possible* at a point A]. The importance of the principle consists in the fact that it gives a condition of equilibrium of the acting forces without the aid of reactions. In the d'Alembertian formulation the principle of virtual work assumes the form: $\sum_{j=1}^{n}(\mathbf{F}_j - m_j\ddot{\mathbf{r}}_j) \cdot \delta\mathbf{r}_j = 0$.

(2) Discovered the scalar wave equation of a vibrating string and found its general solution (*d'Alembert's solution*) (1747).

(3) Calculated the perturbation of the planets on the orbits of the moon and the earth. This theory was previously developed by Newton from a geometrical standpoint. d'Alembert and Clairaut, each in his own way, formulated the result in the form of series solutions of differential equations (1747–1754).

In 1754 d'Alembert developed the mathematical theory of the perturbing effects of the planets (mainly Jupiter) on the motions of earth. He showed that because of these perturbations, the luni-solar precessional period of 26,000 years [known to **Hipparchos** (120 BCE), and shown by Newton (1687) to be caused by the gravitational pull that the sun and the moon exert on the earth's equatorial bulge], must be modified to include *precession of the earth's perihelion* ($47''$ per century) to yield a *general precession of the equinoxes* with a period of 22,000 years.

This must be understood as follows: In the absence of planetary perturbations, the plane of the earth's equator turns in a retrograde direction about the normal to the ecliptic, with the latter regarded as *fixed* relative to the fixed stars. The rate of this nearly uniform rotation of the earth's spin axis is about $50''$ per year, making a full revolution in $\frac{60\times 60\times 360}{50} = 26,000$ years. During this motion, the obliquity of the earth's axis w.r.t. the ecliptic is unchanged, only the orientation of the polar axis relative to the fixed stars varies [however, the lunar forced *nutation* (Bradley, 1737) must, of course, be considered as well].

When planetary perturbations are introduced, the elliptic orbit, as a whole, executes a slow rotation, known as the *precession of the perihelion*. Consequently, as seen from earth, the curve described by the North Pole, is not quite a perfect circle, and it does not close back on itself during 26,000 years but rather sooner, in 22,000 years.

(4) Was first to notice (1754) the unsatisfactory state of the foundation of analysis and see that a theory of *limits* is needed. The actual process of banishing intuitionism and formalism from analysis started in 1797 with **Lagrange**. This led in the 19^{th} century to the *arithmetization of analysis*.

His main lifework was his collaboration with Diderot in preparing the famous *Encyclopédie*, which played a major role in the French enlightenment by emphasizing science and literature and attacking the forces of reaction in church and state.

d'Alembert was a foundling: having been abandoned near the church of St. Jean le Rond, Paris, he was discovered on the 17^{th} of November, 1917.

It afterwards became known that he was the illegitimate son of a Parisian notable. He was called Jean le Rond after the church near which he was found; the surname d'Alembert was added by himself at a later period. In 1730 he entered the Mazarin College, where his exceptional talents were soon noted. His knowledge of higher mathematics was acquired by his own unaided efforts after he had left college.

On leaving college he returned to the house of his foster mother, where he continued to live for 30 years. He studied law and medicine but in 1740 resolved to fully devote his time to mathematics. His association with Diderot in the preparation of the *Dictionaire Encyclopédique* led him to take a somewhat wider range than that to which he had previously confined himself (1754). Apart from contributing mathematical articles to the Encyclopedia, he wrote literary and philosophical works which extended his reputation but also exposed him to criticism and controversy. d'Alembert was interested in music both as a science and as an art. His fame spread rapidly throughout Europe: Frederick the Great, Catherine of Russia, and Pope Benedict XIV each invited him to live in their respective country on lucrative salaries, but he preferred to stick to the quiet and frugal life dictated by his simple tastes.

His latter years were saddened by circumstances connected with a romantic attachment to a noted consort of literary men and savants. On her part there seems to have been nothing more than a warm friendship, but his feelings toward her were of a stronger kind and her death in 1776 deeply affected him.

The chief features of d'Alembert's character were benevolence, simplicity and independence. Though his income was never large, and during the greater part of his life was very meager, he continued to find means to support his foster mother in her old age, to educate the children of his first teacher and to help various deserving students during their college career. His conversation was a singular mixture of feigned malice, goodness of heart and delicacy of wit.

1743–1750 CE **Jean Antoine Nollet** (1700–1770, France). Physicist. Invented the first *electroscope* (1747–1750). Discovered and described the phenomenon of *osmosis* and *osmotic pressure* (1748).

Osmosis is derived from a Greek word, meaning to *push*. A membrane partition separates pure water, say, from a weak solution of a substance (solute) in water (solvent). The membrane is such that the molecules of the water can pass through it, but not those of the solute (a semi-permeable membrane). After some time the level of the pure solvent (water) becomes *lower* than the level of the solution. The process of penetration of a solvent through a semi-permeable membrane is called *osmosis*. The pressure difference created between the two sides of the membrane is called *osmotic pressure*. The new

state of equilibrium can be understood either as due to the entropy gain attendant to the further dilution of the solute (*thermodynamic* argument), or, alternatively, as an impairment of the rate of effusion of water molecules from the *solution* to the *pure-water* side, caused by the presence of solute molecules in the former.

The exact microscopic mechanism of this impairment is not completely understood. Apparently, a complex interaction between the molecules and the semi-permeable membrane is at work. The magnitude of the osmotic pressure depends on the concentration of the solute molecules; the greater the concentration, the higher the osmotic pressure difference. On the other hand, for a given *weight* of solute, the lower the osmotic pressure, the higher the *molecular weight*. This enables one to determine molecular weights, of proteins say, through osmotic pressure measurements.

Nollet was born near Noyon (Oise) to a peasant family. His parents destined him for the clergy, but after finishing his theological studies in Paris, he came under the influence of **Réaumur**, and began the study of the exact sciences.

In the Church he ultimately attained the rank of abbé, but his tastes lay in the direction of experimental physics. In 1734 he was admitted as member of the London Royal Society, and in 1739 he entered the Academy of Sciences at Paris. In 1753 he was appointed to the newly instituted chair of experimental physics in the College de Navarre. He discovered osmosis while experimenting with water diffusing into sugar solution from which it is separated by an animal membrane.

1743–1750 CE **Thomas Simpson** (1710–1761, England). An able self-taught mathematician. Was active in perfecting trigonometry as a science. His name is preserved in the so-called *Simpson's rule* published in his *Mathematical Dissertations on a Variety of Physical and Analytical Subjects* (1743) — a rule for approximate quadrature using parabolic arcs (this result appeared in somewhat different form in 1668 in the *Exeritationes Geometricae* of **James Gregory**).

Simpson's father was a weaver, and, intending to bring his son up into his own business, took little care of the boy's education. Young Simpson was so eager for knowledge that he neglected his weaving, and in consequence of a quarrel was forced to leave his father's house. Until 1743 his life was rather turbulent; he managed to sustain himself through a gamut of odd jobs as fortune-teller, oracle, astrologer and private teacher. After publishing 5 books on mathematics he was finally appointed professor of mathematics in the Royal Military Academy at Woolwich, and in 1745 he was admitted as fellow of the Royal Society of London.

1744 CE **Jean Philippe Löys de Chéseaux** (1718–1751, Switzerland). Astronomer. 'Solved' the riddle of the dark night sky by assuming that starlight is slowly *absorbed* while traveling across the immense gulfs of interstellar space in a boundless universe. In his essay "*On the intensity of light, its propagation through the ether, and the distance of the fixed stars*" (1744) he wrote:

"*The enormous difference between this conclusion and experience demonstrates either that the sphere of fixed stars is not infinite but actually incomparably smaller than the finite extension I have supposed for it, or that the intensity of light decreases faster than the inverse square of distance. This latter supposition is quite plausible, it requires only that starry space is filled with a fluid capable of intercepting light very slightly*".

Chéseaux' calculation, in the updated form given by **Lord Kelvin** (1901), is as follows: we assume that all stars are sun-like, of radius a, and uniformly distributed with density n per unit volume. The number of stars in a shell of radius $q \gg a$ and thickness dq approximately equals $4\pi nq^2dq$, and the *sum of their uneclipsed projected areas* is this number multiplied by πa^2, thus giving $4\pi^2na^2q^2dq$; If we divide this area of stellar disks by the area $4\pi q^2$ of the shell, we find that *the fraction of the sky covered by the stars* is of the shell $n\pi a^2dq = n\sigma dq$, where $\sigma = \pi a^2$ denote the geometric cross-section of a star.

We now integrate out to distance r and find that the fraction of the sky covered is $\alpha = n\sigma r = \frac{r}{\lambda}$, where $\lambda = \frac{1}{n\sigma}$ is the *mean free path* of a light ray traced backwards from a point on earth, terminating on the surface of the star which emitted it. (The mean free path — a term commonly used in the kinetic theory of gases — is the average distance a particle travels between collisions.) When $r = \lambda = \frac{1}{n\sigma}$ (or $\alpha = 1$), the whole sky is covered with stars. The corresponding distance is known as the *background limit*. If $V = \frac{1}{n}$ is the average volume occupied by one star, the background limit in a star-filled universe is $\lambda = \frac{V}{\sigma}$.

Let $N = 4\pi n\frac{r^3}{3}$ stand for the number of stars out to distance r. The number of uniformly scattered stars needed to cover the entire sky is obtained by inserting $r = \lambda = \frac{V}{\sigma}$ in the above equation. Then $N = \frac{4\pi V^2}{3\sigma^3}$. Even in an infinite universe containing an infinite number of stars, we see only out to a *finite distance* and a *finite number of stars*.

Assuming then for simplicity that all stars are similar to the sun in size and luminosity, we take $\sigma = 1.5 \times 10^{12}$ km^2. Noticing that there are

about 10 stars within a distance of 10 light-years from the sun, we obtain $V \approx 100$ (light-years)^3. Consequently[244] $\lambda \approx 6 \times 10^{15}$ light-years, and $N \sim 10^{46}$.

It was the immensity of the background limit that prompted Chéseaux to think that absorption in space, even the slightest, would veil the most distant stars and create the observed dark night sky. Indeed, the above toy model can easily be made to incorporate the effects of both *geometric overlap* by stars of intermediate shells and *absorption*. To this end we multiply our former expression for the fraction of the sky covered by stars in the shell first by $e^{-q/\lambda}$, where $\lambda = \frac{1}{\pi n a^2}$, and then by $e^{-q/\mu}$, where μ is the absorption mean free path. Hence, the fraction of the sky covered by stars in the shell is $\left\{\frac{dq}{\lambda}\right\}e^{-q(\frac{1}{\lambda}+\frac{1}{\mu})}$. Integrating from $q = 0$ to $q = r$, we find $\alpha = \frac{\mu}{\lambda+\mu}\left[1 - e^{(\frac{1}{\lambda}+\frac{1}{\mu})}\right]$, where α is the fraction of the sky covered by unobscured stellar surface of effectively-solar apparent brightness. As $r \to \infty$ in an unlimited and uniform universe, the fraction of the sky covered by stars becomes $\alpha = \frac{\mu}{\lambda+\mu}$. If the *absorption limit* μ is much less than the overlap limit λ, $\alpha \approx \frac{\mu}{\lambda} \ll 1$, and most stars are obscured from view.

Because a system of concentric shells may be constructed about *any* point in space, always yielding the *same* result, we conclude that observers at all places will perceive the sky as consisting of one continuous, though dimmed, stellar surface.

Chéseaux was born in the Swiss village of Chéseaux near Lausanne, the son of a landowner of modest wealth. Educated by his grandfather, the mathematician **Jean-Pierre de Crousaz**, he developed an interest in astronomy while a youth and constructed his own observatory. At age 17 he wrote papers on the physics of collisions, retardation of cannonballs by air resistance, and sound propagation. Never very robust, he died while on a visit to Paris, at the age of 33.

Chéseaux' name is associated with the magnificent comet of 1744, one of the finest of the 18th century. Though not discovered by him, this comet is often referred to as Chéseaux' because he computed its orbit and ephemeris and described its impressive, multiple tails.

ca 1745 CE During the *Silesian wars* between Prussia (Frederick the Great) and Austria (Maria Theresa), the latter saw a disgrace in her loss of Silesia. Deeply frustrated, the Austrian empress blamed her military defeat on the *Jews of Prague*(!) and on Dec. 18, 1744 she ordered their expulsion

[244] Note, however, that in this estimate Chéseaux uses the average star density in our galaxy — which is far higher than the average for the Universe as a whole.

from the country within six weeks. Thus, in the blistering cold of a February day, thousands of people of all ages were clogging the highways and the roadside was lined with the dead and the sick. Eight centuries of continuous Jewish community life and of dynamic intellectual striving were wiped out. It is indeed odd that this most dramatic expulsion should have taken place not during the Dark Ages but in the year 1745 when the Industrial Age and the modern spirit had already made their appearance in Europe.

Three years later the Imperial Treasury in Vienna began to feel acutely the financial loss resulting from the expulsion. This made the empress regret her excessive resentment against the Jews. Consequently, the Imperial Military Council denounced their own previous charges against the Jews and ordered their immediate return to Prague.

ca 1745 CE **Hugh Jones** (1692–1760, North America[245]). Mathematician. An ardent advocate of the octary (radix eight) system (which is used today in connection with certain electronic computers). Jones, a professor of mathematics at the College of William and Mary, was a reformer who asserted that the base eight makes fractional work simpler because octary fractions are just a matter of repeated halving. Moreover, computation is facilitated because the radix eight is a perfect cube, and four, which is one-half of the radix, is a perfect square.

1745–1746 CE **Ewald Georg von Kleist** (1700–1748, Germany) and **Pieter van Musschenbrock** (1692–1761, Holland) independently invented the *Leyden Jar*, an early version of an *electrical capacitor*.

Von Kleist was a German ecclesiastic and scientist. Dean of the Cathedral of Kamin, Pomerania. He discovered it on 04 Nov, 1745. Van Musschenbrock was a Dutch mathematician, physician and physicist. Von Kleist was member of a notable Leiden family[246] of instrument makers (air pumps, microscopes, and telescopes). He was a professor at Duisburg (1719–1723), Utrecht (1723–1740), Leiden (1740–1761). The jar device accumulates electrical charges produced by a static machine. When voltage reaches a critical value, there occurs a discharge through the air-gap. Theirs was the first working model of an electrical storage device.

1745–1785 CE **George Louis Leclerc de Buffon** (1707–1788, France). Naturalist and mathematician. Although not a profound original investigator,

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[245] At this time, a British colony.

[246] To this illustrious family of scientists, soldiers and poets belong also: **Ewald Christian von Kleist** (1715–1759), poet and soldier; **Heinrich Wilhelm von Kleist** (1777–1811), a great dramatist, poet and prose writer; **Paul Ludwig Ewald von Kleist** (1881–1954), army general in WWII.

he is remembered today due to his initiation of some geometrical aspects of probability[247]: he showed how to get *experimental* estimates of π by tossing a needle across a grid a large number of times (1777).

Previously, in 1745, he advanced the first of the so-called *catastrophic theories* which envision a solitary sun disrupted by some singular cataclysmic event. Buffon suggested that a massive body (comet) passed so close to the sun that its gravitational pull drew material out of it, which then condensed to form the planets (1785). He also tried, for the first time, to determine the age of the earth (his result: 74,832 years!).

He regarded (1749) spermatozoa as "living organic molecules" which multiply in the semen.

Buffon was born at Montbard (Côte d'Or). He studied law and mathematics at the Jesuit College at Dijon. Being a rich nobleman he led a life of a scientist-at-large, occupying himself with whatever he liked. His son, an army officer, died by the guillotine at the age of thirty in 1793.

Buffon was a member of all the learned societies of Europe. He was known during his time chiefly due to his great work (44 quarto volumes) on natural history, the publication of which extended over 50 years.

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247 *Buffon's Needle problem*: A table of infinite expanse has inscribed on it a set of parallel lines spaced a units apart. A needle of length $l < a$ is twirled and tossed on the table. What is the probability that when it comes to rest it crosses a line?

What matters is the needle's *angle* θ with the horizontal, and the distance x of the needle's-center from its nearest parallel. Since the needle's-center is equally likely to fall anywhere between the parallels, then for a fixed θ, the chance that the line crosses on of the parallels is $\frac{2x}{2a}$, because the line crosses a parallel if the center falls within x units of either parallel. On account of the twirling, the angle θ might be thought of as uniformly distributed from 0 to $\frac{\pi}{2}$ radians, because crossing that happens for angle θ also happens for angle $(\pi - \theta)$. All we need then is the mean value of $\frac{x}{a}$, or, since $x = l\cos\theta$, the mean value of $\left(\frac{l}{a}\right)\cos\theta$, which is equal to $\frac{l/a}{\pi/2}\int_0^{\pi/2}\cos\theta d\theta = \frac{2l}{\pi a}$. This is indeed the desired result. In the particular case in which $2l = a$, the probability of intersection is $1/\pi$. When the needle is tossed N times onto the ruled plane and on n of these occasions the needle intersects on of the lines, the *Law of Large Numbers* dictates $\frac{n}{N} \approx \frac{1}{\pi}$.

From Divination through the Buffon Needle to Monte Carlo Methods

*Throughout history simple games of chance have been used to communicate with, or seek guidance from, the supernatural. In some primitive societies a person's innocence or guilt was determined by drawing or casting lots [Joshua **7**, 14–18; **18**, 10; Jonah **1**, 7; I.Chronicles **26**, 13–14]. In ancient Greece and Rome oracles based predictions on casts of* astragali *(forerunners of modern dice), and in the Bible there is a reference to an occasion where the direction in which the army was to proceed was determined by shaking arrows in a quiver and observing the direction in which the first one fell [Ezekiel **21**, 26–28: "For the king of Babylon stood at the parting of the way, at the head of the two ways, to use divination"; also in the Talmud; Gittin **5**, 56 p. 1].*

*These early crude attempts to generate random (and sometimes 'rigged'!) events, were later replaced by rolling dice or flipping coins. Yet the harnessing of a gambling device in solving a problem of pure mathematics had to await the year 1777 CE, when the French naturalist **George de Buffon** showed that if a very fine needle of length l is thrown at random on a board ruled with equidistant parallel lines, the probability w that the needle intersect one of the lines is $\frac{2l}{\pi a}$, where $a > l$ is the distance between the parallel lines.*

*The problem and its solution were largely forgotten for the next 35 years, until the great French mathematician **Pierre Simon de Laplace** (1812) called attention to it and gave it a new twist. Writing Buffon's result in the form $\pi = \frac{2l}{aw}$, Laplace realized that it provides a new method of calculating π!*

Indeed, the remarkable thing about this result is that it involves the constant $\pi = 3.1415926\ldots$, which can be thus estimated *by actually tossing a needle on a board suitably ruled with parallel lines. Early experiments of this kind (1850) gave the probability $w = 0.5064$, based on 5000 throws with a needle 36 mm long and a distance of 45 mm between the parallels. This yielded $\pi \approx 3.151496$. Note that a* probabilistic approach *has been used here to solve a* non-probabilistic *problem, very far from the ancient divinations, where other non-probabilistic problems were solved by interpreting chance happenings as divine intent.*

It is not difficult to calculate the probability of obtaining π correct to K decimals in N throws. The result of such calculation show that this method is very inefficient as far as the numerical computation of π is concerned. Yet, Laplace had discovered a powerful method of computation that did not come into its own until the advent of electronic computers. The method that

Laplace proposed consists of finding a numerical value by realizing a random event many times and observing the outcomes experimentally.

A similar procedure, though non-geometrical, was devised for estimating e. It is based on the observation that if $2K$ uniformly distributed numbers x_i are independently drawn in a sequence from a random source, and assuming each draw value to be uniformly distributed, then the probability that they are all in ascending order $x_1, x_2, x_3, \ldots, x_{2K}$ is $\frac{1}{(2K)!}$. This is so because $2K$ numbers can have $(2K)!$ possible orderings and only one ordering will be an ascending one. The probability that a sequence of trials yielding an increasing sequence of x_i's will fail on the odd trial $2K+1$ is the difference $\frac{1}{(2K)!} - \frac{1}{(2K+1)!}$.

Thus, the total probability that a sequence of drawings of random numbers from an equilikely source will produce a rising sequence that ends with an even number of numbers is

$$\sum_{k=1}^{\infty}\left[\frac{1}{(2k)!} - \frac{1}{(2k+1)!}\right] = \sum_{k=0}^{\infty}\left[\frac{1}{(2k)!} - \frac{1}{(2k+1)!}\right] = \frac{1}{e}$$

An experiment using 252 runs gave $\frac{1}{e} = 0.381$, a 3.5 percent error.

This result serves as a basis for a winning strategy of a well-known game in which N cards, assigned with random numbers, are uncovered one at a time by a player. The player must announce his decision of whether or not an uncovered card bears the largest number of the lot. The optimal strategy is to delay decision until after $\left(\frac{100}{e}\right)$ percent of the cards have been uncovered.

In recent decades, the practice of chance happenings [namely, statistical experiments] has become quite respectable through the use of the so-called Monte Carlo methods[248]*; we should add, though, that there is no longer any question of divine intervention. Monte Carlo methods are essentially simulation techniques, and they enable us to study in the classroom or in the laboratory random processes which would otherwise be difficult to observe.*

They have been used, for example, to study the effect of changes in an assembly procedure without actually having to put the changes into operation, the effects of pollution without having to induce them in our environment, as well as complex series of chemical and nuclear reactions (before an engine or reactor is actually built, or to check its design and performance). Very often, the use of Monte Carlo methods eliminates the cost of building and operating experiments; it is thus used in the study of collisions of photons with electrons, the scattering of neutrons, evolution of biological populations, and other complicated phenomena. The chance factors in all these processes

[248] The term "Monte Carlo" was coined by **N. Metropolis** and **S. Ulam** in 1949 [J. Amer. Statistical Assoc. **44**, 241–335].

are simulated by means of appropriate gambling devices, which, most of the time, are themselves simulated by means of electronic computers.

Monte Carlo methods solve certain types of problems through the use of random or pseudo-random numbers[249]*, whose values depend on the outcome of a random or pseudo-random event. The former may be generated by using the outcomes of random physical processes such as throwing of dice, spinning a roulette wheel, scintillation in a Geiger-Müller counter, noise generated by electrical transmission systems, etc.; the latter can be generated via deterministic numerical algorithms.*

Monte Carlo methods offer two types of applications:

- *Sampling: deducing properties of a large set of elements by studying only a small, random subset. Thus an average value of $f(x)$ over an interval may be estimated from its average over a finite, random subset of points in the interval. Since the average of $f(x)$ is actually an integral, this amounts to a Monte Carlo method for approximate integration.*
- *Simulation: providing arithmetical imitations of "real" phenomena. In a broad sense this describes the general idea of applied mathematics. The classical example is the simulation of neutron's motions and absorptions in a nuclear reactor, its zigzag path being imitated by a type of arithmetical random walk.*

Of the mathematical problems to which the Monte Carlo method has been applied, one may mention: solving systems of linear equations, matrix inversion, evaluating multiple integrals, solving the Dirichlet problem, and solving functional equations of a variety of types.

[249] *Random numbers*, in the context of computations and communication *pseudo-random* and *not* numbers generated by a random, analog physical process (such as the flip of a coin or the spin of a wheel). Instead they are numbers generated by a completely deterministic arithmetical process, the resulting set of numbers having various statistical properties which together approximate *randomness*. A typical algorithm for generating pseudorandom numbers is

$$x_{n+1} = rx_n (\mathrm{mod}\ N).$$

An initial element x_0 is repeatedly multiplied by r, each product being reduced modulo N. With decimal computers $x_{n+1} = 7^9 x_n (\mathrm{mod}\ 10^5)$, $x_0 = 1$ is quite satisfactory, while with binary computers a good choice is $x_{n+1} = (8t-3)x_n (\mathrm{mod}\ 2^5)$, $\quad x_0 = 1$, with t being some large number. *Truly* random computer bits can be produced by means of e.g. amplified and discretized electronic noise, or a Geiger counter monitoring nuclear decays.

Suppose, for example, that it is required to evaluate $I = \int_0^1 f(x)dx$*, where* $f(x)$ *is assumed to be bounded above and below so that it can be transformed to satisfy the condition* $0 \leq f(x) \leq 1$*. Monte Carlo integration then proceeds as follows: random points are chosen within the unit square. The integral* I *is then estimated as the* fraction *of random points that fall below the curve* $f(x)$*. The number of points must be sufficiently large and their* uniform distribution *must be truly random in both the* x*- and* y*-dimensions, so that one deals with* n *mutually independent trials. Uniformly distributed random numbers make it possible to break off the procedure at a value* n *for which the successive estimates differ by less than a prescribed limit of accuracy. If* $(k-1)$ *is the number of counting steps executed so far and* I_{k-1} *the resulting estimate of* I*, then the recursive counting scheme*

$$I_k = I_{k-1} + (\xi_k - I_{k-1})/k = [(k-1)I_{k-1} + \xi_k]/k$$

has proved useful, where $\xi_k = 1$ *if the* k^{th} *point falls in the region of* $f(x)$*, and* $\xi_k = 0$ *otherwise.*

If only uniformly distributed random numbers x_i *in the one-dimensional interval* $[0, 1]$ *are chosen for the argument, and* $f(x_i)$ *is calculated for each, then the* statistical mean $M[f(x)]$*, multiplied by the width of the interval 1, is an estimate for the required integral. Because the arithmetic mean is an effective estimate for* $M[f(x)]$*, one obtains*

$$\frac{1}{n}\sum f(x_i) \approx \int_0^1 f(x)dx.$$

Here the deterministic recursive formula

$$I_k = I_{k-1} + [f(x_k) - I_{k-1}]/k = [I_{k-1}(k-1) + f(x_k)]/k$$

has proved suitable (i.e. ξ_k *was replaced by its expectation* $f(x_k)$*).*

This method can, in principle, be extended to multidimensional volumes V*. One picks* N *points, uniformly randomly distributed in* V*. Call them* $x_1, x_2, \ldots, x_N$*. Then the basic theorem of a Monte Carlo integration estimates the integral of a function* f *over the multidimensional volume,*

$$\int f dV \approx V\langle f\rangle \pm V\sqrt{\frac{\langle f^2\rangle - \langle f\rangle^2}{N}}\ .$$

Here $\langle\ \rangle$ *denote taking the arithmetic mean over the* N *sample points,*

$$\langle f\rangle = \frac{1}{N}\sum_{i=1}^{N} f(x_i) \quad ; \qquad \langle f^2\rangle = \frac{1}{N}\sum_{i=1}^{N} f^2(x_i)\ .$$

The "plus-or-minus" form is a one standard deviation error estimate for the integral, which is a rough indication of the probable error range.

1747–1753 CE **James Lind** (1716–1794, Scotland). Surgeon in the Royal Navy. Pioneer of preventive medicine and nutrition. Founder of naval hygiene and promoter of the use of citrus fruits and fresh vegetables to prevent and cure *scurvy*. The discovery was ignored until forty years after he discovered the cure.

1747–1760 CE **Johann Tobias Mayer** (1723–1762, Germany). Mathematician, physicist and astronomer. Made a careful investigation of the *libration of the moon*[250] (1747–1748), and published tables on the positions of the moon which allow determinations of *longitude at sea* (1753). With the tables in hand, a mariner could obtain his longitude through tedious calculations with an accuracy of half a degree. The method was that suggested by **Peter Apian** (1524).

Mayer was born at Marbach, in Würtemburg, and brought up at Esslingen in poor circumstances. A self-taught mathematician, he entered (1746) to work in a cartographic establishment at Nuremberg. Here he introduced many improvements in map-making and gained a scientific reputation which led (1751) to his election to the chair of mathematics in the University of Göttingen. In 1754 he became superintendent of the astronomical observatory of that university. He left behind him an essay on color, in which 3 primary colors are recognized; a memoir on the proper motions of 80 stars; papers on atmospheric refraction (1755), on the motion of Mars as affected by the perturbations of Jupiter and the earth (1756), and on terrestrial magnetism (1760), where he made the first definite attempt to establish a mathematical theory of magnetic action.

The British Government paid his widow a grant of 3000 Sterling for the lunar tables which he submitted to them in 1755.

1747–1784 CE **Benjamin Franklin** (1706–1790, U.S.A.). Scientist, inventor, statesman and diplomat. One of the first men to experiment with electricity (1747–1752). Invented the *lightning rod* (1749) and created such

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[250] The moon slightly *wobbles* as it moves along its orbit. This wobbling, called *libration*, permits us to view 59 percent of its surface.

electrical terms as *armature*, *condenser*, and *battery*. Invented *bifocal eyeglasses* (1784) which allowed both reading and distant lenses to be set in a single frame. Published the first chart of the North Atlantic *Gulf Stream* (1770), based on his own observations.

He was the first to relate the severe Northern Hemisphere winter of 1783/4 to the eruption of the volcano Laki in Iceland in the summer of 1783, speculating that solar heating of the earth is reduced due to the ash and other particles injected by the volcano into the atmosphere. Franklin was instrumental in establishing the *American Philosophical Society*, the first scientific society in the United States (proposed in 1743, established in 1769 at Philadelphia).

Franklin was born in Boston, the 15^{th} child and youngest son in a family of 17 children. His formal schooling ended at the age of 10, but he continued to educate himself throughout his life. From 1723 to 1730 Franklin worked for various printers in Philadelphia and London, England. He became the owner of a print shop in 1730, and began publishing *The Pennsylvania Gazette*, writing much of the material for this newspaper himself. His name gradually became known throughout the colonies. Seeking to improve the poor colonial postal service he became Philadelphia's postmaster in 1737, and in 1753 he became deputy postmaster general for all the colonies.

He started his electrical experiments in 1747, with the discovery that a pointed conductor can draw off electric charge from a charged body. In 1751 he described electricity as a single fluid and distinguished between positive (excess) and negative (deficiency) electricity. He also showed that electricity can magnetize and demagnetize iron needles. In June 1752, Franklin performed his famous kite experiment, showing that lighting is a form of electricity, similar to the discharge from a Leyden jar. This was the first recorded experiment on atmospheric electricity[251], and the first human endeavor to harness this natural source of power.

Franklin became the first scientist to study the movement of the Gulf-Stream. He spent much time charting its course, and recording its temperature, speed, and depth. He hoped that use of his chart would help ships to avoid the current and to speed the mail in crossing from Great Britain to America[252].

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251 In 1912 **Victor Franz Hess** (1883–1964, Austria) discovered, through manned balloon flights, the agent of this phenomenon — cosmic radiation. The bombardment of the earth's surface by massive particles from outer space was named '*cosmic rays*' by **Millikan** in 1925.

252 *Gulf Stream*: a current of warm waters that flows from the Straits of Florida in a north-easterly direction across the Atlantic toward Europe. It flows as fast as 220 km/day and its rate of flow, measured in volume per second, is about 1000

Franklin was a fierce supporter of America's struggle for independence: he played an important part in drafting the declaration of independence (1776) and the United States constitution (1787). During 1776–1785, he served as ambassador to France.

1748–1750 CE **Charles-Louis de Secondat**, **Baron de la Brède et de Montesquieu** (1689–1755, France). Political and social philosopher. His *L'Esprit des lois* (1748) [*The Spirit of the Laws*] is a seminal contribution to political theory which profoundly influenced political thought in Europe and America.

His family belonged to the lesser nobility of Guyenne, with a distinguished tradition of legal service in Bordeaux. He accordingly trained for the law at Bordeaux and Paris, where he was in touch with some of the most emancipated minds. A literary career (1721–1726) led to his election (1727) to the French Academy and in 1728 he began a 3-year European tour which took him to Italy and England and greatly nourished his interest in political and social institutions. He lived in England (1729–1731) and came to admire the English political system.

In his magnum opus he analyzed human institutions and the laws which embody them in terms of their dependence upon forms of government, upon the external relations of the state, upon national temperament, climatic and economic factors. The most influential part of the work, however, has been his analysis of the conditions which create political liberty, and his advocacy (based upon his readings of the English constitution) of a system of equilibrium based upon *a separation of the legislative, executive and judicial powers of the state.*

Montesquieu believed that laws underline all things – human, natural, and divine. One of philosophy's major tasks was to discover theses laws. Man was

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times greater than that of the Mississippi River. It is the second largest ocean current.

The Gulf Stream is partly responsible for the warm southwesterly winds that make the climate of Great Britain and Northwestern Europe much warmer than parts of North America that lie equally far north. These winds pick up heat and moisture from the Sargasso Sea and the Gulf Stream. The Gulf Stream is also an aid to shipping. Many large oil tankers and ore carriers, traveling from South America to Atlantic coastal harbors, attempt to "ride" the current on their northbound journey.

The stream is about 80 km wide and 910 meters deep. It is formed in the Caribbean from the union of the North and South Equatorial currents. These currents, in turn, are generated by trade winds, as suggested by **Benjamin Franklin** already in 1770.

difficult to study because the laws governing his nature were highly complex. Yet Montesquieu believed that these laws could be discovered empirically. Knowledge of the laws would ease the ills of society and improve human life. He maintained that liberty and respect for properly constituted law could exist together.

1748–1768 CE **Johann Joachim Winckelmann** (1717–1768, Germany). One of the fathers of modern archeology and art historian who set the foundation of our modern views on the arts. His writings reawakened the taste for classical art and was responsible for generating the neoclassical movement in the arts.

Born at Stendal in Brandenburg, the son of a poor shoemaker. As a child, Johann was influenced by the ancient Greek culture, especially Homer. He studied theology and medicine at Halle and Jena Universities. In 1748 he discovered the world of ancient Greek art while serving as a librarian near Dresden. There he wrote the essay "Reflections on the Painting and Sculpture of the Greeks" (1755). This was recognized as a manifesto of the Greek ideal in education and art. His other works include "Geschichte der Kunst des Altertums" (1764, "History of the Art of Antiquity").

In 1763 he became superintendent of Roman antiquities, but soon he rose to the position of librarian at the Vatican and later became the secretary to Cardinal Albani, who had an extensive collection of classical art.

In his work, Winckelmann sets forth both the *history of Greek art* and *the principles on which it seemed to him to be based*. He also presents a glowing picture of the political, social and intellectual conditions which he believed tended to foster creative activity in ancient Greece. The fundamental idea of his theories is that the end of art is beauty, and that this end can be attained only when individual and characteristic features are strictly subordinated to the artist's general scheme.

The true artist, selecting from nature the phenomena fitted for his purpose, and combining them through the imagination, creates an ideal type marked in action by "Edle Einfalt und stille Größe" ("*noble simplicity and quiet grandeur*") — an ideal type in which normal proportions are maintained, particular parts, such as muscles and veins, not being permitted to break the harmony of the general outlines.

In the historical portion he used not only the works of art he himself had studied but the scattered notices on the subject to be found in ancient writers; and his wide knowledge and active imagination enabled him to offer many fruitful suggestions as to periods about which he had little direct information.

Many of his conclusions, based on inadequate evidence of Roman copies, have been modified or reversed by subsequent research, but the fine enthusiasm of his work, its strong and yet graceful style, and its vivid descriptions of works of art give it enduring value and interest. It marked an epoch by indicating the spirit in which the study of Greek art should be approached, and the methods by which investigators might hope to attain solid results. To Winckelmann's contemporaries it came as a revelation, *and exercised a profound influence on the best minds of the age.*

On June 8, 1768 on his way back to Rome from Germany and Austria, he was murdered *by a chance acquaintance in Trieste*, Italy, which was where he was buried.

1749–1752 CE **Frederik Hasselquist** (1722–1752, Sweden). Traveler and naturalist; The first modern researcher of the fauna and flora of the Holy Land.

Born at Törnevalla, East Gothland and studied at Uppsala under Linnaeus. On account of the frequently expressed regrets of the latter regarding the natural history of the Holy Land, Hasselquist resolved to undertake a journey to that country. He visited parts of Asia Minor, Egypt, Cyprus and the Land of Israel, making large natural history collections. But his constitution, weakened by chronic consumption, gave way under fatigues of travel, and he died near Smyrna on his way home.

His collections reached home in safety, and five years after his death his notes were published by Linnaeus under the title *Resa till Heliga Landet, 1749–1752.* It was translated into French (1762) and English (1766). Among his discoveries: the fig-wasp (*Blastophaga psenes*), St. Peter's fish (*Thilapia galilaeae*), the common jerboa (*Jaculus jaculus*). His herbal collection *Flora Palestina* (1763) includes 600 species.

The Honeycomb — or, How to Hold the Most Honey for the Least Wax

One of the most beautiful hexagonal arrays is the honeycomb constructed by bees. The walls of the main body of connected cells form regular hexagonal prisms. The bottom of each cell is shaped like a concave triangular pyramid and constructed from three equilateral rhombs. The cell walls are slightly tilted toward the rim, which prevents honey from running out before the cells are closed.

The first question that comes to mind is, why the hexagonal cross-section? After all, the bees might have built their cells with rounded walls as the bumblebees do or as they themselves build for the cradles of their queens. Or they could base their architectural style on some other geometrical configuration. However, if the cell were round or, say, octagonal or pentagonal, there would be empty spaces between them. This would not only mean a poor utilization of space; it would also compel the bees to build separate walls for all or part of each cell, and entail a great waste of material.

These difficulties are avoided by the use of triangles, squares, and hexagons. But of those three geometrical figures with equal area (and for equal-depth cells — also with equal volume) the hexagon has the smallest circumference. This means that the amount of building material required for cells of the same capacity is the least in the hexagonal construction. The geometry of the cell-bottoms and the manner in which they dovetail into each other, contributes to the stability of the comb. A comb measuring 37 *by* 22.5 *centimeters can hold two kilogram of honey. Yet in the manufacture of such a comb, the bees use only* 40 *grams of wax.*

This natural architectural marvel must have attracted the attention and excited the admiration of mathematicians from time immemorial.

The writings of **Pappos of Alexandria** *(ca 300* CE*) inform us that the ancient Greeks had already tried to explain the regularity of beehive cells by means of an optimum principle. He has left us an account of its hexagonal plan, and drew from it the conclusion that the bees were endowed with "a certain geometrical forethought... There being, then, three figures which of themselves can fill up the space around a point, viz. the triangle, the square and the hexagon, the bees have wisely selected for their structure that which contains most angles, suspecting indeed that it could hold more honey than either of the other two".*

Erasmus Bartholinus *(1669) was the first to suggest that the hypothesis of 'economy' was not warranted, and that the hexagonal cell was no more than*

the necessary result of equal pressures, each bee striving to make its own little circle as large as possible.

The understanding of the particular shape of the bottom of the cell was a more difficult matter than that of its sides, and came later. ***Kepler*** *was first to deduce from the space-filling symmetry of the honeycomb that its angles must be those of the* rhombic dodecahedron; and **Swammerdam** *(1673) also recognized the same geometrical figure in the base of the cell. But Kepler's discovery passed unnoticed, and to the Italian astronomer* ***Giacomo Filippo Malardi*** *[(1665–1729), a nephew of D. Cassini; lived in Paris] goes the credit of ascertaining the shape of the rhombs and the solid angle which they bound, while watching the bees in the garden of the Paris Observatory (1712).*

He found the angles of the rhomb to be 110° *and* 70°. *He later observed that the angles of the three rhombs at the base of the cell depend on the basal angles of the* 6 *trapezia which form its sides. It then occurred to him to ask what must these angles be, if those on the floor and those of the sides are equal to one another. The solution to this geometrical problem yielded the theoretical values of* $70^\circ 32'$ *and* $109^\circ 28'$. *Thus, invoking the two* principles of simplicity and mathematical beauty, *Malardi obtained a theoretical result very close to the observed values!*

The next step, taken by the French physicist and naturalist ***René-Antoine Ferchault de Réaumur*** *(1734), had been foreshadowed long before by Pappos. Though Euler had not yet published his famous discussion on curves,* maximi minimive proprietate gaudentes, *the idea of* maxima *and* minima *was in the air as a guiding postulate, a heuristic method, to be used as Malardi used his principle of simplicity.*

So it occurred to Réaumur that the hexagonal structure of the bee's honeycomb should follow from a minimum principle: the bee would build its cells with the greatest economy in order to use as little wax as possible; and that, just as the closed-packed hexagons gave the minimal extent of boundary in a plane, so the figure determined by Malardi, namely the rhombic dodecahedron, might be that which employs the minimum of surface for a given volume; or which, in other words, should hold the most honey for the least wax[253].

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253 Consider a right prism of height h, having regular hexagonal base *abcdef*, top $ABCDEF$, both with side s (volume $= \frac{1}{2}3\sqrt{3}s^2h$). At the top, we cut off the corners B, D, F by planes through the lines AC, CE, EA. Using these lines as 'hinges', we rotate the so-formed three tetrahedrons such that they all meet at a common vertex V. The new body with top faces $AXCV$, $CYEV$, $EZAV$ (rhombuses), is the bee's cell and has the same volume as the original prism. The hexagonal base at the opposite end is the open end. One parameter

Réaumur posed his conjecture to ***Samuel Koenig****, a young Swiss mathematician: Given a hexagonal cell terminating with three similar and equal rhombs, what is the configuration which requires the least quantity of material for its construction? Koenig (1739) found that the angle* $109^\circ 24'$ *followed from the minimum principle proposed by Réaumur [Koenig's own paper, sent to Réaumur, remained unpublished and was lost and his method of solution is unknown]. Thereupon* ***Bernard Le Bovier de Fontenelle*** *(1657–1757), the perpetual secretary of the French Academy, declared that bees had no intelligence; yet they were "blindly using the highest mathematics by divine guidance and command".*

In spite of the striking success of the calculus in explaining the cell's geometry in terms of 'wax economy', a line of mathematicians since Bartholinus doubted the philosophical implications of this theory. ***Glaisher*** *(1873) summed up the matter as follows:*

"As the result of a tolerably careful examination of the whole question, I may be permitted to say that the economy of wax has played a very subordinate part in the determination of the form of the cell. I should not be surprised if it were found that the form of the cell had been determined by other considerations, into which saving wax did not enter, although I would not go as far as to say that the amount of wax required was a matter of absolute indifference to the bees".

D'Arcy Thompson *(1860–1948) commended in the same spirit that it makes more sense to suppose:*

"that the beautiful regularity of the bee's architecture is due to some automatic play of the physical forces" than to suppose "that the bee intentionally seeks for a method of economizing wax".

But all this assumes that the bees have somehow hit upon the optimal honeycomb. Have they? This question was investigated by the Hungarian

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is, however, left to our choice: the angle through which we cut-off the three tetrahedrons, or alternatively, the angle θ which the vertical at V makes with the line VX. The bees form the faces by using wax. When the volume is given, it is economic to spare wax and, therefore, to choose the angle of inclination θ in such a way that the *surface area S of the bee's cell is minimized*.
Simple geometrical considerations reveal that $S(\theta) = 6hs + \frac{3}{2}s^2\big(\frac{\sqrt{3}}{\sin\theta} - \cot\theta\big)$. The derivative $S'(\theta)$ vanishes if, and only if, $\cos\theta_0 = \frac{1}{\sqrt{3}}$, yielding $\theta_0 = 54.7^\circ$ or $2\theta_0 = 109^\circ 24'$, independent of the choice of h and s. It is worth comparing the result with the actual angle chosen by the bees. It is difficult to measure this angle. However, the average of all measurements does not differ significantly from the theoretical value of $2\theta_0 = 109^\circ 24'$. Thus, the bees strongly prefer the optimal angle. It is rather unlikely that the result is due to chance.

mathematician **Fejes Tóth** *(1964). In his paper "What the bees know and what they don't know", he considered honeycombs, which he defined as a set of congruent convex polyhedra called cells, filling the space between two parallel planes without overlapping and without interstices in such a way that:*

(1) each cell has a face (called a base or opening) on one and only one of the two planes; and

(2) every pair of cells is congruent in such a way that their bases correspond to each other.

The cells built by the bees are prismatic vessels, the openings (and cross sections) of which are regular hexagons, whereas their bottoms consist of three equal rhombi.

The bees construct their honeycomb in such a way that the hexagonal openings of the cells are attached to one of the two planes. Is the zigzagged bottom surface constructed by the bees the most economical one? (It is certainly more advantageous than a plane.)

In order to state the problem precisely, we formulate (following Tóth) the isoperimetric problem for honeycombs: Given any two numbers V and W, find a honeycomb of width W whose cells have smallest surface area and yet enclose the volume V. (The width W is the distance between the two parallel planes that bound the honeycomb.)

We don't know yet what the solution is, but definitely it cannot be the bee cell, because Fejes Tóth found another cell that yields a slightly better result. The bottom of this cell consists of two hexagons and two rhombi. The advantage of Tóth's cell amounts to less than 0.35% of the area of an opening (and a much smaller percentage of the surface area of a cell). Hence we can state that the bees do a pretty good but not a perfect job, although their practical result, taking the margin of error into account, might still be optimal.

Evolution of Minimum and Variational Principles[254]

Many fundamental ideas in science were conceived in antiquity, and our present way of thinking owes a great deal to our predecessors. One essential idea that modern science has inherited from the classical world is the concept of a fundamental order and harmony to the universe, a harmony that could be reflected in the beauty of mathematical structures. Because Greek mathematics was mainly restricted to geometry, the ancient scientists used geometric models to describe nature.

Thus, since the nascence of the Milesian school (ca 600 BCE), Greek philosophers and scientists sought to reduce the manifold phenomena of nature to a basic set of unifying laws. This quest for simplicity was continued by ***Pythagoras*** *(ca 540 BCE), but whereas the Ionian physicists postulated a single* substance *from which all substances comprising the cosmos were derived, Pythagoras put the emphasis on mathematical reasoning and believed that the concepts of* harmony *and* number *(positive integers) embrace the whole structure of the universe (mathematics and physics were in his time indistinguishable).*

Plato *(427–347 BCE) continued the Pythagorean legacy and upheld the view that number rules the universe. Although he rejected the experimental*

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[254] For further reading, see:

- Elsgolts, L., *Differential Equations and the Calculus of Variations*, Mir Publishers: Moscow, 1980, 440 pp.
- Yourgrau, W. and S. Mandelstam, *Variational Principles in Dynamics and Quantum Mechanics*, Dover: New York, 1968, 201 pp.
- Weinstock, R., *Calculus of Variations*, Dover: New York, 1974, 326 pp.
- Gelfand, I.M. and S.V. Fomin, *The Calculus of Variations*, Prentice Hall, 1965, 232 pp.
- Lanczos, C., *The Variational Principles of Mechanics*, University of Toronto Press, 1964, 367 pp.
- Woodhouse, R., *A Treatise on Isoperimetrical Problems, and the Calculus of Variations*, Chelsea Publications: New York.
- Todhunter, I., *A History of the Progress of the Calculus of Variations in the Nineteenth Century*, Chelsea Publications: New York.

method, he still adhered to the conceptual representation of the phenomenal world through ideas of simplicity, uniformity, order and perfection.

Aristotle *(384–322 BCE) mentioned the fact that of all curves enclosing a given area, the circle possesses the shortest perimeter. This marks the transition from the belief in simplicity to a* minimum *principle, explicitly stated for the first time. This minimum hypothesis was not dictated by any appeal to quantitative measurement and was not subject to rigorous scrutiny.* ***Hero of Alexandria*** *(ca 150 BCE) gave a geometrical demonstration of the principle of shortest optical path (distance) for light rays reflected from a plane mirror. No further development of this idea was made until the advent of Fermat's least-time principle in the* 17^{th} *century.*

In the interim period, however, the claim for the simplicity of nature was strongly advocated by **William of Ockham** *(1285–1349). His 'razor' principle, while indicating a viewpoint similar to the simplicity hypothesis of Aristotle, differs from it in the sense that while the Greek philosopher held that nature possesses an immanent tendency to simplicity, Ockham demanded that in* describing *nature one should avoid unnecessary complications. Both doctrines appear simultaneously in the writings of* **Copernicus** *(1473–1543),* ***Galileo*** *(1564–1642) and* **Kepler** *(1571–1630) in the form of Pythagorean-Platonic mysticism and deep-rooted convictions of a simple, harmonious and ordered universe.*

Both **Newton** *(1642–1727) and* **Leibniz** *(1646–1716) reformulated the principle of simplicity: "We are to admit no more causes of natural things than such as are both true and sufficient to explain their appearances" (Newton); "The perfectly acting being... can be compared to a clever engineer who obtains his effect in the simplest manner one can choose" (Leibniz). The French philosopher* **Nicolas de Malebranche** *(1638–1715) replaced the word* simplicity *by* economy *and arrived at a similar view which he called the 'Economy of Nature'*[255]*.*

In 1621, ***Snell*** *derived empirically the law of light refraction across a boundary between two homogeneous media in which the velocity of light has the values* c_1 *and* c_2*, respectively. If the ray passes from one region to the other, then it must consist of two straight-line-segments which satisfy the*

[255] There were, however, other more prosaic motives for the preoccupation of scientists with extremum problems; the question of shortest and quickest connections became especially important to the European powers during the 15^{th} and 16^{th} centuries, when they were searching for the best routes to the Far East and to the New World. Faster sailing routes promised greater profits. The well-known expeditions of Vasco da Gama and Christopher Columbus must be seen mainly in economic terms.

law of refraction, $\sin\alpha_1/\sin\alpha_2 = c_1/c_2$, *where* α_1 *and* α_2 *are the angles between the normal to the boundary and the two line-segments at the point of intersection.*

In 1657, ***Fermat*** *succeeded in deriving Snell's law from a new principle — the principle of least time. It stated that a light ray requires less time along its actual path between two points than it would require along any other conceivable ('virtual') path satisfying the given condition.*

It is remarkable that Fermat demonstrated the principle using elementary algebra only (no derivatives!). He later generalized his result to curved surfaces separating the two media and also for inhomogeneous media. He thus arrived at the general Fermat principle of geometrical optics:

"In an inhomogeneous medium, a light ray traveling between two points follows a path along which the time taken is minimum w.r.t. paths joining the two points"[256].

Minimizing the travel-time t *between two points* P *and* Q *means minimizing the integral* $I = \int_P^Q \frac{ds}{v}$, *where* s *is the arc-length along the ray and* $v = v(s) = v(\mathbf{r}(s))$ *is the velocity at a general point* $\mathbf{r}(s)$ *on the ray. The principle then states that* $\delta \int_P^Q \frac{ds}{v} = 0$, *meaning that the variation between the time taken to travel along the actual path and that needed to cover an infinitesimally adjacent virtual path is zero.*

Thus, 1800 *years had to pass before Hero's observation could be improved upon and generalized. The ideas unfolded by Fermat have had a tremendous influence on the development of physical thought in and beyond the study of classical optics; including its analogues in classical and quantum mechanics as well as both classical and quantum field theories. Fermat's principle provided science with an insightful and highly useful way of anticipating the behavior of light, matter and energy. Note that Fermat's principle is not so much a computational device as it is a concise way of thinking about the propagation of light. It is a statement about the grand scheme of things, without reference to any underlying causal mechanisms.*

The first real justification of Fermat's principle was given by ***Huygens*** *who, in* 1678, *deduced the laws of reflection and refraction on the basis of the wave theory of light (Huygens' principle). Furthermore, he demonstrated that*

[256] Fermat's principle is a *true minimum principle* (and not merely a stationary value principle) if we make comparisons in the *local* sense. However, it is required that all along the trajectory the wave surfaces shall be well defined, single-valued with definite normals (no intersection of ray trajectories!). The mathematical machinery needed to derive the equations of the rays for a given $v(\mathbf{r})$ was not known to Fermat and had to await for another 100 years.

the travel-time of light upon refraction was a real minimum. Fermat's achievement, with Huygens' support, stimulated a great deal of effort to supersede Newton's laws of mechanics with a similar variational formulation.

In 1740, *the French mathematician* ***Maupertuis*** *(1698–1759) announced le principe de la moindre quantité d'action — the famous principle of least action.*

According to this principle all events in nature take place such that a certain quantity, called "action", is rendered minimum. He postulated that the action must depend on the mass and the histories of the velocities and displacements; he therefore defined action as an integral of the product of these three factors (dimensionally it is also equal to the product of energy and time or to angular momentum). The bold universality of this assumption is admirable and well in line with the spirit of the 18^{th} *century. It conforms with the spirit of the Platonic-Pythagorean cosmology, as well as with the natural philosophy of Leibniz, and follows in the footsteps of Hero and Fermat.*

However, Maupertuis' original definition of action (as a product of mass, velocity and distance without integration) was very obscure, owing to the fact that the distance covered by the moving body varies with time, and his failure to specify the time interval for which the product is to be computed. For these reasons Maupertuis could not establish satisfactorily the quantity to be minimized. He applied his principle to the derivation of the laws of elastic collision. This phenomenon is very intricate if treated as a minimum problem and requires great skill in handling (the mathematical powers of Maupertuis were far behind the high standards of his period); he obtained the correct result by an incorrect method. More satisfactory was his treatment of the law of refraction, in which he showed how Fermat's principle of least time can be replaced by the principle of least action (this result was earlier recognized by ***Johann Bernoulli****).*

Thus, the original statement by Maupertuis was vaguely theological and could hardly pass muster today. The integral formulation which today bears his name is actually due to ***Euler*** *who discovered the principle in* 1743 *in an entirely correct form (he may have been inspired, in part at least, by Maupertuis'* 1740 *paper). In particular, Euler knew that both actual and virtual motions have to satisfy the law of conservation of energy. Without this auxiliary condition, the action quantity of Maupertuis, even if corrected from a sum (the form in which he used it) to an integral, loses all significance.*

Euler, who confined himself to a single particle moving on a plane curve, asserted:

"When a particle travels between two fixed points, it takes the path for which $\int v ds$ *is a minimum",* v being the velocity of the particle and ds the

corresponding element of the curve. Euler also gave an alternative formulation through which the actual path can be mathematically evaluated:

"A particle travels between two fixed points in such a way that the difference between the integral $\int v ds$ taken along the actual path and that taken along any neighboring virtual path between the two points, is an infinitesimal quantity of second order; the particle is supposed to travel along the virtual path with the velocity for which the energy is equal to the given energy" (virtual path is one along which the particle may be imagined to move *without* satisfying Newton's laws of motion).

The condition is thus

$$\delta \int_P^Q v ds = 0$$

where P and Q are the initial and final points and δ denotes the variation *of the integral under the aforementioned restrictions.*

The nascence of the calculus of variations was in **Euler***'s work (1744) "Methodus Inveniendi lineas Curvas Maximi Minimive proprietate gaudentes" (a method to find curved lines that enjoy a maximum or minimum property).*

He was seeking an admissible function $y(x)$ that extremalizes the functional given by the integral $I = \int_{x_1}^{x_2} f(x, y, y')dx$. He showed that if $y(x)$ exist, it must obey the differential equation

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0.$$

Euler applied the calculus of variations to the study of elasticity, examining the bending, buckling and vibrations of bands and plates. Note that since in Euler's equation $f = f\big(x, y(x), y'(x)\big)$, $\frac{\partial f}{\partial y'}$ is in general an explicit *function of x as well as an* implicit *function of x via $y(x)$ and $y'(x)$. Therefore*

$$\frac{d}{dx} = \frac{\partial}{\partial x} + \frac{dy}{dx}\frac{\partial}{\partial y} + \frac{d^2y}{dx^2}\frac{\partial}{\partial y'}.$$

Consequently the second order ODE *for $y(x)$ is found to be*

$$\left(\frac{\partial^2 f}{\partial y'^2}\right)\frac{d^2y}{dx^2} + \left(\frac{\partial^2 f}{\partial y \partial y'}\right)\frac{dy}{dx} + \left(\frac{\partial^2 f}{\partial x \partial y'} - \frac{\partial f}{\partial y}\right) = 0.$$

The solution of this equation constitute a two-parameter family of curves, and among these, the stationary functions are those in which the two parameters are chosen to fit the given boundary conditions.

In the case of three independent variables, the integral in question in

$$I[u] = \iiint\limits_R F\left(x, y, z; u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) dxdydz,$$

where $u = u(x, y, z)$, *the local stationary point of the functional* $I[u]$ *obeys the Euler equation*

$$\frac{\partial I}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial I}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial I}{\partial u_y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial I}{\partial u_z}\right) = 0.$$

Suppose we wish to find $u(x, y, z)$ *which has a minimum average value of the square of the gradient in a certain region in space, i.e.*

$$F = (\nabla u)^2 = u_x^2 + u_y^2 + u_z^2.$$

The resulting Euler equation is simply the Laplace equation $\nabla^2 u = 0$ *which must be satisfied, for instance, by the electric potential in free space.*

The Laplace equation is therefore the necessary condition that the average electrostatic field energy be minimized in a given volume. If the same quantity is to be made stationary, but with the additional requirement that $\int u^2 dxdydz$ *shall have a fixed value, another interesting equation results. In that case we define*

$$F = (u_x)^2 + (u_y)^2 + (u_z)^2, \qquad F_1 = u^2.$$

If we extremalize the integral $\iiint (F - \lambda^2 F_1) dxdydz$, *with* λ^2 *the Lagrange multiplier of the constraint* $\int u^2 dxdydz = \text{constant}$, *Euler's equation then reads*

$$\nabla^2 u + \lambda^2 u = 0$$

which is the Helmholtz wave equation for monochromatic waves. Such a wave may therefore be characterized as a disturbance in which the displacement u *has a fixed mean square value and at the same time a minimum mean square gradient.*

Another example of the power of the variational calculus is the propagation of light in an inhomogeneous medium: Let $v(x, y, z)$ *be the velocity of light at each point of the medium. The square element of distance between two points on a light ray* (x, y, z) *and* $(x + dx, y + dy, z + dz)$ *is* $ds^2 = dx^2\left[1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2\right]$, *where* $y = y(x), z = z(x)$ *constitute the equations describing the ray.*

Therefore, the travel time along a path between two fixed points P *and* Q *is* $t = \int_P^Q \frac{\sqrt{1+y'^2+z'^2}}{v(x,y,z)} dx$. *It is required to simultaneously find two functions* $y = y(x)$, $z = z(x)$ *such that the functional* t *is the smallest.*

Writing the system of Euler equations for this functional, i.e.

$$\frac{\partial v}{\partial y}\frac{\sqrt{1+y'^2+z'^2}}{v^2}+\frac{d}{dx}\frac{y'}{v\sqrt{1+y'^2+z'^2}}=0;$$

$$\frac{\partial v}{\partial z}\frac{\sqrt{1+y'^2+z'^2}}{v^2}+\frac{d}{dx}\frac{z'}{v\sqrt{1+y'^2+z'^2}}=0,$$

we obtain the two coupled ordinary differential equations for the curve along which light propagates. Once $y(x),\ z(x)$ *are known, the path of light from* P *to* Q *is given in the parametric form* $\{x, y(x), z(x)\}$.

Another important application of Euler's equations is the determination of the path of shortest distance between two points on a surface $\mathbf{r}=\mathbf{r}(u,v)$. *If such a path exists, we call it a geodesic. The line-element on the surface is*

$$ds^2 = d\mathbf{r}\cdot d\mathbf{r} = \left(\frac{\partial \mathbf{r}}{\partial u}du + \frac{\partial \mathbf{r}}{\partial v}dv\right)^2 = Edu^2 + 2Fdudv + Gdv^2$$

where

$$E=\frac{\partial \mathbf{r}}{\partial u}\cdot\frac{\partial \mathbf{r}}{\partial u},\qquad F=\frac{\partial \mathbf{r}}{\partial u}\cdot\frac{\partial \mathbf{r}}{\partial v},\qquad G=\frac{\partial \mathbf{r}}{\partial v}\cdot\frac{\partial \mathbf{r}}{\partial v}.$$

A curve on the surface has the parametric representation $u=u(t),\ v=v(t)$. *Hence, the distance between* $P(t_1)$ *and* $Q(t_2)$ *on the surface and along the curve is given by the line integral*

$$J[u,v]=\int_{t_1}^{t_2}\sqrt{Eu'^2+2Fu'v'+Gv'^2}\,dt,$$

where $u'=\frac{du}{dt},\ v'=\frac{dv}{dt}$. *Writing Euler equations for the functional* J, *we obtain the simultaneous differential equations for the parametric functions* u *and* v:

$$\frac{A_u}{\Delta}-\frac{d}{dt}\frac{C}{\Delta}=0;\qquad \frac{A_v}{\Delta}-\frac{d}{dt}\frac{D}{\Delta}=0,$$

where

$$\Delta=\sqrt{Eu'^2+2Fu'v'+Gv'^2};$$

$$A_u=\frac{\partial E}{\partial u}u'^2+2\frac{\partial F}{\partial u}u'v'+\frac{\partial G}{\partial u}v'^2;$$

$$A_v=\frac{\partial E}{\partial v}u'^2+2\frac{\partial F}{\partial v}u'v'+\frac{\partial G}{\partial v}v'^2;$$

$$C=2(Eu'+Fv');\qquad D=2(Fu'+Gv').$$

Although Euler was first to implement Maupertuis' conjecture, the credit for having given the correct formulation of the principle of least action goes to **Lagrange** *(1736–1813). It is true that Euler was first to introduce the concept of variation and* stationarity *(*minimum *or* maximum*) instead of an exclusive* minimum*, but he still held to the conviction that some sort of* maximum *or* minimum *law prevails throughout nature. Lagrange, on the other hand, showed that the principle of least action together with the law of conservation of energy is fully equivalent to Newton's law of motion and may, indeed, be employed as an alternative formulation of the principles of dynamics.*

Lagrange himself remarked, in conformity with his general outlook on natural philosophy, that the principle of least action was to be considered not as a metaphysical postulate, but as a simple and general consequence of the laws of mechanics (1788).

For the next half century, the principle of least action was thought of as interesting rather than important, and no use at all was made of it. Indeed, as late as 1837*, it was described as "only a useless rule" by Poisson, who failed to read* **Hamilton***'s* 1835 *paper. There, Hamilton gave the first exact formulation of the principle of least action for systems which are not necessarily conservative (showing it to be equivalent to the Lagrange equations of motion) and stated his principle*

$$\delta \int L dt = 0.$$

This effort culminated in the celebrated Hamilton-Jacobi equation (1828–1837) which brought about the geometrization of dynamics and the mathematical analogy between optical rays and mechanical paths of point-masses. A bridge had finally been established between Fermat's principle of least time and Hamilton's principle of least action. Thus, a long line of thinkers from **Hero** *through* **Fermat***, and even* **Euler***, believed in an underlying metaphysical optimum law of one kind or another. But such postulates were transformed by* **Lagrange***,* **Hamilton** *and* **Jacobi** *into exact analytic instruments capable of solving concrete problems in physics, mathematics and astronomy.*

The striking similarity between the principles of Fermat and Hamilton played an important role in **Schrödinger***'s development of quantum mechanics. In* 1942 **R.P. Feynman** *showed that quantum mechanics can be formulated in an alternative way using a variational approach*[257]*. And so, the*

[257] Feynman used a continuous sum (*path integral*) over all virtual trajectories between two given particle positions at two given times: the *classical* path, and its neighboring trajectories, determine the path integral in the classical limit. In this way, the Huygens principle is seen to apply to quantum mechanics.

continuing evolution of variational principles take us back to optics via the modern formalism of the matter waves of quantum mechanics.

So far we have concentrated mainly on dynamical problems. In the geometrical vein matters had a history of their own, and in order to see it in the right perspective we must return to the period 1690–1701*, namely to the emergence of the calculus of variations due to the efforts of the brothers* ***Jakob and Johann Bernoulli****.*

The pivotal year is 1696*, in which two problems were proposed: In December of that year Johann Bernoulli challenged the mathematicians of his age in the journal Acta Eruditorum to solve the problem of the brachistochrone by Easter 1697: He asked to determine a curve of the quickest descent of a massive particle moving between two given points in a homogeneous gravitational field. In time, three mathematicians solved the problem: Johann and Jakob Bernoulli, and Leibniz. The path, which happens to be a cycloid, is known as the brachistochrone*[258]*.*

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[258] Let the particle start from rest at the origin; the terminal point of the motion is (x_2, y_2). It is convenient to extend the y-axis to the right and to measure x downwards. From the energy equation

$$mgx = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{ds}{dt}\right)^2 = \frac{1}{2}m\left[\frac{\sqrt{dx^2+dy^2}}{dt}\right]^2,$$

we find $dt = \left[1+(y')^2\right]^{1/2}(2gx)^{-1/2}dx$. The integral to be minimized is therefore

$$\sqrt{2g}\; t = \int_0^{x_2}\left(\frac{1+y'^2}{x}\right)^{1/2} dx.$$

Euler's equation reads

$$\frac{d}{dx}\frac{y'}{[x(1+y'^2)]^{1/2}} = 0;$$

hence $y' = x\left(\frac{x}{c} - x^2\right)^{-1/2}$, with c a constant. If we introduce $2a = \frac{1}{c}$, integration leads to

$$y = a\cos^{-1}\left(1-\frac{x}{a}\right) - \sqrt{2ax - x^2}.$$

It represents an inverted cycloid with its base along the y-axis and the cusp at the origin. The constant a must be so adjusted that the cycloid passes through the point (x_2, y_2).

This problem must not be confused with another problem, the *tautachrone*, proposed by **Jakob Bernoulli** in 1690: to determine a curve on a vertical plane, along which a massive particle will arrive at a given point of the curve in the same time interval, no matter from what initial point of the curve it

The other problem to which attention was called in 1696 *by Jakob Bernoulli is the isoperimetric problem (iso = equal, perimetron = circumference), of which the Greek mathematicians were well aware. It has been one of the most stimulating and influential problems in the history of mathematics. Its origins lay in the ancient legend associated with the founding of the city of Carthage (ca 900 BCE). Dido, a Pheonician princess, fled from the city-state of Tyre when her ruthless brother Pygmalion murdered her husband to usurp her possessions. She bought a parcel of land from the King of Numidia under the condition that she would obtain only as much land as she could enclosed by the skin of an ox. To maximize the land she cut the hide in thin strips and tied them together to form a cord of some* 1500 *meters, and then formed with it a semi circle with the Mediterranean coast as its diameter.*

The Greek thus knew that among all closed lines of the same perimeter, the circle has the maximal area[259]*. An incomplete proof of the isoperimetric property of the circle was given by* ***Zenodoros*** *(ca 180 BCE).* ***Jakob***

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started (This curve, too, is a cycloid!). It was solved earlier by Huygens (1673) and Newton (1687) and applied by Huygens in the construction of pendulum clocks. The isochronous property of the cycloid is this: a pendulum constrained to swing between two successive arches of an inverted cycloid must oscillate such that the time to the lowest point is $\pi\sqrt{\frac{a}{g}}$, where a is the radius of the circle that generates the cycloid, and irrespective of the oscillation's angular amplitude.

259 The Greeks, who held some rather impressive notions of beauty and perfection, came to the conclusion that the circle was the most beautiful curve. After all, the sun and moon were round, the horizon was round, and the planets (so they thought) orbited in circles. The circle must be the perfect figure, for the architect of the universe would certainly not deal with imperfect creations. Some views about the circle were even more sweeping. The philosopher **Empedocles** held that the nature of God is a circle whose center is everywhere and whose circumference is nowhere.

But this theory received a blow even at ancient times. When astronomers were able to measure the paths of the planets accurately, they were found not to travel in true circles. Ptolemy invented an ingenious but ad hoc system of epicycles — that is, of circles and circles upon circles — to generate the paths of the planets. In the 17^{th} century, when **Kepler** found that the planets moved in ellipses around the sun, the circle was dethroned from its position as the most perfect curve. Yet, no other curve shares the following list of characteristics:

- Every point on the circle is at the same distance from the center.
- Every diameter of the circle is an axis of symmetry.
- A circle is a figure with constant width.
- Every tangent to a circle is perpendicular to the radius drawn from the center to the point of tangency.

Bernoulli *gave a proof in* 1701*. A complete proof, however, was first given by* **Weierstrass** *in* 1865*.*

Denoting the given perimeter of the curve by L*, all permissible areas* A *formed by* L *obey the isoperimetric inequality* $A \leq \frac{L^2}{4\pi}$*. This inequality has the following theorem as a consequence:*

Among all figures of equal area, the circle has minimal perimeter.

This can easily be seen, because the area A *of a circle of perimeter* $\overline{L}$ *equals* $\overline{L}^2/4\pi$*. If there were a plane figure with the same area but a smaller perimeter* $L < \overline{L}$*, we would have* $A > \frac{L^2}{4\pi}$*, which contradicts the isoperimetric inequality.*

This explains the circular shape of oil slicks: the molecular forces generate a figure of least potential energy (least surface area) for a given amount of oil. There are several other optimum properties of the circle. For instance: among all plane domains of a given area, the disc can support the largest sand pile; among all cross-sections of a perfectly elastic column of equal area, the disc can withstand the largest torsional moment; out of all drums with a given cross-sectional area, the circular membrane has the lowest tone [this last result was conjectures by **Rayleigh** *(1877) on the basis of experiments, but it was proved by* **Faber** and **Krahn** *(1923–1924)].*

The term isoperimetric problem is usually extended beyond its classical content to include the general case of finding extremals for one integral (subject to constraints) requiring a second integral to take on a prescribed value[260]*.*

- The curvature of a circle is constant at every point.
- Of all curves that enclose the same area, the circle has the least perimeter, and of all curves with the same length of perimeter, the circle encloses the greatest area. Thus, given any plane figure of area A and perimeter L, then $\frac{4\pi A}{L^2} \leq 1$ (equality holds only for the circle). For a semicircle, this ratio is about 0.75, for a square 0.79 and for a perfect hexagon 0.91.

  The 3-*dimensional* isoperimetric property can be expressed by the inequality $\frac{36\pi V^2}{A^3} \leq 1$ between the surface area A and the volume V, the equality holding only for the sphere.

260 If the curve is expressed parametrically by $x(t)$ and $y(t)$ and it is traversed once counterclockwise as t increases from t_1 to t_2, then the enclosed *area* is known to be $A = \frac{1}{2}\int_{t_1}^{t_2}(x\dot{y} - y\dot{x})dt$, [where $\cdot = \frac{d}{dt}$], which is an integral depending on two unknown functions [this integral expression for A is a special case of Green's theorem]. Since the length of the curve is $L = \int_{t_1}^{t_2}\sqrt{\dot{x}^2 + \dot{y}^2}\,dt$, the problem is to maximize A subject to the constraint that L must have a constant value. Using the method of *Lagrange multipliers*, the problem reduces to maximizing the unconstrained functional $I = \int_{t_1}^{t_2} F(x, y; \dot{x}, \dot{y})dt$ where $F = \frac{1}{2}(x\dot{y} - y\dot{x}) + \lambda\sqrt{\dot{x}^2 + \dot{y}^2}$.

In August 1697*, Johann Bernoulli again publicly posed the problem of finding the shortest line between two given points on a convex surface. This was meant as a challenge to his brother Jakob, with whom he was publicly feuding. The unfortunate rivalry of the two brothers eventually became so intense, and their polemics so ugly, that the scientific journals of the time declined to publish them. Anyway, the challenge was met by Jakob who solved the problem in* 1698 *for all surfaces of revolution.*

Johann then announced that he had found a solution of the shortest connection problem for an arbitrary *surface (1698). His unpublished solution appeared in the form of a geometric theorem:*

"at each point P of a shortest line C, the corresponding osculating plane *of C intersects the tangent plane to the surface in a right angle (the osculating plane includes the tangent to C at P and the principal normal at P)".*

Thirty years later, in December 1727*, Johann again posed the problem to his student Euler! Euler published his solution in* 1728 *under the title* Da linea brevissima in superfice quacunque duo quaelibet puncta jungente *("On the shortest line on an arbitrary surface connecting any two points whatsoever").*

In contradistinction to the geometric solution of Johann Bernoulli, Euler reduced the problem to the solution of a differential equation. Euler stated that we can easily solve the problem of shortest connection between two points on a convex surface by a simple mechanical artifice: we fix a string at one of the points and pull it taut in the direction of the other. The string then yields the shortest connection between the two points.

The Euler equations for this case then read:

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) - \frac{\partial F}{\partial x} = 0$$

and

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right) - \frac{\partial F}{\partial y} = 0.$$

The integration of these equations yields the *circle*:

$$(x - c_1)^2 + (y - c_2)^2 = \lambda^2.$$

Early Theories of Cosmic Evolution

1750–1785 CE *Descartes' idea of a universe evolving by natural processes of separation and combination was the source of a succession of theories of cosmic evolution by* ***Swedenborg*** *(1688–1772, Sweden; 1734),* ***Thomas Wright*** *(1711–1786 , England; 1750),* ***Immanuel Kant*** *(1724–1804, Germany; 1755),* ***Johann Heinrich Lambert*** *(1728–1771, Germany; 1761),* ***Georges Louis Leclerc de Buffon*** *(1707–1788, France; 1785) and others.*

Their theories related to the formation of the solar system and the phenomenon of the 'Milky Way' [galaxy is the Greek word for 'milk']. Kant held to the idea that in the beginning all matter was in a gaseous state and was spread more or less uniformly throughout the universe (his interpretation of Genesis ***I****, 1–2). He assumed that we live in an evolutionary universe in the sense that the past was essentially simpler than the present. Subsequently a giant cloud of gas, contracting under its own gravitation, began to rotate and shed matter from its center, to in turn form the planets by further gravitational contraction.*

The phenomenon of the 'Milky Way' was interpreted by similar speculations of ***Wright*** *and* ***Lambert*** *who came very close to the truth. Wright suggested that the Milky Way consisted of a flattened distributions of stars forming a disc, which rotates about its center on an axis normal to the disc plane. He also suggested that what appear to be nebula are actually galaxies and that the solar system comprises a small portion of one of the universe's endless galactic structures.*

These ideas remained speculative until 1785*, when they were confirmed by the observations of* ***Frederick William Herschel*** *(1738–1822, England).* ***Kant*** *suggested that the universe was hundreds of millions of years old, and that it was in a state of continuous dynamical evolution that is manifested through motion, creation and disintegration. Kant's theory liberates time from its earthly confinement and links it with cosmic processes.*

1750 CE **Gabriel Cramer** (1704–1752, Switzerland). Mathematician. Widely known among students of mathematics for his rule for solving a system of linear equations by determinants[261].

Cramer was born in Geneva. He belonged to an ancient Holstein family known first in Strasbourg, and then in Geneva, where his father and grandfather were physicians. Cramer was educated at the University of Geneva, and in 1724 was given an appointment there as a professor of mathematics. In 1727 he took a two-year leave for travel, during which time he made the acquaintance of **Jean Bernoulli** in Basel. He died in Bagnols near Nimes in the south of France, where he sought to restore his failing health.

1750 CE **Maria Gaetana Agnesi** (1718–1799, Italy). Mathematician and philosopher. Became the first woman to occupy a chair of mathematics in modern times. It happened at the University of Bologna, Italy. The plane curve $y(x^2 + ya^2) = 8a^3$, now known as the *witch of Agnesi*[262], is named after her.

ca 1750 CE **Eugene Aram** (1704–1759, England). Self-taught philologist. Recognized in advance of scholars the Indo-European affinities of *Celtic* and disputed the derivation of Latin from Greek. But he was not destined to live in history as a pioneer of philology, as he should; In 1759 he was convicted of murdering his wife's lover (1745) and executed. This was the subject of a romance by Bulwer Lytton *Eugene Aram* (1832).

1750–1784 CE **John Michell** (1724–1793, England). Geologist, astronomer and the founder of the science of seismology. Expounded novel and farsighted ideas on a wide range of subjects:

- Made accurate magnetic observations, described a method of *magnetization* and gave a lucid exposition of the nature of *magnetic induction* (1750).
- Invented the *torsion balance* (1784) independently of **Coulomb** (1777). Michell described it in his proposal of a method for obtaining the mean density of the earth. He did not live to put his method into practice;

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[261] This rule was discovered independently by **Colin Maclaurin** (1698–1746, Scotland) in 1742.

[262] The name is a misnomer; It seems that Agnesi confused the old Italian word "*versorio*" [given earlier (1703) to the curve by **Guido Grandi** (1671–1742, Italy)] which means 'free to move in any direction' with *versiera* which means 'Devil's wife' or 'goblin'. This curve was treated earlier by **Fermat** (1663). A similar curve was studied by **James Gregory** (1658) and used by **Leibniz** (1674) in deriving the series $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$.

this was done by **Henry Cavendish**, who made, by means of Michell's apparatus, the celebrated determination that now goes by the name of *Cavendish's experiment* (*Phil. Trans.*, 1798).

- In his geological essay entitled *Conjectures concerning the cause and observations upon the phenomena of earthquakes* (*Phil. Trans.* **60**, 1760), he recognized that earthquakes originate *within* the earth and send out elastic waves through the earth's interior.
- Originated the concept of a *black hole*[263] in his essay *On the means of discovering the distance, magnitude etc. of the fixed stars* (*Phil. Trans.*, 1784), 12 years ahead of **Laplace**. Reasoning, a la Newton, that light is composed of particles, he calculating that a star with the same density as the sun but with a radius 500 times larger could, due to its gravitation alone, prevent the escape of light and consequently be invisible to the rest of the universe.

Michell was educated at Queens' College, Cambridge. He became M.A. in 1752, received his doctor's degree in 1761, and taught mathematics, theology, Greek, Hebrew and philosophy there. Appointed Woodwardian professor of geology in 1762, and in 1767 became rector of Thornhill in Yorkshire. He was elected a fellow of the Royal Society in the same year as Henry Cavendish (1760). Michell had a wide circle of scientific friends, among them **Joseph Priestley**, **John Smeaton** and **William Herschel**.

1750–1820 CE The *Classical Period* in music. Its leading composers are:

- Johann Wilhelm Hertel 1727–1789
- **Joseph Haydn** **1732–1809**
- Luigi Boccherini 1743–1805
- Domenico Cimarosa 1749–1801
- Carl Stamitz 1745–1801
- Giovanni Batista Viotti 1755–1824
- **Wolfgang Amadeus Mozart** **1756–1791**

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263 The velocity of escape v from the gravitational influence of a massive star of mass M and radius R is given by $v^2 = \frac{2GM}{R} = \frac{8\pi}{3}G\rho R^2$, where ρ is the star's density and G is the universal gravitational constant. If we require $v > c$ (velocity of light), light will be trapped inside the star; this happens whenever $\rho R^2 \geq \frac{3c^2}{8\pi G}$. Inserting the numerical values: $c = 3\times 10^{10}\ \frac{\text{cm}}{\text{sec}}$, $G = 6.672\times 10^{-8}$ cgs, ρ (sun's mean density) $= 1.41\ \frac{\text{g}}{\text{cm}^3}$, we find: $R \geq 500R_\odot$, where $R_\odot$ (sun) $= 6.96\times 10^{10}$ cm. The entity $R = \frac{2GM}{c^2}$ (obtainable from the equality $v = c$, i.e. $\frac{GM^2}{R} = \frac{1}{2}Mc^2$) is known today as the radius of the *event horizon* of a black hole of mass M. It is remarkable that already **Newton** (1687) hypothesized that light is attracted by massive bodies.

- Luigi Cherubini 1760–1842
- Johann Hoffmann 1760–1820
- **Ludwig van Beethoven 1770–1827**
- Johann Nepomuk Hummel 1778–1837

1751 CE **Johann Andreas von Segner** (1704–1777, Germany). Physicist and mathematician. Introduced the concept of *surface tension* of liquids.

Segner was a professor at Jena (1732–1735), Göttingen (1735–1755), and Halle (1755–1777). Invented (1750) a simple reaction waterwheel, later developed by Leonhard Euler into a crude turbine.

1751–1753 CE **Joseph-Jérome Le Francais de Lalande** (1732–1807, France) and **Nicolas-Louis de Lacaille** (1713–1762, France) obtained for the earth-moon distance the figure of 60 earth radii. The two French astronomers arrived at this result from measurement of the moon's parallax at the Cape of Good Hope and Berlin. Their result provided a more exact value than the estimates known since antiquity.

Lacaille was born at Rumigny, in the Ardennes. He studied theology at the Collège de Lisieux in Paris, and after taking his deacon's orders, he devoted himself exclusively to science. Through the patronage of **G.D. Cassini** he was employed in remeasuring the French arc of the meridian (1739). Subsequently, he was appointed a professor of mathematics in Mazarin College. His desire to observe the Southern heavens led him to propose (1750), an astronomical expedition to the Cape of Good Hope.

Lalande was born at Bourg. He studied law in Paris but was accidentally drawn to astronomy. On the completion of his legal studies he was about to return to Bourg to practice there as a lawyer, when **Lemonnier** sent him to Berlin to make observations on the lunar parallax in concert with those of Lacaille at the Cape of Good Hope.

1751–1772 CE **Jean-Jacques Rousseau**[264] (1712–1778, France). Philosopher of history and social reformer, whose ideas had great influence on Western civilization. He was the first to diagnose, from secular aspects, the symptoms of the crisis of modern civilization, that has not yet come to an end in the age of two World Wars. Both modern civilization and the entire history that shaped its features were condemned by Rousseau as deviations from nature.

[264] For further reading, see:

- De Beer, G., *Rousseau*, G.P. Putnam's sons: New York, 1972, 117 pp.

He asserted that *progress in arts and sciences was disastrous for mankind*; cultural life is degenerating more and more because vital needs of the human heart are neglected. He demanded a radical reform that does not mean return to primitive barbarism, but rather a restitution of the natural order in which reason and sentiment become harmonized. Free and equal men with inalienable wills have a right to institute a State through mutual agreement, by engaging in a social contract.

Rousseau became the precursor of the French and American Revolutions and caused a literary turmoil that started soon after the publication of his principal works [*Discours sur les arts et sciences* (1750); *Du contract social* (1762); *Emile* (1762)]. This religious creed is a deism that relies more on feelings than on reason, without excluding rational principles. In the field of education, his ideas were adopted by **Pestalozzi** (1746–1827).

Among the philosophers, his teachings are reflected in the works of **Kant** (1724–1804), **Fichte** (1762–1814), **Hegel** (1770–1831), and **Karl Marx** (1818–1883). His literary influence remained strong from the times of **Goethe** (1749–1832) and **Byron** (1788–1824) to the days of **R.L. Stevenson** (1850–1894) and **D.H. Lawrence** (1855–1930). Notwithstanding the excesses of the French Revolution [**Maximilien Robespierre** (1758–1794) was one of Rousseau's most devoted followers!], Rousseau continued to be regarded the apostle of democracy, although it it was discovered that some aspects of his philosophy favor totalitarian dictatorship.

The tormented soul of Jean-Jacques Rousseau was born to a **Huguenot** family of watchmakers in Geneva. Since the age of 10 he lived as an orphan. His unstable temperament, insatiable need for love, and deep sense of guilt explain his restless, wandering existence. An engraver's apprentice from 1727, he fled Geneva (1728), and in Annecy encountered Mme de Warens, to whom he would return at intervals in the following years. She sent him to Turin where he became a Catholic convert. Thereafter he was a footman, a seminarist, a music master (with little knowledge of music, he undertook to revolutionarize musical notation!), a tutor.

During 1732–1740 he settled near Mme Warens at Chambéry and made up for many gaps in his education by voracious readings. But from 1840 he was rootless again. He went to Paris (1741) and entered on a career in society, which included sojourn abroad as a secretary to the French ambassador in Venice (1743–1744), but otherwise he lived in Paris. He became attached (1746) to an illiterate inn-servant by whom he had 5 children, all placed in a foundling hospital.

At the same time he collaborated with Diderot in the *Encyclopedia*, almost exclusively on musical subjects. The reputation which he enjoys today as a writer is entirely founded on work written from 1750 onwards. The 1750's

saw quarrels with philosophers like **Voltaire** and **Diderot**, who had formerly befriended him, and the pattern was to be repeated until his death.

With tireless energy he wrote operas, plays, novels, essays, political tracts, autobiography and social discourses, enough to fill 47 volumes. Nearly everything that came from his pen was controversial and combative enough to make for him many distinguished enemies, which he constantly fled from. These miseries fed his persecution complex.

Several years of his life were spent in exile. It was in England (1766–1767) that he wrote his *confessions* and quarreled with **Hume**. However, some respite was granted in 1770, when he found humble but quiet lodging in Paris, where he wrote his latest works. He was undoubtedly partly insane during the 10–15 last years of his life.

Rousseau's *social contract* was not only an influence on his time and his country, but also on the revolutionary founders of democracy in America.

1751–1776 CE **Denis Diderot** (1713–1784, France). Encyclopedist. One of the first *evolutionary* thinkers. He was the editor of the great *Encyclopedia of the Sciences and Crafts*, whose publication in 35 volumes, 1751–1776, impeded by government censorship, was the culminating event of the Enlightenment. Diderot himself wrote many articles on industrial and technical processes, studying them first hand for this purpose. He gained thereby a genuine appreciation for practical and experimental knowledge that led him to urge *the founding of scientific laboratories*. Diderot followed **Montesquieu** (1689–1755, France) in breaking the chains of Biblical chronology of nature. He thought that the age of the universe was a matter of several millions of years.

1752–1756 CE **James Dodson** (1705–1757, England). Mathematician, actuary[265], and innovator in the insurance industry. Published (1756) "First Lectures on Insurance".

[265] An actuary is a business professional who deals with the financial impact of risk and uncertainty.

Actuaries have a deep understanding of financial security systems, their reasons for being, their complexity, their mathematics, and the way they work. They evaluate the likelihood of events and quantify the contingent outcomes in order to minimize losses, both emotional and financial, associated with uncertain undesirable events. Since many events, such as death, cannot be totally avoided, it is helpful to take measures to minimize their financial impact when they occur. These risks can affect both sides of the balance sheet, and require asset management, liability management, and valuation skills. Analytical skills, business knowledge and understanding of human behavior and the vagaries of information systems are required to design and manage programs that control risk.

Dodson's pioneering work on the level premium system led to the formation of the 'Society for Equitable Assurances on Lives and Survivorship' (1762) which used the actuarial principles that Dodson had developed over the previous decade. This was the first life insurance company to use premium rates which were calculated scientifically for long-term life policies.

Actuaries' insurance disciplines may be classified as life; health; pensions, annuities, and asset management; social welfare programs; property; casualty; general insurance; and reinsurance. Life, health, and pension actuaries deal with mortality risk, morbidity, and consumer choice regarding the ongoing utilization of drugs and medical services risk, and investment risk. Products prominent in their work include life insurance, annuities, pensions, mortgage and credit insurance, short and long term disability, and medical, dental, health savings accounts and long term care insurance. In addition to these risks, social insurance programs are greatly influenced by public opinion, politics, budget constraints, changing demographics and other factors such as medical technology, inflation and cost of living considerations.

Casualty actuaries, also known as non-life or general insurance actuaries, deal with catastrophic, unnatural risks that can occur to people or property. Products prominent in their work include auto insurance, homeowners insurance, commercial property insurance, workers compensation, title insurance, malpractice insurance, products liability insurance, directors and officers liability insurance, environmental and marine insurance, terrorism insurance and other types of liability insurance. Reinsurance products have to accommodate all of the previously mentioned products, and in addition have to properly reflect the increasing long term risks associated with climate change, cultural litigiousness, acts of war, terrorism and politics.

Actuaries use skills in mathematics, economics, finance, probability and statistics, and business to help businesses assess the risk of certain events occurring, and to formulate policies that minimize the cost of that risk. For this reason, actuaries are essential to the insurance and reinsurance industry, either as staff employees or as consultants, as well as to government agencies such as the Government Actuary's Department in the UK or the Social Security Administration in the US. Actuaries assemble and analyze data to estimate the probability and likely cost of the occurrence of an event such as death, sickness, injury, disability, or loss of property. Actuaries also address financial questions, including those involving the level of pension contributions required to produce a certain retirement income and the way in which a company should invest resources to maximize its return on investments in light of potential risk. Using their broad knowledge, actuaries help design and price insurance policies, pension plans, and other financial strategies in a manner which will help ensure that the plans are maintained on a sound financial basis.

In ancient Rome, the title of *actuarius* was given to the secretary of the senate, responsible for compiling the *Acta Senatus*. Prior to 1762, the use of the term had been restricted to an official who recorded the decisions (or 'acts') of ecclesiastical courts.

The 17^{th} century was a period of extraordinary advances in mathematics in Germany, France and England. At the same time there was a rapidly growing desire and need to place the valuation of personal risk on a more scientific basis. Independently from each other, compound interest was studied and probability theory emerged as a well understood mathematical discipline. Another important advance came in 1662 from a London draper named **John Graunt**, who showed that there were predictable patterns of longevity and death in a defined group, or cohort, of people, despite the uncertainty about the future longevity or mortality of any one individual person. This study became the basis for the original life table. It was now possible to set up an insurance scheme to provide life insurance or pensions for a group of people, and to calculate with some degree of accuracy, how much each person in the group should contribute to a common fund assumed to earn a fixed rate of interest. The first person to demonstrate publicly how this could be done was Edmond Halley. In addition to constructing his own life table, Halley demonstrated a method of using his life table to calculate the premium someone of a given age should pay to purchase a life-annuity (Halley 1693). Dodson built on these statistical mortality tables.

In the eighteenth and nineteenth centuries, computational complexity was limited to manual calculations. The actual calculations required to compute fair insurance premiums are rather complex. The actuaries of that time developed methods to construct easily-used tables, using sophisticated approximations to facilitate timely, accurate, manual calculations of premiums. In the 1930s and 1940s, however, rigorous mathematical foundations for stochastic processes were developed. Actuaries could now begin to forecast losses using models of random events instead of the deterministic methods. Computers further revolutionized the actuarial profession. From pencil-and-paper to punchcards to microcomputers, the modeling and forecasting ability of the actuary has grown exponentially.

1752–1773 CE **Victor Albrecht von Haller** (1708–1777, Switzerland). Physician, naturalist, anatomist, physiologist, botanist, historian of science and poet.

One of the founders of experimental physiology. Elucidated the mechanism of respiration. Discovered the function of *bile*. First to distinguish and relate muscle irritability and nerve sensibility and show transmission of nerve impulse (1752). His book *Elementa physiologiae* was the demarcation line between modern physiology and whatever preceded it (9 volumes, 1759–1776).

He was first to show (1766) that nerves stimulate muscles to contract and that all nerves lead to the spinal cord and brain.

Haller was born in Bern. He was known as a child prodigy and at age 10 already mastered Latin, Greek and Hebrew. He studied at the Universities of Tübingen (1723), Leiden (1725–1727), Paris (1728) and Basel (1728). He earned his medical degree in 1727 and studied mathematics under **John Bernoulli** in Basel. He served as a professor of medicine and botany in Göttingen (1736–1753) and practiced medicine at Bern (1753–1777). Haller published 650 articles on almost every branch of human knowledge and wrote books, treatises and bibliographies on physiology, medicine, history of science, botany, philosophy and poetry. He topped this prolific literary and research activity with three marriages, having 8 children.

In 1773 the state of his health rendered necessary his entire withdrawal from public business; for some time he supported his failing strength by means of opium; it is believed that the excessive use of the drug hastened his death.

1753–1763 CE **Carolus Linnaeus** (Carl von Linné, 1707–1778, Sweden). Naturalist and botanist. Established the modern scientific method of classification and naming of plants, animals, minerals and diseases. In this system, each living thing has a name with two parts; the first part is the *genus* (group), and the second part is for the *species* (kind). Linnaeus' book *Species Plantarum* (1753) forms the basis for plant classification. His *Systema Naturae* (1758) covers animal classification, while his *Genera Morborum* (1763) classifies diseases.

Linné was born in Råshult, in the province of Småland, Sweden. In 1726 his father destined him to be an apprentice to a shoemaker. He was, however, saved from this fate through his town physician, who expressed his belief that he would yet distinguish himself in medicine, and who further instructed him in physiology.

In 1728 he entered the University of Uppsala and studied botany. In 1732 he undertook to explore Lapland; with the equivalent of 50 dollars given to him by the Royal Society of Science, he spent 5 months collecting plants while walking nearly 1600 kilometers. Linnaeus then went to The Netherlands, where he earned his medical degree in 1735. He returned to Stockholm in 1738 to practice medicine as a naval physician. In 1741 he was appointed to the chair of medicine at Uppsala, but soon changed it for that of botany (1742). In 1761 he was granted a title of nobility with the name **Carl von Linné**.

When Linné appeared upon the scene, new plants and animals in increasing numbers were daily discovered thanks to the increase in trade. To him belongs the honor of having first enunciated the principles for defining genera

and species. No naturalist has impressed his own character with greater force upon his pupils than did Linné. He imbued them with his own intensive acquisitiveness, taught them in an atmosphere of enthusiasm, trained them to close and accurate observation, and then dispatched them to various parts of the globe.

1754–1761 CE **Jean Etienne Montucla** (1725–1799, France). Historian of the mathematical sciences. His book *Histoire des mathématiques* (2 volumes, 1758; second edition, 4 volumes, 1795–1802) is essentially a history of science from a mathematical viewpoint. It is the first comprehensive modern evaluation of the evolution of mathematical thought, especially with reference to the 17^{th} and 18^{th} centuries.

Montucla was born in Lyon. He received his first education in the Jesuit College of Lyon. It included a thorough training in mathematics, Greek and Latin. He later picked up sufficient understanding of Italian, English, German and Dutch. In 1745 he studied law in Toulouse and a few years later established himself in Paris. There he came under the influence of **Diderot**, **d'Alembert**, **Lalande** and others and began his investigations on the history of mathematics. His first publication concerns the history of the attempts to square the circle (*Histoire des recherches sur la quadrature du cercle*, 1754).

After an ill-fated trip to French Guiana (1764–1765), where he was appointed royal astronomer of that colony, he lived for the rest of his life in Versailles, where he was superintendent of royal buildings, gardens, manufactures, and academies. During that period of peaceful activity, Montucla devoted his leisure to historical studies. In spite of the fact that he has been a clerk in the royal administration, he had good friends among the revolutionaries who kept him unharmed and unaffected by the revolution.

1754–1798 CE **Immanuel Kant** (1724–1804, Germany). Idealist philosopher and speculative scientist. Established a system of thought that dominated the philosophy of the 19^{th} century. No other philosopher of modern times has been throughout his work so imbued with the fundamental conceptions of physical science; no other has been able to hold with such firmness the balance between empirical and speculative ideas.

The early writings of Kant are almost without exception on questions of physical science. It was only by degrees that philosophical problems began to engage his attention, and that the main thrust of his literary activity turned toward them. The following are the most important of his works which bear directly on physical science:

- *The Nebular Hypothesis* (1755) was motivated by the faint patches of light which telescopes revealed in large numbers. A particularly troublesome

item was a cloudy patch of light in the constellation *Andromeda*[266]. In his book *Universal Natural History and Theory of the Heavens*[267], Kant hypothesized a primeval, slowly rotating cloud of gas (nebula) which in some unspecified fashion condenses into a number of discrete globular bodies. The rotation of the parent nebula is preserved in the rotation of the sun, the revolution of the planets about the sun, and the rotation of the planet about their axes — all in the same direction.

According to Kant and Laplace, the original mass of gas cooled and began to contract. As it did, the rotational speed increased until successive rings of gaseous material spun off from the central mass by centrifugal forces. In the final stages the rings condensed into planets. While Laplace considered the Andromeda Nebula to represent a planetary system in the process of formation, Kant did not accept the Andromeda as a visible support of his own theory.

Instead, he suggested that Andromeda and similar bodies, might represent immensely large conglomerations of stars, which appeared as small, fuzzy

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[266] Visible to the naked eye as a small object of the 4^{th} magnitude that looks like a fuzzy star. Some Arab astronomers had noted it in their maps, but the first to describe it in modern times was the astronomer **Simon Martin** (1570–1624, Germany) in 1612.

[267] Published anonymously. The publisher went bankrupt and the stock was seized by the creditors, so that very few copies reached the public.

Laplace proposed essentially the same theory in 1796, without the mathematical formulation which he was incapable of providing. Had he been able to provide it, he might have discovered some serious flaws. Indeed, **Maxwell** and **Jeans** showed about 100 years later that there was not enough mass in the rings to provide the gravitational attraction for condensation into individual planets. The coup de grâce was delivered in 1906, when **Forest Ray Moulton** (1872–1952, U.S.A.) showed that the nebular hypothesis violated the observation that the planets carry 99 percent of the angular momentum (the sun, which collected 99.9 percent of the mass should have gathered most of the angular momentum of the system). Nevertheless, *recent* theories tend to be neo-Kantian in the sense that they revive the idea of primordial, rotating cloud of gas and dust whose shape and internal motions were determined by gravitational and rotational forces. At some moment, gravitational attraction became the dominant factor, contraction began, and the rotation speeded up. The cloud tended to flatten into a *disk*; matter began to drift toward the center, accumulating into the *proto-sun*. The proto-sun collapsed due to its own gravitation, ending with the known scenario of sustained thermonuclear reactions. The formation of the planets and how they picked up the necessary angular momentum is, however, still poorly understood.

patches only because they were immensely far away. He felt they might represent "*island universes*", each one a separate galaxy, so to speak. However, this suggestion of Kant's was not based upon any observational data available to the astronomers of the time. It made very few converts, and was dismissed as a kind of a science fiction[268].

- *Secular retardation of the earth's rotation* (1754). Pointed out that the tide-generating forces of the moon might act through the oceans to produce a breaking effect on the earth's rotation. (The attendant acceleration in the *orbital* motion of the moon had been suggested by Halley in 1695.) First to suggest that tidal friction would cause a lengthening of the day.
- Calculated (1754) that if the sun's light came from ordinary combustion, it would have burned out in 1000 years.
- Conjectured (1786) that the major forces of nature are manifestation of a single force, and can be converted one into the other.
- *Theory of winds* (1756). Independently of Hadley (1735), pointed out how the varying velocity of rotation of the successive zones of the earth's surface furnishes a key to the phenomena of periodic winds.

Consideration of these works is sufficient to show that Kant's mastery of the science of his time was complete and thorough, and that his philosophy is to be dealt with as having throughout a reference to general scientific conceptions.

Trained in the philosophy of **Leibniz**, he was influenced by the mathematical theories of **Newton**, by the psychological theories of **John Locke** (1632–1704, England), and especially by the philosophy of **David Hume** (1711–1776).

His own system was rooted in a rationalistic outlook, but sought to implant a comprehensive method and doctrine of experience that would improve upon mere intellectual idealism. His *Critique of Pure Reason* (*Kritik der Reiner Vernunft*) (1781) cost him fifteen years of critical analysis of human thought. Like other philosophers before him he maintained that only part of our knowledge is based on experience. The world we observe is only a part of a reality that we are able to conceive. Another part is *not* inferred inductively from our experience but is acquired by our senses, then filtered and elaborated by

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268 The nebular hypothesis was given observational support in 1983, when an orbiting telescope in space, the *Infrared Astronomical Satellite*, found the first evidence for disks of particles orbiting stars.

To date, such disks of gas and dust, which may be proto-solar-systems and/or debris left over from the formation of the planets, have been found around as many as a quarter of all nearby stars.

our reason, thinking and intelligence. Here, reason brings laws, order and regularity into the observed phenomena.

Thus, the laws conceived are the result of the process of reason, which does *not* derive from nature. We understand these phenomena because we approach them with certain notions and concepts for which Kant uses the term *a priori*. Among such necessary notions, or categories, are space and time. They are prerequisite and basic structures into which we must fit all our perceptions. We cannot imagine that there could be no space, even if we can imagine that there should be nothing in the space. The same reflections apply to time. Without these two *a priori* notions we should be unable to perceive a well-ordered universe.[269]

The law of *causality* is another *a priori* notion: When an event is observed, it must be determined by a preceding event. For Kant the *a priori* category of the law of causality forms an absolute necessity of all science; it is not an empirical assertion that can be proved or disproved by experiment. Rather it forms the *basis* of all experience.

These *a priori* categories are based on Newtonian mechanics, which strongly influenced not only Kant's philosophy but that of the 19^{th} century. These laws of physics had absolute validity for Kant and were not subject to any question.[270]

Kant argued that no description of the World can free itself from the reference to human experience. Although the world that we know is not of

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[269] Kant's proposition that the human mind inevitably imposes order on the world so as to make sense of it ceased to impress scientists of the 20^{th} century. Kant knew nothing of atomic or nuclear structure, yet the study of the atom revealed the same sort of mathematical regularities — many more of them in fact — that occur in the organization of the solar system. This fact has nothing to do with the way we choose to perceive the world. Moreover, it is difficult to be convinced that the deep and complex mathematical symmetries evinced in the operation of the fundamental forces is of no significance except as a tribute to the tidy nature of the human mind.

[270] Their limited applicability only became apparent through the results of modern physics. Moreover, in Newtonian physics, the geometry that formed the essential basis in his concepts was that of Euclid. Not till the 19^{th} century was a new geometry developed, particularly by the pioneer work of **Gauss** (1777–1855), which then greatly influenced thinking in physics and philosophy. Kant obviously could not have foreseen the startling developments of modern physics, neither the theory of relativity nor quantum theory. The former forced the change of the *a priori* concepts of time and space. The latter demonstrated that the law of causality is not strictly applicable to events in the atom.

our creation, it cannot be known except from the point of view that is ours. All attempts to break through the limits imposed by experience, and to know the world '*as it is in itself*', from the absolute perspective of 'Pure Reason' – end in contradiction. 'Ideas' of reason can never be coherently applied, and although we may have intimations of an 'absolute' or 'transcendental' knowledge, that knowledge can never be ours: to be sure, we know only appearances, colors, sounds and the like, never the *thing-in-itself* ("Ding an Sich").

Thus Kant maintained that true knowledge cannot transcend experience. The temptation of Pure Reason, Kant argued, can never be overcome. It is part of our nature as rational beings that we should aspire towards the 'transcendental' perspective. This yearning of reason toward the eternal is at the root of morality. Transformed into practical imperative, the Ideas of Reason provide a moral law which guides us. Kant was certain that there cannot be morality without some belief in God or immortality. This obliged one to presuppose the existence of God as a necessity.

Kant's '*Idealist*' philosophy was the exact opposite to 17^{th} century *materialist* philosophy. The materialists wanted to reach an absolute truth through *science*. Kant claimed that this truth is subordinate to our senses and for this reason science is unable to discern the *essence* of things independently of the process of understanding. Reason enables man to conceive the universe but his senses prevent him from doing so. To establish universal laws one must go beyond all possible experience. The objective of knowledge is just a myth, although Kant still accepts the objective *existence* of things, which he called 'Being'. 'Being' is the very essence of things, and independent of the way in which things appear to us.

Kant was born at Königsberg. His grandfather was an emigrant from Scotland, and the name Cant is not uncommon in the north of Scotland. In his youth he studied theology and his inclination at this time was towards the classics. During his university course, which began in 1740, Kant was principally attracted towards mathematics and physics. During 1746–1755 he was much disturbed by poverty and was compelled to earn his own living as a private tutor. But with the aid of friends he was able to resume his studies, and during 1755–1770 he slowly and patiently worked his way from the rank of privatdocent to that of professor of logic and metaphysics at Königsberg.

In the course of 1781–1793, the Kantian philosophy made rapid progress in Germany, and the *Critique of Pure Reason* was expounded in all the leading universities, and even penetrated the schools of the Church of Rome. Young men flocked to Königsberg as to a shrine of philosophy.

In 1792, Kant was involved in a dispute with the government on the question of his religious doctrines, since his *moral rationalism* could not be reconciled to the literal doctrines of the Lutheran Church. The government, influenced by hatred and fear of the French Revolution, banned his writings in Berlin, and exacted from him a pledge not to lecture or write at all on religious subjects in the future. Consequently, in 1794, he withdrew from society and in 1797 he ceased altogether his public affairs, after an academic career of 42 years.

His stature was small, and his appearance feeble. He was little more than 5 feet high; his breast was almost concave, and he had a deformed right shoulder. His senses were quick and delicate, and though of weak constitution, he stayed healthy through a strict regimen. His life was arranged with mechanical regularity; and, as he never married, he kept the habits of his studious youth to old age. His man-servant, who woke him summer and winter at 5 o'clock, testified that he had not once failed in 30 years to respond to the call.

After rising, he studied for 2 hours, then lectured for another two, and spent the rest of the morning, till one, at his desk. He then dined at a restaurant (which was his only regular meal), and often held prolonged conversation until late in the afternoon. He then walked out for at least one hour in any weather, always at the same time (the burghers used to set their watches when he passed under their windows!). The evenings were spent in lighter reading[271], except for an hour or two devoted to the preparation of his next day's lectures, after which he retired between 9 and 10.

His acquaintance with books of science, general history and travels was boundless. He was fond of newspapers and works on politics. As a lecturer, Kant avoided altogether that rigid style in which his books were written. He sat behind a low desk, with a few jottings on slips of paper or book margins, and delivered an extemporaneous address, opening up the subject by partial glimpses and many anecdotes or familiar illustrations, until a complete idea of it was conveyed. His voice was extremely weak, but sometimes rose into eloquence, and always commanded perfect silence. Though kind to his students, he refused to remit their fees, as this, he thought, would discourage independence. Another of his principles was that his chief exertions should

[271] At 70 he wrote an essay "*On the Power of the Mind to Master the Feeling of Illness by Force of Resolution*". One of his favorite principles was to breathe only through his nose, especially when outdoors; hence, in autumn, winter and spring he would permit no one to talk to him on his daily walks; better silence than a cold. He applied philosophy even to holding up his stockings — by bands running up his trousers to the pockets, where they ended in springs contained in small boxes. He remained a lifelong bachelor; he felt that marriage would hamper him in the honest pursuit of truth.

be bestowed on the intermediate class of talent, as the geniuses would help themselves and the dunces were beyond remedy.

Truthful, kind-hearted and high-minded as Kant was in all moral respects, he was somewhat deficient in sentiment. He held little enthusiasm for the beauties of nature, and indeed never sailed into the Baltic, or traveled more than 60 km from Königsberg; shunned music and poetry, and held the female sex in low esteem. Though faithful in a high degree to the duties of friendship, he could not bear to visit his friends in sickness, and after their death he repressed all allusion to their memory. His engrossing intellectual efforts no doubt tended to harden his character, and in his zeal for rectitude of purpose he forgot the essential part which affection and sentiment play in human affairs.

Yet, the influence of Kant on Europe was enormous: the entire philosophic thought of the 19th century revolved about his speculations. After Kant, all Germany began to talk metaphysics: **Schiller** and **Goethe** studied him; **Beethoven** quoted with admiration his famous words; and **Fichte**, **Schelling**, **Hegel** and **Schopenhauer** produced in succession systems of thought reared upon the idealism of Kant. His criticism of reason, and his exaltation of feeling, prepared for the teachings of Schopenhauer, **Nietzsche**, **Spencer**, **Bergson** and **William James**. His identification of the laws of thought with the laws of reality gave to **Hegel** a whole system of philosophy.

Immanuel Kant made disparaging statements about Jews and non-whites. However, because of the magnitude of his achievement, scholars have tended to downplay his unwholesome writings on Jews and non-white people.

Clearly, Kant did not generate his anti-Semitism out of thin air: As with other figures of the Enlightenment (e.g. **Voltaire** and **Thomas Paine**), his mind was furnished with the medieval thinking he intended to refute. Going back to at least the 12th century, European culture had developed a distorted image of the Jews as grasping materialists and as slaves to pedantic legality. These perceived traits (encapsulated in the Shakespearean figure of Shylock) were contrasted with an idealized revision of Christianity committed to otherworld values and spiritual freedom – providing the structure for Kant's world view[272].

[272] The historian **Michael Mack** in his study "*German Idealism and the Jews*" (University of Chicago Press) argues for a deep affinity between modern anti-Semitism and the philosophy of Immanuel Kant. By Mack's account, Kant's contempt for the Jew is intimately related to the central themes of his world view, and sheds light on the limits of Enlightenment thinking. According to Mack, all the positive traits of Kantian philosophy (freedom, autonomy, reason) are formed by being contrasted with a negative image of unenlightened

Worldview XIV: Immanuel Kant

* * *

"Human reason is burdened by questions which, as prescribed by the very nature of reason itself, it is not able to ignore, but which, as transcending all its power, it is also not able to answer."

* * *

"It is impossible to prove the existence of God through any normal means."

* * *

"Every intent, whether scientific or religious, to define reality is nothing other than pure hypothesis."

* * *

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humanity. *He saw Judaism as an inherently materialist religion, based upon a quid pro quo between God and His chosen people.*

"In order to fully define the formal structures of his philosophy (autonomy, reason, morality and freedom), Kant almost unconsciously fantasized about the Jews as its opposite," Mack notes. "*He posited Judaism as an abstract principle that does nothing else but, paradoxically, desire the consumption of material goods.*"

As portrayed in Mack's book, Kant is a pivotal figure in Western thought because he *took this earlier religious hostility toward Jews and reformulated it in philosophic language.* By showing that the traditional critique of the Jews could be made by an Enlightenment philosopher, *Kant set the stage for modern secular anti-Semitism.* In the central chapters of his book, Mack argues that what he believes is Kant's fundamental antinomy (free enlightened humanity versus *Jews enslaved to materialism*) provided the framework for future anti-Semites, notably the philosopher **G.W.F. Hegel** and the musician **Richard Wagner**. Since Wagner in particular was a cultural hero for Adolf Hitler, Kant's own anti-Semitism can be seen as having a far-reaching effect.

"Every attempt to apprehend transcendental knowledge is vain, since for every thesis the mind produces, one can oppose an equally valid anti-thesis."

* * *

"Give me matter and I will construct a world out of it."

* * *

"I call it the 'thing in itself'. I differentiate it from phenomena, that is, the world as it appears to us."

* * *

"Seek not the favor of the multitude; it is seldom got by honest and lawful means. But seek the testimony of the few, and number not voices but weigh them."

* * *

"Two things fill the mind with ever new increasing admiration and awe: the starry heavens above me, and the moral law within me."

* * *

"Concepts without factual content are empty; sense data without concepts are blind: the understanding cannot see, the senses cannon think. By their union only can knowledge be produced."

* * *

"One must understand that the greatest evil that can oppress civilized peoples derives from wars, not, indeed, so much from actual present or past wars, as form the never-ending arming for future war. To this end all the nation's powers are devoted, as are all those fruits of its culture that could be used to build a still greater culture."

* * *

"Understanding is the knowledge of the general.
Judgment is the application of the general to the particular.
Reason is the power of understanding the connection between the general and the particular."

* * *

"Intelligence divorced from judgment produces nothing but foolishness."

* * *

"The mark of a mature man is to live for a cause, that of an immature man to die for a cause."

* * *

"Memory should only be occupied with such things as are important to be retained, and which will be of service to us in real life."

* * *

"Science is organized knowledge. Wisdom is organized life."

* * *

1755 CE, Nov. 01 A major earthquake destroyed two-thirds of the city of Lisbon and killed more than 60,000 people in Portugal.

1755 CE **Samuel Johnson** (1709–1784, England). Writer, moralist and scholar. Composed the first comprehensive authoritative *Dictionary of English Language*[273], including definitions for some 114,000 words. In the preface he declared that

"The Dictionary was written *with little assistance of the learned, and without any patronage of the great; not in the soft obscurities of retirement, or under the shelter of academic bowers, but amidst inconvenience and distraction, in sickness and in sorrow*".

Though plagued by ill health and stricken by the death of his wife, he produced the work just $8\frac{1}{2}$ years after he had begun. He legislated standard English into existence – by the power of a printed dictionary.

In his preface Johnson explained that language was inevitably changed by conquests, migration, and commerce, and by the progress of thought and knowledge; "*No dictionary of a living tongue ever can be perfect, since while it is hastening for publication, some words are budding and some falling away*".

1755–1783 CE **Ruggiero Giuseppe Boscovich** (Rudjer Josip Bošcović; 1711–1787, Croatia and Italy). Mathematician, astronomer and physicist. One of the earliest continental savants to adopt Newton's gravitation theory and apply it to the calculation of orbits and rotations of celestial bodies, and to the figure of the earth. Advanced an atomistic theory of matter (1758) in which atoms possess inertia and mutual interaction. He considered that chemical elements result from combination of point atoms, and chemical compounds from combination of chemical elements. These ideas influenced both **Humphry Davy** and **Michael Faraday**.

He published many remarkable memoirs, among them solutions of the problem to determine the orbit of a comet from three observations, and the *achromatic telescope* (1778).

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[273] The first English dictionary appeared in 1604, authored by **Robert and Thomas Cawdrey**, schoolmaster father and his son, and entitled: "*A Table Alphabeticall, conteyning and teaching the true writing and understanding of hard usuall English wordes, borrowed from the Hebrew, Greeke, Latine, or French, etc.*" Next came *The New World of English Words* (1658) by **Edward Phillips** (1630–1696). It was followed by *A New English Dictionary* (1702) by **John Kersey**.

Two other famous dictionaries of the English language are **Noah Webster**'s *American Dictionary of the English Language* (1828) and **James A.H. Murrey**'s (1837–1915) *Oxford English Dictionary* (1925).

Boscovich was born at Ragusa in Dalmatia. Joined the Jesuits (1725), and on completing his noviciate at Rome, studied mathematics and physics at the Collegium Romanum. Taught in Rome (1740), Pavia (1764) and Milan (1770), and became director of optics for the French navy (1773–1883). He took part in the Portuguese expedition for the survey of Brazil, and the measurement of a degree of the meridian (1743). He also measured an arc of two degrees between Rome and Rimini.

In 1783 he returned to Italy. But his health was failing, his reputation was on the wane, and his works did not sell. He fell into melancholy, and finally madness, with lucid intervals, and died in Milan.

1755–1788 CE **Joseph Louis Lagrange** (1736–1813, France). One of the greatest mathematicians of the 18^{th} century. He belongs to that brilliant group of mathematicians whose magnanimous rivalries helped to accomplish the task of generalization and deduction reserved for the post-Newtonian era. Indeed, it is by no means easy to distinguish and apportion the respective merits of the competitors. This is especially the case between Lagrange and **Euler** on the one side and between Lagrange and Laplace on the other. Lagrange's mathematical career can, however, be viewed as a natural extension of the work of his older and greater contemporary **Euler**, which in many respects he furthered and refined.

In 1755 Lagrange communicated to Euler his method of multipliers for solving isoperimetric problems. He is justly regarded as the inventor of the *calculus of variations* (the name given by Euler in 1766).

During 1773–1784, Lagrange undertook the demanding task of verifying Newton's universal gravitation via the observed motions of the planets and comets of the solar system. Using the method of planetary perturbations and transferring the origin of coordinates from the center of the sun to the center of gravity of the sun-planet system, he was able to achieve great simplification.

With **Alexandre Vandermonde** he introduced in 1770 the notion of a 'group' (though not the term). In 1773 he originated the idea of scalar gravitational potential.

Lagrange took conspicuous part in the advancement of almost every branch of pure mathematics. In the theory of numbers he furnished proofs of many of Fermat's theorems, and added some of his own. In algebra he discovered the method of approximating the real roots of cubic and quartic equations by means of continued fractions. [*Traité de la résolution des équations numerique de tous degrés* (1767).] To the calculus of finite differences he contributed the beautiful formula of *interpolation* which bears his name (although substantially the same result seems to have been previously obtained by Euler) and the *Lagrange expansion* (1770).

Lagrange's contributions to the theory of equations were doubtless the most potent anticipations of Galois' later breakthrough (1831). In a 1770–1771 memoir, Lagrange attempted to find a uniform procedure for solving equations of all degrees. He analyzed the methods that had yielded general solutions for degrees 2, 3, 4, and found that in each case the technique involved the use of a *resolvent* equation. Although the latter was of lower degree than the original for $n = 2, 3, 4$, Lagrange discovered that application of the previously successful pattern to the quintic $(n = 5)$, led to an irreducible sextic $(n = 6)$, and the problem became more difficult instead of being resolved. He then hinted at the impossibility of solution by radicals, and let the matter drop.

His greatest achievement was the transformation of mechanics [defined by him as a "*Geometry of four dimensions*"] into a branch of analysis, by exhibiting mechanical principles as simple results of the calculus: instead of following the motion of each individual mass, he determined their collective configuration by a sufficient number of dynamical variables, whose number is that of the scalar motional degrees of freedom, there being as many equations as the system has degrees of freedom. The kinetic and potential energy of the system can then be expressed in terms of these, and the differential equations of motion follow by simple differentiations.

Lagrange gave the solution of isoperimetric problems quite independently of Euler, and with entirely new methods. He developed for this purpose the new *calculus of variations*.

His work had deep influence on later mathematical research, for he was the earliest first-rank mathematician to attempt a rigorization of the calculus. His cardinal idea was the representation of a function $f(x)$ by a Taylor's series. The notation $f'(x)$, $f''(x)$ is due to Lagrange. But he failed to give sufficient attention to matters of convergence and divergence, which were later taken up by his pupil **Cauchy**.

Lagrange[274] was born at Turin of mixed French-Italian ancestry. His interest in mathematics was aroused through the reading of a paper by Halley on the uses of algebra in optics. An intensive self-study for two years placed him on a level with the greatest of his contemporaries and at the age of 19 he was appointed professor of geometry in the Royal Artillery School in Turin (1754). At the age of 26, Lagrange found himself at the summit of European fame (1762). In 1764 he carried off the prize offered by the Paris Academy of Sciences for the best essay on the *librations of the moon.* He won four more such prizes: *theory of Jovian systems* (1766), *restricted 3 body problem* (1772),

[274] He was born with the name *Lagrangia*.

secular equation of the moon (1774) and *the theory of cometary perturbations* (1778).

In 1776, when Euler left Berlin for St. Petersburg, he suggested to Frederick the Great that Lagrange be invited to take his place. The invitation conveying the wish of the "greatest king in Europe" to have the "greatest mathematician" at his court, was sent to Turin. Lagrange accepted and lived in Berlin for twenty years (1766–1786) until the death of Frederick. There he had ample leisure for scientific research, and royal favor sufficient to secure him respect without exciting envy. During this period he introduced the concept of *velocity potential*, and made the first use of the *stream function* in the analysis of fluid motion (1781). In 1788 he wrote the treatise *Mécanique Analytique* in which he unified and developed analytical mechanics, introducing the '*Lagrangian*'[275] and the '*Lagrange equation*'[276]. In this book Lagrange created a new and powerful tool which could solve any mechanical problem on

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[275] For further reading, see:

- Doughty, N.A., *Lagrangian Interaction*, Addison-Wesley, 1990, 569 pp.

[276] Consider the dynamical system composed of n mass points located at vector positions $\mathbf{r}_j$ $(j = 1, 2, \dots, n)$ w.r.t. some origin. The resultant external force on the j^{th} mass is denoted $\mathbf{F}_j$. We shall designate by q_k $(k = 1, 2, \dots, m)$ the generalized coordinates necessary to describe the system; in general $m \le 3n$ due to possible constraints. Since $\mathbf{r}_j = \mathbf{r}_j(q_1, q_2, \dots, q_m)$, we have: $\delta\mathbf{r}_j = \sum_{k=1}^{m} \frac{\partial \mathbf{r}_j}{\partial q_k}\delta q_k$; $\dot{\mathbf{r}}_j = \sum_{k=1}^{m} \frac{\partial \mathbf{r}_j}{\partial q_k}\dot{q}_k$ $\left(\text{dot} = \frac{d}{dt}\right)$. We confine our attention to *holonomic* systems, that is, systems in which the δq_k and $\delta \dot{q}_k$ are independent. The virtual work is then

$$\delta w = \sum_{j=1}^{n} \mathbf{F}_j \cdot \delta\mathbf{r}_j = \sum_{k=1}^{m} Q_k \delta q_k,$$

where

$$Q_k = \sum_{j=1}^{n} F_j \cdot \frac{\partial \mathbf{r}_j}{\partial q_k}$$

are defined as *generalized forces*. The d'Alembertian principle of Virtual Work then reads:

$$\sum_{k,j} (\mathbf{F}_j - m_j \ddot{\mathbf{r}}_j) \cdot \frac{\partial \mathbf{r}_j}{\partial q_k}\delta q_k = \sum_{k=1}^{m}\left[Q_k - \sum_{j=1}^{n} m_j \ddot{\mathbf{r}}_j \cdot \frac{\partial \mathbf{r}_j}{\partial q_k}\right]\delta q_k = 0.$$

In this expression

$$m_j \ddot{\mathbf{r}}_j \cdot \frac{\partial \mathbf{r}_j}{\partial q_k} = \frac{d}{dt}\left[m_j \dot{\mathbf{r}}_j \cdot \frac{\partial \mathbf{r}_j}{\partial q_k}\right] - m_j \dot{\mathbf{r}}_j \cdot \frac{\partial \dot{\mathbf{r}}}{\partial q_k}. \tag{1}$$

the basis of pure calculation, without any reference to physical or geometrical considerations, provided that the kinetic and potential energies of the system were given in analytical form.

He returned to Paris in 1787 and accepted a professorship at the newly established École Polytechnique[277]. Marie Antoinette warmly patronized him, he was lodged at the Louvre and received a generous pension. He emerged unscathed from the turmoil of the French Revolution, since he was respected and held in affection by all political parties: the revolutionary tribunals overlooked his association with the aristocracy, and even his pension was continued by the National Assembly. Lagrange, however, was revolted by the cruelties of the Terror. When the great chemist **Lavoisier** went to the guillotine, Lagrange expressed his indignation at the stupidity of the execution: "*It took the mob only a moment to remove his head; a century will not suffice to reproduce it*".

Later in life Lagrange was subject to fits of loneliness and despondency. He was rescued from these, when he was 56, by a young and beautiful girl nearly forty years his junior — the daughter of his friend, the astronomer

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The last term in (1) is equivalent to

$$-\frac{\partial}{\partial q_k}\left[\frac{1}{2}m_j\dot{\mathbf{r}}_j\cdot\dot{\mathbf{r}}_j\right]=-\frac{\partial}{\partial q_k}\left(\frac{1}{2}m_jv_j^2\right).$$

The term preceding this takes a similar form when $\frac{\partial \mathbf{r}_j}{\partial q_k}$ is replaced by $\frac{\partial \dot{\mathbf{r}}_j}{\partial \dot{q}_k}$, allowed by $\frac{\partial \dot{\mathbf{r}}_j}{\partial \dot{q}_k}=\frac{\partial \mathbf{r}_j}{\partial q_k}$. Putting these forms into (1) and summing over the particles (j) yields:

$$\sum_k\left[Q_k-\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)+\frac{\partial T}{\partial q_k}\right]\delta q_k=0.$$

With all the δq_k arbitrary, this can vanish if and only if, for each degree of freedom $k=1,2,\dots,m$,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)-\frac{\partial T}{\partial q_k}=Q_k,$$

where T is the total kinetic energy. In cases of forces derivable from a potential $V(q_1\cdots q_k;t)$ such that $Q_k=-\frac{\partial V}{\partial q_k}$, there follow *Lagrange's equation* of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)-\frac{\partial L}{\partial q_k}=0,$$

for $L=T(q,\dot{q},t)-V(q,t)$ and $k=1,2,\dots,m$.

277 A famous school in the history of mathematics, where many of the great mathematicians of modern France were trained and held professorships.

Lemonnier (1715–1799). She was so touched by Lagrange's unhappiness that she insisted on marrying him (1792). He had no children by this or his former marriage.

Lagrange was nominated president of the Academic Commission for the reform of weights and measures and for the establishment of the metric system. Napoleon loaded him with personal favors and official distinctions. He became a senator, a count of the Empire, and a grand officer of the Legion of Honor.

Toward the end of his life, Lagrange felt that mathematics had reached a dead end and that the physical and biological sciences would attract the ablest minds of the future. His pessimism might have been relieved if he had been able to foresee the coming of Gauss and his successors, who made the 19^{th} century the richest in the long history of mathematics.

He was buried in the Pantheon on April 10, 1813. The funeral oration was pronounced by Laplace. Hamilton called Lagrange the "Shakespeare of Mathematics" on account the extraordinary beauty, elegance and depth of his methods.

The astronomy historian **Agnes Mary Clerke** (1842–1907) succinctly summarized his life work in the statement:

"*His treatises are not only storehouses of ingenious methods, but models of symmetrical form. The Clearness, elegance and originality of his mode of presentation give lucidity to what is obscure, novelty to what is familiar, and simplicity to what is abstruse. His genius was one of generalization and abstraction, and the aspirations of the time towards unity and perfection received, by his serene labors, an embodiment denied to them in the troubled world of politics*".

Lagrange and the '3-Body Problem' (1772)

Euler *(1760) seems to be the first to have studied the general problem of three bodies' motion under their mutual gravitation, although at first he only considered the restricted three bodies problem when one of the bodies has a negligible mass.*

It is then assumed that the motions of the other two can be solved as a two body problem, the body of negligible mass having no effect on the other

two. Then the problem is to determine the motion of the third body attracted to the other two bodies which orbit each other.

Even in this form the problem does not lead to exact solutions. ***Euler****, however, found a particular solution with all three bodies in a straight line.*

The Paris Academy Prize of 1772 for work on the orbit of the Moon was jointly won by ***Lagrange*** *and* ***Euler****.* ***Lagrange*** *submitted Essai sur le problème trois corps in which he found another solution where three bodies were at the vertices of an equilateral triangle.*

The motion of an isolated system of two attracting point masses[278] *is solvable exactly in the framework of Newtonian dynamics. Lagrange considered*

[278] "Point mass": a model for a spherically-symmetric mass distribution, which for purposes of Newtonian attraction is considered to be concentrated at the sphere's center. It is tacitly assumed in astronomical applications that the mass' dimensions are small compared to the inter-mass distance (no other restriction on size of mass).

While this is only a very good first approximation for real masses in the universe, it nevertheless underlines the mathematical theory of nearly all problems in celestial mechanics. In the following, the word 'mass' or 'body' will mean 'point-mass', unless otherwise stated.

The attraction (external gravitational potential) of a finite body that is *not necessarily symmetric*, at a point P a distance r from the body's mass-center O, is given by the expression

$$\phi(P) = -G\left[\frac{M}{r} + \frac{\mathfrak{Q}\colon \mathbf{e}_r\mathbf{e}_r}{2r^3} + O\left(\frac{1}{r^4}\right)\right].$$

Here $M = \int \rho(\mathbf{r}')dr'$ is the total mass ($r' \ll r$), $\rho(r)$ the density, $\mathfrak{Q} = \int \rho(\mathbf{r}')(3\mathbf{r}'\mathbf{r}' - {r'}^2 I)dr'$ is the *mass quadrupole moment tensor*, and $\mathbf{r} = r\mathbf{e}_r$ is the position vector of the field point relative to the mass center.

The potential ϕ can be represented in a number of alternative forms:

(I) *Maccullagh's formula*: same as the above with $\mathfrak{Q}\colon \mathbf{e}_r\mathbf{e}_r = A + B + C - 3J$, where $\{A, B, C\}$ are the principal moments of inertia of the body relative to the *mass center*, and J is the body's moment of inertia about the *axis* OP.

(II) *Multipole expansion*:

$$\phi(r > r') = -G\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{4\pi}{2\ell+1}\left[\frac{\widehat{Y}_{\ell m}(\theta,\varphi)}{r^{\ell+1}}\right]Q_{\ell m}$$

where

$$Q_{\ell m} = \int \rho(\mathbf{r}'){r'}^{\ell}\widehat{Y}^*_{\ell m}(\theta',\varphi')d^3\mathbf{r}';$$

$$\widehat{Y}_{\ell m}(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-|m|)!}{(\ell+|m|)!}}\; P_\ell^{|m|}(\cos\theta)e^{im\varphi}\delta_m;$$

the motion of an isolated system of three masses, free to move under their mutual Newtonian attractions. It is governed by 3 2^{nd} order vector ODE's, or equivalently by a system of 18 first-order ODE's. An explicit analytic solution does not exist[279] *for the general case. This is so because only* 10 *out of the* 18 *constants of integration (i.e. coordinates and velocities of each mass at a common fiducial time) are expressible as conservation laws. They are: the system's total mechanical energy content,* 6 *components of the mass-center position and velocity vectors, and* 3 *components of the total angular momentum vector at any chosen time.*

To see this, we write the equations of motion in the form:

$$\ddot{\mathbf{r}}_1 = G\left[\frac{m_2}{r_{12}^3}\mathbf{r}_{12} - \frac{m_3}{r_{31}^3}\mathbf{r}_{31}\right];$$

$$\ddot{\mathbf{r}}_2 = G\left[\frac{m_3}{r_{23}^3}\mathbf{r}_{23} - \frac{m_1}{r_{12}^3}\mathbf{r}_{12}\right];$$

$$\ddot{r}_3 = G\left[\frac{m_1}{r_{31}^3}\mathbf{r}_{31} - \frac{m_2}{r_{23}^3}\mathbf{r}_{23}\right],$$

where (m_1, m_2, m_3) *are the three masses in question,* G *is the universal constant of gravitation,* $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ *are the time-dependent respective position-vectors of the masses w.r.t. an inertial frame of reference and* $\mathbf{r}_{jk}$ *is a vector with origin at* m_j *and terminus at* m_k*. Simple algebraic and differential manipulations of the above equations yield the following results:*

$$\delta_m = \begin{Bmatrix} 1 & m \geq 0 \\ (-)^m & m < 0 \end{Bmatrix}.$$

When this expansion is applied to a nonspherical earth (bulged at the equator, flattened at the poles but azimuthally symmetric), it reduces to

$$\phi = -\frac{GM}{r}\left[1 - 10^{-6}\sum_{\ell=2}^{\infty} J_\ell \left(\frac{r_e}{r}\right)^\ell P_\ell(\sin\lambda)\right],$$

where $r_e =$ equatorial radius, $\lambda =$ geocentric latitude $= \sin^{-1}\frac{z}{r}$, and $J_2 = 1082.64 \pm 0.03$; $J_3 = -2.5 \pm 0.1$; $J_4 = -1.6 \pm 0.5$; $J_5 = -0.15 \pm 0.1$; $J_6 = 0.57 \pm 0.1$; $J_7 = -0.44 \pm 0.1$.

[279] Although there is no closed-form analytical solution, the problem is still solvable numerically by the use of computers, since the number of equations and unknowns is compatible. However, the computer cannot give answers to questions about the behavior of the system which require an *infinite time* to answer, such as whether some member of the system will escape from it or eventually collide with another member.

(1) *The center of mass of the system either remains at rest or moves uniformly in space on a straight line.*

(2) *The total angular momentum of the system is fixed in magnitude and direction at all times.*

(3) *The sum of potential and kinetic energies of the system is constant.*

Result (1) is mathematically expressed as

$$\mathbf{R} \equiv \frac{1}{M}(m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + m_3\mathbf{v}_3) = \frac{\mathbf{a}_1}{M}t + \frac{\mathbf{a}_2}{M}$$

where $M = \sum_{i=1}^{3} m_i$ *and* $\{\mathbf{a}_1, \mathbf{a}_2\}$ *are two constant vectors. This equation therefore yields* 6 *scalar equations satisfied by the mass coordinates.*

Result (2) has the mathematical form

$$\frac{d}{dt}\mathbf{H} = 0, \qquad \mathbf{H} = (\mathbf{r}_1 \times m_1\mathbf{v}_1) + (\mathbf{r}_2 \times m_2\mathbf{v}_2) + (\mathbf{r}_3 \times m_3\mathbf{v}_3).$$

It consists of 3 *additional scalar relations linking the positions and velocities of the masses.*

Result (3) reads

$$\frac{1}{2}\sum m_i\dot{\mathbf{r}}_i^2 - G\left(\frac{m_1m_2}{r_{12}^2} + \frac{m_2m_3}{r_{23}} + \frac{m_3m_1}{r_{31}}\right) = E = const.$$

It yields one additional relation (integral of motion). Thus, 10 *of the* 18 *integrals which are necessary for complete solutions are known, and they are all* algebraic *functions when expressed in rectangular coordinates. In 1887,* ***H. Bruns*** *showed that when rectangular coordinates are chosen as the independent variables in the 3-body problem, the* 10 *integrals (constants) described above are the only integrals to be expected, and no more such* algebraic *integrals exist.* ***Poincaré*** *showed, in the same year, that there are no new* transcendental *integrals, even when the masses of all bodies except one are small.*

Knowing all this (although lacking Brun's proof), ***Lagrange*** *sought* particular *exact solutions to the 3-body problem which do not require more than* 10 *constants of integration. These particular solutions were discovered by him in a prize memoir in* 1772.

Lagrange considered two separate problems. The first, known as the restricted three-body problem, *is of particular importance in discussions of space probes moving in the gravitational fields of the earth and the moon. In this*

case one of the three masses, referred to as the 'particle', is so small in comparison with the other two that its gravitational effects on these two masses can be neglected. The particle is thus considered as a test mass *whose motion is the object of calculation. [The earth-moon system together with a small artificial satellite or spacecraft constitutes such a system, if we ignore the presence of the sun, the lack of sphericity of the earth and the eccentricity of the moon's orbit.]*

It is then advantageous to set up the following model: two massive bodies (masses m_1 and m_2 respectively) move in circular *orbits about their center of mass, which is taken as the origin of a coordinate system that revolves around its z-axis (such that the xy axes are in the plane of motion of the two finite masses) with an angular velocity that is equal to their orbital velocity, $\omega = \frac{G\sqrt{m_1+m_2}}{a^{3/2}}$, $(m_1 \geq m_2)$ by Kepler's third law (a = radius of circular orbit).*

In this special rotating system, the equation of motion of the particle is

$$\mathbf{a} + 2(\boldsymbol{\omega} \times \mathbf{v}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = -G\left[\frac{m_1}{\rho_1^3}\boldsymbol{\rho}_1 + \frac{m_2}{\rho_2^3}\boldsymbol{\rho}_2\right],$$

where $\boldsymbol{\omega} = \omega\mathbf{e}_z$; $\mathbf{a} = \ddot{x}\mathbf{e}_x + \ddot{y}\mathbf{e}_y + \ddot{z}\mathbf{e}_z$, $\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$; $\{\boldsymbol{\rho}_1, \boldsymbol{\rho}_2\}$ are the vector distances of the particle from the masses m_1 and m_2 respectively and $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ are the three unit vectors along the axes of the rotating system.

The general problem of determining the motion of the test particle requires six integrals for its complete solution. Simple manipulations of the above equation of motion yield the first integral in the form which resembles the energy integral in a two-body system. Choosing the unit of mass such that $m_1 + m_2 = 1$ [i.e. $m_1 = 1 - m$, $m_2 = m$], and furthermore choosing the units of distance and time such that $G = 1$, $a = 1$ $(\omega = 1)$, the first integral reads

$$\dot{x}^2 + \dot{y}^2 = (x^2 + y^2) + \frac{2(1-m)}{\rho_1} + \frac{m}{\rho_2} - c.$$

There remain five integrals to be found. However, by restricting the motion of the particle to the xy plane, *it is possible to reduce the total number of the needed constants to a total of two*[280]. *One new integral is therefore needed, but cannot in principle be found, on the strength of the above-mentioned Brun's theorem. Nevertheless, the first integral is sufficient to deduce the*

280 This was shown by **C.G.J. Jacobi**, in 1844. The first 'energy-like' integral is also due to him and known as *Jacobi's integral*. Lagrange arrived at his results in another way.

salient features of the motion of the particle, as was done by Lagrange: he derived the contours of zero velocity, by examining the equipotential contours

$$x^2 + y^2 + \frac{2(1-m)}{\sqrt{(x-x_1)^2 + y^2}} + \frac{2m}{\sqrt{(x-x_2)^2 + y^2}} = c.$$

It turns out that there are five points in the xy plane, known as the Lagrangian points[281]*, which are of special significance in the three body problem. At each of these* 5 *positions (relative to the two masses in mutual circular revolution) the particle, once placed, will also move on a circular orbit, always maintaining a fixed orientation w.r.t. to the other two masses. Three of these points, called the Lagrangian points* L_1, L_2 *and* L_3, *along the line joining the two masses, are unstable, in the sense that if the particle is displaced slightly from one of them, it will leave its circular orbit.*

Because small perturbations are always likely to occur, we would not expect to find many examples in nature in which three bodies revolve exactly in those configurations.

The two remaining points, known as L_4, *and* L_5 *("L" in honor of Lagrange) are, however, stable. A particle at one of those positions cannot be*

[281] The equipotential contours display the combined gravitational fields of the two massive objects. The field has a constant strength at each point (x, y) along the curve. Thus, we can think the equipotential contour map as a sort of topographical map, showing 'hills' and 'valleys' in the gravitational field. A small object like an asteroid can be permanently trapped at one of the stable Lagrange points.

In 1776, *asteroids* had not yet been discovered, and Lagrange knew no actual case that would demonstrate the existence of the $\{L_1, L_2, L_3\}$ points. However, 90 years after Lagrange's theoretical work, **Daniel Kirkwood** (1866) showed that it applied perfectly to Jupiter and the asteroids. Those places between Mars and Jupiter where no asteroid would be found have been known as *Kirkwood's gaps* ever since.

The *Trojan asteroids* at L_4 and L_5 along Jupiter's orbit have been known since 1906 and provided the first proof of Lagrange's theoretical ideas about these points. Since then, stable Lagrange points have been found to exist at many places in the solar system; while passing Saturn, the *Voyager* spacecraft (1980) discovered tiny satellites at the L_4 and L_5 points of the Saturn-Tethys and the Saturn-Dione systems. A group called the L_5 *society* argues that the L_5 point on the Earth-Moon system would be an ideal location for a huge space station with a permanent human population. Despite careful searches, no asteroids have been found at the stable Lagrange points of the Earth-Sun and Saturn-Sun systems.

forced away by slight perturbations. It can be shown that in these configurations, the particle and the two masses are at the corners of an equilateral triangle. *We do, in fact, find natural examples of this kind of motion: The best known is the equilateral configuration defined by the sun, the planet Jupiter and the two groups of* Trojan asteroids *(the sun and Jupiter move in nearly circular paths around their mutual COM, and the minor planets have negligible mass in comparison).*

Lagrange's solution to the restricted 3-body problem also specifies the regions of space within which the particle can *move relative to the two larger ones. In recent years his theory found another application in the theory of evolution of massive stars: there are many* binary star systems *in which the two stars revolve about each other in nearly circular orbits. If the two stars are relatively close together and if one evolves to a large enough size, the atoms of its outer distended layers, having negligible mass, move about (in the role of particles) in a manner predicted by Lagrange.*

We thus find that during the evolution of stars in binary systems, matter can flow from one star to another, or can flow in an orbit around one or both stars, or can even flow into space, escaping the two stars altogether (from the inner Lagrangian points). This mass exchange, *believed to occur between many stars in closed binary systems, can have profound effects of the evolution of the stars in a system, possibly accounting for such phenomena as* novae *and* supernovae. *It can also lead to the formation of a large circumstellar disk or ring of matter around the binary system, and even be involved in the creation of* neutron stars *and* black holes.

The second problem considered by Lagrange was that of special stationary solutions of the three-body problem for arbitrary *masses. By a stationary solution we mean one in which the geometric configuration of the three masses remains self similar w.r.t. time. If the motion of the masses is such that their mutual distances from each other remains unchanged, the configuration simply* rotates in its own plane *around the center of mass. On the other hand, an expansion or contraction may take place which does not alter the* shape *of the patterns of points.*

Lagrange showed that there are only two such configurations: one in which the three masses lie on a straight line, *and the other in which the masses form an* equilateral triangle *whose base is the segment a between two masses. In this latter case the motion is such that the plane through the three masses is fixed in space while the plane rotates with fixed angular velocity $\omega = \left\{\frac{G(m_1+m_2+m_3)}{a^3}\right\}^{1/2}$ onto itself. The resultant Newtonian forces on each of the three masses passes through their common mass center. Finally, the three points describe conic sections similar to each other, with the common mass-center lying at the focus. In a coordinate system rotating with angular velocity*

ω in the plane of the masses, the Lagrangian points (masses) are fixed: at these points, the gravitational and the centrifugal forces just balance each other.

In the straight-line solution, the masses will be located at Lagrangian points (also known as libration points) if they are arranged on a line (the x-axis, say) with coordinates $\{x_1, x_2, x_3\}$ such that if $x_2 - x_1 = 1$, then $x_3 - x_2 = p$, where p is the only positive root of the quintic equation:

$$(m_1 + m_2)p^5 + (3m_1 + 2m_2)p^4 + (3m_1 + m_2)p^3$$

$$-(m_2 + 3m_3)p^2 - (2m_2 + 3m_3)p - (m_2 + m_3) = 0.$$

The angular velocity for this case[282] *is*

$$\Omega = \omega \left[\frac{m_1 p^2 - m_3}{m_1 p^2 - m_3 p^3} \right]^{1/2}.$$

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[282] To translate this into a mathematical language, we write the total coplanar acceleration of any one of the mass points $\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta$, where $\mathbf{r} = r\mathbf{e}_r$ is its position vector (drawn from the system's mass-center) and $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ are unit vectors in radial and transverse directions respectively. Since we have assumed $\dot{r} = \ddot{r} = 0$, $\dot{\theta} = \omega =$ the constant angular speed of revolution about the mass-center, the accelerations become $\mathbf{a}_i = -r_i\omega^2\mathbf{e}_i$ where $(\mathbf{e}_r)_i = \mathbf{e}_i$ $(i = 1, 2, 3)$. Using these values in the general equations of motion given at the beginning of this section, we obtain the *differential equations*

$$\ddot{r}_i = -\omega^2 r_i \qquad (i = 1, 2, 3),$$

where general solutions are conic sections. If the orbits are ellipses, where ε_s is the *eccentricity* and E_s is the *eccentric anomaly*, then $r_i = a_i(1 - \varepsilon_s \cos E_s)$. Thus the various conics described by the three bodies are all similar and the masses occupy corresponding positions in their orbits at any given instant. To derive the dependence on the angular velocity ω on the constants of the system we must solve the three simultaneous *algebraic vector equations*

$$-\omega^2\mathbf{r}_1 = \frac{m_2}{r_{12}^3}\mathbf{r}_{12} - \frac{m_3}{r_{31}^3}\mathbf{r}_{31};$$

$$-\omega^2\mathbf{r}_2 = \frac{m_3}{r_{23}^3}\mathbf{r}_{23} - \frac{m_1}{r_{12}^3}\mathbf{r}_{12};$$

$$-\omega^2\mathbf{r}_3 = \frac{m_1}{r_{31}^3}\mathbf{r}_{31} - \frac{m_2}{r_{23}^3}\mathbf{r}_{23}$$

with the additional center of mass condition

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3 = 0,$$

where $\mathbf{r}_{12} = \mathbf{r}_2 - \mathbf{r}_1;\quad \mathbf{r}_{23} = \mathbf{r}_3 - \mathbf{r}_2;\quad \mathbf{r}_{31} = \mathbf{r}_1 - \mathbf{r}_3.$

In conclusion, the n-body problem $(n > 2)$ can be solved in general only by laborious numerical calculations. There are, however, some special circumstances in which there exist solutions, or partial solutions, in the form of algebraic equations. Usually, these solutions apply only when the mass-system has a very particular and most unlikely configuration. Nevertheless, the Lagrange's theory can serve as an approximate model for certain important and interesting astrophysical phenomena.

There is yet another aspect of the three-body problem that one should not overlook: consider the case of two spherical bodies that move under the influence of their mutual attractions, each describing a conic section w.r.t. their center of mass as a focus. If there is a third body attracting the other two under consideration, their orbits will cease to be exact conic sections. The difference between the coordinates and components of velocity in the actual orbits *and those which the bodies would have had if the motion had been undisturbed are the* perturbations.

For example: in the solar system, to first approximation, each planet moves as though it and the sun constituted a two-body system. This suggests, therefore, that we first study the motion of a planet as a part of a two-body system. Then we determine the deviations from a purely two-body motion that will result from the presence of other disturbing bodies, namely — the perturbations. This approach supplements the above results of Lagrange, who was interested in the motion of the test particle and ignored its perturbing effect on the large masses.

It is of interest to note that Lagrange considered his solutions for the 3-body problem as inapplicable to the solar system. We now know, however, that both earth and Jupiter have asteroids sharing their orbits in the equilateral triangle solution configuration discovered by Lagrange. For Jupiter, these bodies are called Trojan planets, the first to be discovered being Achilles (1908).

The first comet to have a calculated elliptical orbit which was far from a parabola was observed (1769) by **Charles Messier** *(1730–1817). The elliptical orbit was computed by* **Lexell** *(1740–1784) who correctly realized that the small elliptical orbit had been produced by the perturbation of Jupiter. The comet made no reappearance and Lexell correctly deduced that Jupiter had changed the orbit so much that it was thrown far away from the sun.*

The Lagrange theory found interesting and important applications in recent times. It was shown that the forces at the Lagrangian balance-points could capture objects and keep them orbiting. The European Space Agency has taken advantage of one balance point by launching a sun-observatory called SOHO that currently orbits at L_1. The orbits of objects at these

points are exotic, often tadpole-shaped and rarely horseshoe-like. The horseshoe orbit involves movement around L_3, L_4 and L_5 points.

In 1986 astronomers discovered a new Near-Earth asteroid, named 3753 Cruithne[283] *(also known as 1986 TO). At that time no one had tracked its path thoroughly enough to detect its rare orbit. Then, in 1997, it was found that Asteroid 3753 follows a spectacular horseshoe orbit and has characteristics never before seen or even anticipated, either in theory or in computer simulations.*

Cruithne is co-orbital with the earth (meaning that it shares the earth's orbit). In a co-rotating frame with the earth (in which earth is stationary) Cruithne is on an spiraling horseshoe orbit: every year, the asteroid traces out a kidney bean. Over time, this kidney bean drifts along the earth's orbit, tracing out a spiral which, when complete (after 385 years) fills in an overlapping horseshoe.

Cruithne avoids collision with earth: at its closest approach it only gets to within 15 million km. Each year, it is at its closest in the autumn, and at this point it will pass almost directly beneath the earth's South Pole.

The asteroid is about 5 km wide and takes 770 years to complete its horseshoe-shaped orbit around the earth. It is believed that it is a temporary companion, remaining in a suspended state around the earth for at least 5000 years.

The Lagrange theory for this 3-body problem (earth-Cruithne-sun) provides for new dynamical channel through which free asteroids become temporarily moons of the earth and stay there for periods ranging from a few thousand years to several tens of thousands of years. Thus, some asteroids that cross the earth's orbit may be trapped in orbits caused by the gravitational dance between earth and sun.

It is believed that the laws of nature would make it very difficult for an asteroid to have entered into this orbit recently. The asteroid may be as old as the solar system itself, and it might have found its way into this orbit when the solar system was forming. On the other hand, if it joined us more recently, the mechanics and physics that would have been needed to get this asteroid into orbit in recent times are akin to threading a needle.

Asteroid 3753 is following the most complicated horseshoe orbit ever seen, and it is unique in our solar system. It has unique characteristics, including: a spiraling motion; a big inclination (titled path) and an overlap at the end of the horseshoe.

[283] The *Cruithne* (=croo-een-ya) were the first Celtic tribal group to come to the British Isles between about 800 to 500 BCE.

1754–1778 CE **Joseph Black** (1728–1799, Scotland). Physician, chemist and physicist. Rediscovered *carbon dioxide* (1754). First to introduce the concepts and theories of *heat capacity* and *latent heat* (1760). These theories contributed substantially to Watt's development of the steam engine. He visualized heat as a certain imponderable fluid (called "*calor*"), which can penetrate all material bodies and thus increase their temperature. Mixing a gallon of boiling water with a gallon of ice cold water, he noticed that one finds the temperature of the mixture just halfway between two initial temperatures, and he interpreted this fact by saying that, after the mixing, the excess of "calor" in hot water is equally distributed between the two portions.

He defined the unit of heat as the amount necessary to raise the temperature of 1 lb of water by 1 °F (in the modern metric system we speak of *calorie*, which is the amount of heat it takes to raise the temperature of 1 gm of water by 1 °C). He concluded that equal weights of different materials heated to the same temperature contain different amounts of "calor" since, indeed, by mixing equal weights of hot water and cold mercury, one gets a temperature which is much closer to the original temperature of water than of mercury. Therefore, he argued, cooling a certain amount of water by 1 ° liberates more heat than is necessary to heat an equal weight of mercury by 1 °.

This led him to the notion of the *heat capacity* of different materials, characterized by the amount of heat needed to raise their temperature by 1 °. Another important notion introduced by Black was that of *latent heat*, which is the amount of heat needed for a change of phase, e.g. to turn ice into ice water (both at 0 °C), or to burn boiling water into water vapor (both at 100 °C). He thought that adding a given amount of the imponderable heat fluid to a piece of ice loosens up its structure, making it liquid, and that, in a similar way, adding more heat to the hot water further loosens its structure, turning it into vapor.

Black was born at Bordeaux, where his father, a native of Belfast but of Scottish descent, was engaged in the wine trade. He studied medicine in Glasgow. James Watt, at the University of Glasgow, later absorbed the thermal theories of Joseph Black (by then professor of medicine at the university) and applied them in his invention of the improved condensing steam-engine (1765).

1756–1763 CE *Seven Years' War*. Prussia and Austria fought for control of Germany in a war that involved nearly every nation in Europe. It pitted Prussia (Frederick the Great) against Austria, Sweden and France. The war ended exactly where it began, with no territorial changes in Europe. In North

America, however, France gave up Canada to Britain and also yielded its colonies in North America.

1756–1774 CE **John Smeaton** (1724–1792, England). Civil engineer and inventor. Founder of the civil engineering profession during the early days of the Industrial Revolution in England. Improved instruments used in navigation and astronomy. His major achievements were

- Rediscovered (1756) *hydraulic cement*, unknown since the fall of Rome.
- Made improvements on windmills and watermills (1759).
- Design large pumping engines; improved diving bell; rebuilt *Eddystone* lighthouse (1759).
- Constructed Ramsgate harbor (1774), Forth and Clyde Canal, and Perth, Banff, and Coldstream bridges.
- Improved the steam-engine of James Watt (1775).

Smeaton was born at Austhorpe Lodge, near Leeds. Left the grammar school of Leeds in his 16 th year to become apprentice to an instrument maker and in 1750 set up his own business. In 1759 he read a paper before Royal Society entitled '*An Experimental Inquiry concerning the Native Powers of Water and Wind to turn Mills and other Machines depending on a Circular Motion*' for which he received the Copley medal.

In 1754 he made a tour of the Low Countries to study the great canal works there. He died at Austhorpe and was buried in the old parish church of Whitkirk.

The Watch[284] ***and Modern Time-Culture*** (1502–1760)

The invention of portable timepieces dates from the end of the 15^{th} *century, and the earliest manufacture of them was in Germany. It is known that* **Peter Henlein** *(1480–1542), a locksmith of Nuremberg, built, during* 1502–1510, *a small round clock with steel mainspring enclosed in a box. It was known as the Nuremberg Egg. Being too large for the pocket it were frequently hung from the girdle. It was the first pocket watch ever made. Before Henlein invented the watch, time was told by clocks that used heavy weights. The mainspring supplied the power to turn the wheels. The manufacture of watches by hand soon spread throughout Europe. The difficulty with these early watches was the inequality of action of the mainspring.*

An attempt to remedy this was provided through 1525–1540. *In early watches, the escapement was the same as in early clocks, namely, a crown wheel and pallets with a balance ending in small weights. Such an escapement was, of course, very imperfect; since the force moment acting on the balance does not vary with the displacement, the time of oscillation varies with the arc, and this in turn varies with every variation of the driving force. An immense improvement was therefore effected when the hair-spring was added to the balance, which was replaced by a wheel. This was done about the end of the* 17^{th} *century.*

During the 18^{th} *century a series of escapements were invented to replace the old crown wheel, ending in the chronometer escapement. Though great improvements in detail have since been made, the modern mechanical watch may nevertheless be called an* 18^{th}*-century invention.*

Early watches had only an hour hand. The minute hand was developed in 1687. *In the* 1800*'s, new machinery made it possible to produce accurate watches cheaply.*

The invention of the clock in the 14^{th} *century and its technical improvements during the* 15^{th} *and* 16^{th} *centuries, rendered a useful means for the fulfillment of religious and social functions: it was regularly fixed on the front walls of churches and town halls, or placed in city squares, where its chimes*

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284 From the Old English word *waecce* = a keeping guard or watching, from *wacian* = to guard, watch, *wacan* = to wake. Hence watch = that which keeps watchful or wakeful observation or attention over anything. The term was used for persons who patrolled the streets, called the hours, and performed the duties of modern police. The term was later applied to a period of time marked by the change of sentries or ship crews.

served to assemble the burghers to prayer and public meetings. Only in the 16^{th} *century, when clocks became part of the household, and more so in the* 17^{th} *century, when portable watches were in the private use of individuals, did the* modern time-culture *begin. This was the time when man's concept of time underwent a* revolutionary *change.*

Prior to the invasion of the metric time keeper into the household, the day- and night-cycle and the annual cycle of the seasons dominated the conduct of human life. In the agrarian pre-industrial society, all activities were predetermined by the calendar*, by the constant march of generations and the ages of man, and the periodic change of the seasons. Beyond that was the consciousness of the existence of an eternity beyond life, granted by faith. This routine was totally disrupted when life according to the calendar changed to life according to the watch; Western civilization has come to be* dominated *by the clock and the timetable, and Westerners have had little sympathy with people who have escaped this domination.*

Hours, days and periods that were of unique value to societies and cultures, and previously sanctioned by their calendars, were absent from the indifferent faces of the new timekeepers, which from now on became the only measure of homogeneous, universal and objective time. During the next 300 *years, there occurred a gradual but perpetual subjugation of all norms, concepts and values to clock time; Westerners adopted a new puritan world outlook, known by its motto:* TIME IS MONEY *(**Benjamin Franklin**, 1733).*

The message was clear: no more spontaneous prayers, easygoing work, communal togetherness and mutual aid; life became more mechanized and more personal. Cooperation gave way to mere synchronization. People became more punctual, more pedantic, more purposeful. Even basic functions such as eating and sleeping became mundane; Europeans dined not out of hunger but when the clock said so, and they turned in to sleep not when tired but when the time came. Puritans turned from a life of meditation and abstinence to a life of creativity and labor. The harnessing of inanimate physical forces in the Industrial revolution made it possible for work to be carried on for 24 *hours a day throughout the year — under cover, by artificial light, and at a controlled temperature.*

But that was not all: clock-time intensified man's consciousness of the fleeting moment, the discretized unit of time, and increased his fear of death[285]*. Time became less abstract and more real, symbolized by the per-*

[285] Great poets and artists, as always, are first to feel the deep implications of social changes.

The clock as a harbinger of death is clearly portrayed in the sonnets of **Shakespeare** (1609) and the woodcuts of **Hans Holbein**, "*The Dance of Death*" (1538).

petual moving hands of the clock. It became the idol of a new mercantile-industrial society. The value of goods was measured by the time needed to produce it and vice versa: the value of time was measured by the amount of goods produced. The clock thus became a machine that produced time!, and obviously, like any other material object whose value is measured by it usefulness, the collective time became redundant after its use; it lost its moral value as soon as it passed and there was no motivation whatsoever in its keeping. The clock turned time into a one dimensional disposable entity that is not accumulated in any cultural collective consciousness or tradition. It did not turn anymore into a significant past — it became a historic time.

1757–1776 CE **Johann Heinrich Lambert** (1728–1777, Germany). Physicist, mathematician and astronomer. Came close to being the founder of non-Euclidean geometry. His mathematical discoveries were extended and overshadowed by the work of his contemporaries.

In 1770 he derived the continued fraction representation

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cdots}}}}.$$

It yields as partial fractions the historical approximations $\frac{3}{1}$, $\frac{22}{7}$, $\frac{333}{106}$, $\frac{355}{113}$, $\frac{103993}{33102}$, $\frac{104348}{33215}$, $\cdots$ where the 4^{th} fraction yields π with an error of at most 3 units in the 7^{th} decimal place[286]. Later (1776), Lambert proved that π and e are *irrational*[287].

286 This expansion does not seem to have any regularity. Apparently, it was obtained by transforming the decimal fraction for π into a continued fraction. The first 23 terms are: $[3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, 2, \ldots]$.

287 Lambert's proof is rather hard. The following simple proof, of unknown origin, appeared in the *Mathematics Preliminary Examination* at Cambridge in 1945: Consider the integral $I_n = \int_{-1}^{1}(1-x^2)^n \cos\left(\frac{\pi}{2}x\right)dx$. Two integrations by parts yield $I_n = \left(\frac{2}{\pi}\right)^{2n+1} n! P_n$, where P_n is a polynomial in $\left(\frac{\pi}{2}\right)$ of degree $\leq 2n$ and with integral coefficients depending on n. Assuming $\frac{1}{2}\pi = \frac{b}{a}$, where a and b are integers, it follows that $\frac{b^{2n+1}}{n!} I_n = P_n a^{2n+1}$. The right side is an integer. But $0 < I_n < 2$, and as $n \to \infty$, $b^{2n+1}/n! \to 0$. Hence for some m and

In 1761 he loosely speculated that various solar systems might revolve about a common center, that such systems might in turn revolve about another system[288] and "where shall we stop?" In 1776 he argued in favor of developing a non-Euclidean geometry by building a logically consistent system through the explicit rejection of the parallel postulate, while keeping all other postulates intact.

Among his other contributions is his series solutions of the equation $x^m + px = q$ (1757), which was extended by **Euler** and **Lagrange**, and the first systematic development of the theory of hyperbolic functions. He also contributed to the mathematics of descriptive geometry, the determination of cometary orbits and the theory of *map projections*, some of which bear his name.

Lambert was born at Mulhausen, Alsace (then part of Swiss territory), to a poor family. He was self-educated and worked his way up patiently. In 1764 he removed to Berlin, where he received many favors at the hand of Frederick the Great and was elected a member of the Royal Academy of Sciences. He died of consumption.

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all $n > m$, $\frac{b^{2n+1}}{n!} I_n < 1$, and therefore $= 0$ since it is a non-negative integer. Hence $I_n = 0$ for $n > m$.
But for $-1 < x < 1$, $\cos \frac{1}{2}\pi x$ is positive, and $1-x^2$ is positive. Hence $I_n > 0$ for all n. This contradiction shows that $\frac{1}{2}\pi$ cannot be of the form b/a.

The irrationality of e is even easier to demonstrate; let $e = \frac{p}{q}$ be rational (p and q integers). We may write $e = e_n + R_n$, where $e_n = \frac{p_n}{q_n} = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$ or $n!\frac{p_n}{q_n} = 1 + nf(n)$. Since $R_n < \frac{1}{n!}$ we can replace $\frac{p_n}{q_n}$ by $\frac{p}{q}$ for a large enough n. Then $n!\frac{p}{q} = 1 + nf(n)$. But we can choose $n > q$ in which case the l.h.s. is divisible by n whereas the r.h.s. is not. The irrationality of e^π was proved by **Gelfond** (1929).

[288] This idea is borne out by our current knowledge of the *hierarchical structure of the universe: stars clusters, spiral arms, galaxies, galaxy clusters and superclusters, etc.*

The 1758 return of Comet Halley

In 1705, ***Halley*** *predicted his comet return in late 1758 or early 1759. The prediction was confirmed by* ***Cheseaux*** *(1744) and* ***Euler*** *(1746). In 1757,* ***Lalande*** *suggested that the comet would be most easily seen during the month of November. What was needed was an improvement in knowing the time of the comet's perihelion passage. This task was entrusted in the hands of the French mathematician* ***Alexis-Claude Clairaut****.*

Applying the first approximation to the 3-body problem, Clairaut began his computations in June 1757. Since the return was imminent, he was racing against time. Initially the plan was to compute the comet's motions around the sun over the 1607 to 1759 interval, taking into account perturbative effects of both Jupiter *and* Saturn. *To assist him in the lengthy computations Clairaut enlisted the aid of his young colleague* ***Lalande****, who in turn enlisted the aid of Madame Lepaute, wife of the clock maker to King Louis XV. The three of them made calculations from morning to night over six months. The discrepancy of 33 days is only a modest error considering the uncertainty in the planetary masses, the perturbations from neglected or undiscovered planets, and the approximations that had to be made in the method itself.*

Clairaut's first paper on the predicted return was read to the Academy of Sciences in Paris on Nov 14, 1758, thus winning the race between himself and the comet. Had he waited with his announcement after the paper was published and the comet recovered prior to the announced result, their work might have been perceived as a mere footnote in astronomical history rather than the classic work it turned to be. Indeed, the published version of his prediction did not appear until January 1759 – well after the first sighting of the comet by ***Palitzsch*** *on December 25, 1758.*

In 1760, after the comet was recovered, Clairaut corrected some errors in the earlier work, made more comprehensive perturbation calculations for Saturn, and suggested a perihelion passage of April 4, 1759. His essay two years later moved this date back further, to March 31, 1759, which Clairaut considered to be within 19 days of the observed perihelion passage. *A competition made in 1985 between Clairaut's work and computer results based on modern astronomical data, showed that 6 days of this remaining error as due to the planets* Uranus *and* Neptune, *which had not yet been discovered; another 6 days due to neglected effects of Mercury, Venus, Earth, and Mars; and 4 days from errors in the masses of Jupiter and Saturn that Clairaut adopted.*

The arduous work left Clairaut with an unspecified malady that changed his temperament for the rest of his life.

For mid-European observers, the comet's apparition was broken into 3 phases:

- *The first phase:* *From December 24, 1758, through February 14, 1759. It ended when the comet disappeared into the evening twilight.*
- *The second phase:* *Rounding perihelion on March 13, 1759, the comet again became visible in early April, before it sank below the local horizon.*
- *The third phase* *of visibility was from early May to when it was last seen on June 22. The comet passed within 0.12 AU of the earth on April 26, 1759, and became a rather impressive naked-eye object.*

The bold prediction and successful recovery of comet Halley in 1758 and 1759 was the most visible confirmation of Newtonian dynamics in the 18th century.

Apart from the man vs. nature aspect of the 1758 apparition of comet Halley, there is another side to the story: ***Johann Georg Palitzsch*** *(1723–1788) was a German farmer and amateur astronomer. Palitzsch lived in Prohlis, a small town near Dresden in Saxony. On the nights of 24–26 Dec 1758 he observed the comet with his eight-foot telescope but did not identify it with Halley's. His observations were however published in a Dresden newspaper. To their chagrin, members of the Paris Academy (Clairaut included) learned about the comet recovery more than 3 months later (April 1, 1759), unable to understand how a German farmer beat them to it.*

Palitzsch also observed the June 6, 1761 transit of Venus. *He observed a black band linking Venus and the sun near the beginning and end of the transit and correctly interpreted this as evidence that* Venus *possessed an atmosphere. He also found that the brightness of Algol varied with a period of 2 days, 20 hours, 53 min.*

1759 CE **Franz Aepinus** (**Franz Maria Ulrich Theodor Hoch** 1724–1802, Germany). Physicist. Professor at St. Petersburg (1757–1798).

Discoverer of the *pyroelectric effect* in the gemstone tourmaline[289]. He also noticed that when tourmaline is subjected to a mechanical stress it can generate electric charges. Conversely, it can change its shape when voltage is applied to it.

Aepinus was first to apply mathematics systematically to phenomena of electricity and magnetism. He constructed the first condenser with parallel plates.

1759–1772 CE **James Brindley** (1716–1772, England). Canal engineer of remarkable mechanical ingenuity. Pioneered in construction of canals and aqueducts at the dawn of the industrial revolution.

Brindley was born at Thornsett, Derbyshire. His innate ingenuity more than compensated for his lack of training.

1759–1787 CE **Jean-Sylvain Bailly** (1736–1793, France). Astronomer, historian of science and statesman. Computed the orbit of Comet Halley (1759); studied major satellites of Jupiter (1766). Author of histories of ancient and modern astronomy (1775–1787).

Bailly was born in Paris; originally intended for the profession of a painter, he preferred writing tragedies until attracted to science by the astronomer **Nicolas de Lacaille**. Gained high literary reputation by his writings on **Moliere**, **Corneille** and **Leibniz** (1770); admitted to all three French Academies (1784–5).

The cataclysm of the French Revolution interrupted his studies. He became president of the National Assembly (1789) and first mayor of Paris (1789). Imposed martial law and called out the National Guard to keep order, leading to massacre of Champ de Mars (1791). Late in 1793, Bailly quited his Nantes home to join his friend Pierre Simon de Laplace at Melun;

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[289] Sodium aluminum borosilicate

$$Na(Mg, Fe, Mn, Li, Al)_3Al_6(Si_6O_{18})(BO_3)_3 \cdot (OH, F)_4.$$

Commonly used as a gemstone. It crystallizes in rhombohedral, *hexagonal* system as prisms. It consists of six-membered silica rings and 3-membered borate rings held together by sodium, aluminum, and other positive ions.

Transported to Europe from Sri Lanka by Dutchmen (1703). Its name comes from the Sinhalese 'tormalli'. The columns of tourmaline are charged when heated, positively at one end, and negatively at the other.

The Sri Lanka variety is reddish to vividly red (*rubellite*). The less conspicuous black tourmaline (*schorl*) was known in Europe long before, but its dull coloring did not arouse their curiosity.

but was there recognized, arrested and brought before the Revolutionary Tribunal at Paris, where he was guillotined amid the insults of a howling mob. He met his death with patient dignity.

1760 CE **John Harrison** (1693–1776, England). Horologist. Solved the greatest scientific problem of his time: built the first mechanical marine chronometer, accurate to within 0.1 sec/day, leading to longitude determinations accurate to within ca 1.3 minutes of arc. Harrison's instrument was tested on a voyage to Jamaica, and on its return to Portsmouth in 1762 it was found to have lost just under 2 min. The function of the clock was to keep Greenwich time, needed in celestial navigation to determine the *longitude* at sea from the time-dependent star position. [*Latitude* could be determined by the stars alone, through the identification of new constellations as the ships moved south, as well as the elevation of Polaris.]

Harrison, the son of a carpenter, was born at Faulby in Yorkshire. At first he learned his father's trade and worked at it for several years, but later became interested in mechanical devices, and during 1715–1726 made ingenious clocks. In 1714 Queen Ann authorized a public reward of $ 20,000 to any person who should construct chronometers that would determine a ship's longitude in the open sea within 30 minutes of arc[290].

In this connection **Isaac Newton** said: "*That, for determining the longitude at sea, there have been several projects, true to the theory, but difficult to execute*: *one is, by a watch to keep time exactly; but, my reason of the motion of a ship, the variation of heat and cold, wet and dry, and the difference in gravity in different latitudes, such a watch hath not yet been made*".

Harrison applied himself vigorously to the task, and in 1735 went to the *Board of Longitude* with a watch which he also showed to **Edmund Halley** and others. Through their influence he was allowed to proceed in a king's ship to Lisbon to test it. The result was so satisfactory that he was paid $500 to carry out further improvements. Harrison continued to work on the subject with the utmost perseverance for the next 25 years. In 1762, Harrison claimed the full reward of $ 20,000, but it was not until 1773 that he was paid in full. He was never able to express his ideas clearly in writing for lack of formal education, although in conversation he could give a very precise and exact account of his many intricate mechanical contrivances.

In Harrison's watches, compensation for changes in temperature was applied for the first time by means of a "*compensation-curb*", designed to alter

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290 Since the earth rotates once every 24 hours, the time at noon changes by one hour every $\frac{360^\circ}{24} = 15^\circ$ of longitude. To be accurate to $\frac{1}{2}^\circ$, a clock must not vary by more than 2 minutes at the end of the voyage.

the effective length of the balance-spring in proportion to the expansion or contraction caused by variations in temperature. Harrison's timekeeper was used on Cook's last voyage of 1776, and Cook had nothing but praise for it.

Further improvements were made in the 18^{th} century, especially in the development of the escapements. The best solution to the temperature-compensation problem was ultimately proposed by **Pierre le Roy** (1717–1785, France) in 1765, and perfected in 1785 by **Thomas Earnshaw** (1749–1829, England). Their idea was to diminish the inertia of the balance-wheel in proportion to the increase of temperature, by means of the unequal expansion of the metals composing the rim. Earnshaw's chronometer made the voyage of the *Bounty* with Captain William Bligh in 1791.

The Longitude Problem[291]*, or John Harrison against the Admiralty* (1714–1760)

"For lack of a practical method of determining longitude, every great captain in the Age of Exploration became lost at sea despite the best available charts and compasses. From Vasco da Gama to Vasco Núñez de Balboa, from Ferdinand Magellan to Sir Francis Drake — they all got where they were going willy-nilly, by forces attributed to good luck or the grace of God.

As more and more sailing vessels set out to conquer or explore new territories, to wage war, or to ferry gold and commodities between foreign lands, the wealth of nations floated upon the oceans. And still no ship owned a reliable means for establishing her whereabouts. In consequence, untold numbers of sailors died when their destinations suddenly loomed out of the sea and took them by surprise. In a single such accident, on October 22, 1707, at the Scilly Isles near the southwestern tip of England, four homebound British warships ran aground and nearly two thousand men lost their lives".

"The active quest for a solution to the problem of longitude persisted over four centuries and across the whole continent of Europe. Most crowned heads

[291] Includes quotations from: '*Longitude*' by D. Sobel, Penguin Books, 1996, New York, 184 pp.

of state eventually played a part in the longitude story, notably King George III of England, and King Louis XIV of France. Seafaring men such as Captain William Bligh of the Bounty *and the great circumnavigator Captain James Cook, who made three long voyages of exploration and experimentation before his violent death in Hawaii, took the more promising methods to sea to test their accuracy and practicability.*

Renowned astronomers approached the longitude challenge by appealing to the clockwork universe: ***Galileo Galilei***, ***Giovanni Domenico Cassini***, ***Christiaan Huygens***, ***Isaac Newton***, and ***Edmund Halley***, *all entreated the moon and stars for help. Palatial observations were founded in Paris, London, and Berlin for the express purpose of determining longitude by the heavens. Meanwhile, lesser minds devised schemes that depended on the yelps of wounded dogs, or the cannon blasts of signal ships strategically anchored – somehow – on the open ocean.*

In the course of their struggle to find longitude, scientists struck upon other discoveries that changed their view of the universe. These include the first accurate determinations of the mass of the earth, the distance to the stars, and the speed of light".

As time passed and no method proved successful, the search for a solution to the longitude problem assumed legendary proportions. The governments of the great maritime nations — including Spain, the Netherlands, and certain city-states of Italy — periodically roiled the fervor by offering jackpot purses for a workable method. Finally, in 1714 Queen Ann (through the famed Longitude Act of Parliament) authorized a public reward of £20,000 to any person who should construct chronometers that would determine a ship's longitude in the open sea within 30 minutes of arc.

There lay the problem; and as it often happens in the history of science, or history in general for that matter, the nation that produced the problem, also produced the individual that was equal to the challenge, one ***John Harrison***, *clockmaker, a mechanical genius who pioneered the science of portable precision timekeeping, who devoted his life to the quest. He accomplished what Newton had feared was impossible: he invented a clock that would carry the true time from the home port to any remote corner of the world. He would build the first mechanical marine chronometer, accurate to within 0.1 sec/day, leading to longitude determination accurate to within 1.3 minutes of arc, thus solving the greatest scientific problem of his time.*

There is yet another side to this story which must be told. As in a Shakespearean play, heroes go with villains, and the archvillain in our epic is ***Nevil Maskelyne*** *(1732–1811). Reverend and the 5^{th} astronomer royal, who contested his claim to the coveted prize money, and whose tactics at certain junctures can only be described as foul play. As a member of the longitude-prize*

board, he made all possible obstructions to prevent Harrison from getting the prize. In addition, he took all Harrison chronometers from him (1766) and placed them in a damp cellar at Greenwich, untouched until 1920. Moreover, the commissioners charged with awarding the longitude prize (orchestrated by Maskelyne) changed the contest rules whenever they saw fit, so as to favor the chances of professional astronomers over the likes of Harrison and his fellow 'mechanics'.

Thus, in 1767, Maskelyne published a nautical almanac *(1767) which gave the positions of each heavenly body for exact time and dates. By observing the direction of several stars and measuring their angles above the horizon, the navigator could* roughly *estimate his longitude at sea. The method was much inferior to Harrison's clocktime determination and could not meet the prize conditions.*

"With no formal education or apprenticeship to any watchmaker, Harrison nevertheless constructed a series of virtually friction-free clocks that required no lubrication and no cleaning, that were made from materials impervious to rust, and that kept their moving parts perfectly balanced in relation to one another, regardless of how the world pitched or tossed about them. He did away with the pendulum, and he combined different metals inside his works in such a way that when one component expanded or contracted with changes in temperature, the other counteracted the change and kept the clock's rate constant.

But the utility and accuracy of Harrison's approach triumphed in the end. His followers shepherded Harrison's intricate, exquisite invention through the design modifications that enabled it to be mass produced and enjoy wide use.

An aged, exhausted Harrison, taken under the wing of King George III, ultimately claimed his rightful monetary reward in 1773 – after forty struggling years of political intrigue, international warfare, academic backbiting, scientific revolution, and economic upheaval".

1760–1797 CE **Eliahu ben Shlomo Zalman** (1720–1797, Lithuania; known as THE 'VILNA GAON'; acronym: HA'GRA). Scholar, teacher and leader. One of the greatest Jewish scholars of the 2^d millennium CE. Sought to lead the Jews out of their mental ghetto into the wide world of general culture without doing harm to their specifically Jewish culture. He felt that Jewish learning had excluded too much of the secular knowledge which could be helpful in understanding the world as well as Judaism.

Born in Vilna to a family of distinguished rabbis, he turned out to be a prodigy: at the age of six he had completed the study of the Bible and was deep in the Talmud[292]. By thirteen he had mastered most rabbinic and mystical literature. He steadfastly refused to undertake the responsibilities of active rabbinate, but preferred to live exemplary life on a meager stipend left him in a relative will, so that he might have more time for study. His reputations, however, grew despite his seclusion and before long he was recognized as the unofficial spiritual head of all the communities of Eastern Europe.

The Vilna Gaon was the last of the great Jewish scholars of Talmudism, revered by the orthodox but ignored by the moderns. Through his interest in science, he had shown the Talmudic students the way to Western Enlightenment. The seeds for the coming massive Jewish involvement in modern science were sown when the Vilna Gaon had encouraged not only his but other Talmud students to study and translate scientific works into the language of the prophets.[293]

History repeats itself: as in the Greco-Roman and Islamic days, when these 18^{th} century Jewish youth came in contact with new ideas, they also became imbued with them.

The Vilna Gaon left no written works, but over 40 volumes of his textural notes and his student's notes have been published.

1761–1766 CE **Joseph Gottlieb Kölreuter** (1733–1806, Germany). Botanist. Pioneer of hybridization experiments with plants. Recognized the importance of insects and the wind in pollinating flowers: Published reports describing 136 quantitative experiments in artificial hybridization, foreshadowing the work of Mendel. Professor of natural history and curator of Botanical Gardens at Karlsruhe (1764–1786).

1764–1770 CE **James Hargreaves** (c. 1722–1778, England). Inventor, weaver and carpenter. Invented the '*spinning jenny*', the first machine to spin many threads at a time. He turned the spindles of several spinning wheels upright and placed them in a row. He then added a frame which alternately held and pulled the *rovings* (crude twists of cotton) from which threads were made. He patented the 'jenny' in 1770. Earlier, (1733) **John Kay** invented

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[292] At the age of ten he wanted to become a scientist, but his horrified father turned him from science to Talmud. He never forgot his early interest in science. Had he been born in the 12^{th} century he would have been a great philosopher. At the 18^{th} century he was an anachronistic man.

[293] The Gaon urged one of his pupils, who knew German, to translate Euclid's *Geometry*, for example, which he felt ought to be studied by Jews.

the *flying shuttle loom* which doubled the amount of cloth that weavers could make, but Hargreaves' invention supplied the weavers with more thread.

Hargreaves was a weaver in Standhill, England, and first used the '*jenny*' at home. He then sold some machines. The sales made his patent invalid, and he was never rewarded for his invention. Local spinners worried that the increased amount of yarn the '*jenny*' spun might cost them their jobs. They burned Hargreaves' machine and drove him from the town.[294] He moved to Nottingham (1768) and helped found a prosperous spinning mill. His machine was used in the mill. Other manufacturers used the '*jenny*' without paying him. No one really knows the origin of the term *jenny*.

1765 CE *The Lunar Society* of Birmingham, an informal club of technologists, was founded in England. The society included men such as **Erasmus Darwin** (1731–1802) and **James Watt**. Its members, consisting of Midland scientists and manufacturers, met once a month on the occasion of the new moon to discuss technology and other subjects of shared interest, such as chemistry of clays and glazes, surveying, geology and the developing science of climate and weather. They projected plans for new canals, and devices for harnessing the power of wind and steam. The Lunar Society was the intellectual seedbed for the industrial revolution.

1765–1774 CE **James Watt** (1736–1819, England). A Scottish engineer whose improved engine design first made steam power practicable.

Crude steam engines were used before Watt's time but burned large amounts of coal and produced little power. Their lateral motion restricted their use to operating pumps. Watt's invention of a separate condenser made steam engines more efficient, and his further development of crank movement enabled them to turn wheels and made possible their wider application (patented, 1769). The first primitive steam-engine to convert heat into mechanical energy (used to drain mines) was invented by **Thomas Newcomen** (1663–1729, England) in 1712. His machine was improved by **John Smeaton** (1724–1792, England).

[294] Wool weavers afraid of loosing their job destroyed **John Kay**'s loom in 1733 and sent him packing to France. Nevertheless, about 1750, cotton workers started using the flying shuttle.

Table 3.7: THE EVOLUTION OF SURFACE TRANSPORTATION (1769–1997) (TRAINS, AUTOMOBILES, AIRPLANES, SHIPS, SUBMARINES)

1769 CE	**James Watt** (Scotland) patented his improved *steam engine*.
1770 CE	**Nicolas Cugnot** (France) built first *steam-powered wagon*.
1787 CE	**John Fitch** (USA) built first successful *steamboat*.
1802 CE	**John Stevens** (USA) constructed steamboat that uses *screw propeller*. **Richard Trevithick** (England) built the first *steam railway locomotive*.
1807 CE	**Robert Fulton** (USA) directed the building of the 'Clermont', the first steamboat to become a practical and financially successful (20 HP engine).
1814–1829 CE	**George Stephenson** (England) built the first *reliable railway locomotive*. Completed the adaptation of the steam engine to the railroad.
1815 CE	**John McAdam** (England) introduced a new method of road-building, using crushed rocks.
1830 CE	**Robert L. Stevens** (USA) invented the *railroad rail* (inverted "T").
1834 CE	**Thomas Davenport** (USA) built the *electric streetcar*.
1836 CE	The *screw propeller* for ships was patented.
1838 CE	The *Sirius* (England), first *steamship* to cross the Atlantic Ocean without sails. It made crossing in 18 days.
1839 CE	**Charles Goodyear** (USA) invented the process of *rubber vulcanization*.
1845 CE	**Robert W. Thomson** (England) invented the *pneumatic rubber tire*.
1860 CE	**Etienne Lenoir** (France) invented first practical gas engine for a road vehicle.
1863 CE	First successful subway built in London.

1865 CE **Pierre Lallement** and **Ernest Michaux** (France) constructed the pedal-powered *bicycle*.

1869 CE The *Suez Canal* opened.
First *transcontinental railway* completed in the U.S.

1874 CE Ocean liners cross the Atlantic in only 7 days.

1875 CE **Siegfried Marcus** (Germany) built the first successful 4-cycle petrol driven engine and carriage.
Nickolaus Otto achieved this feat a year later.

1879 CE First *electric locomotive* demonstrated in Berlin.

1881 CE First *electric streetcar* built in Berlin.

1885 CE **Karl Benz** (Germany) built a *gasoline-powered automobile*.

1887 CE **J.B. Dunlop** (Scotland) invented the *air-inflated rubber tire*.

1896 CE **Rudolph Diesel** (Germany) built the first successful *diesel engine*.

1896 CE **Samuel P. Longly** (USA) made first successful powered flight of an unmanned heavier then-air-plane. The craft, weighting 12 kg, is powered by a small steam engine.
Otto Lilienthal (Germany) was killed while flying one of his experimental gliders after making hundreds of successful flights. His pioneering work heavily influenced the Wright brothers' airplane design.

1900 CE *Electrical ignition* system invented for internal combustion engine.

1903 CE The **Wright brothers** (USA) made the first successful *airplane flight*.

1907 CE **Louis Bréquet** and **Paul Cornu** (France) made the first successful *helicopter flight*.

1908 CE *Gyroscopic compass* was invented.
Henry Ford introduced the Model T car.

1914 CE The *Panama Canal* opened.
Red and green traffic lights utilized for the first time in Cleveland, Ohio.

1927 CE **Charles A. Lindbergh** (USA) completed first nonstop solo transatlantic flight: flew 5180 km from New York to Paris in $33\frac{1}{2}$ hours.

1930 CE **Frank Whittle** (England) patented the first *jet engine*.

1932 CE First successful *synthetic rubber* became available commercially.
Diesel-electric trains were introduced.

1935 CE The French ocean liner 'Normandie' crossed the Atlantic in only 4 days.
Gas-turbine engines patented in England and Germany contributed to the development of the *jet aircraft engine*.

1939 CE First flight of jet-powered aircraft in Germany.

1947 CE An experimental rocket-plane broke the *sound barrier* in the United States.

1949–1952 CE First commercial *turbo-jet airliner* (the De Havilland 'Comet') was unveiled in Great Britain. Went into *regular service* in 1952.

1950 CE Jet aircraft made its first transatlantic flight.

1954 CE U.S. launched '*Nautilus*', world's first nuclear submarine.

1955 CE First practical *hovercraft* was built.

1958 CE U.S. launched '*Savannah*', world's first nuclear-powered cargo ship.

Boeing 707, first American jet air-liner, begun regular commercial service.

1964 CE Boeing 727 commercial airliner was introduced.

1969–1970 CE 'Concorde' – *supersonic jet-liner* (French-British) and the Soviet Tu-144, fly at supersonic speeds for the first time.
Pan-American World Airlines began commercial flights of the *362-passenger* Boeing 747 jet.

1986 CE Lightweight airplane 'Voyager' completed record *round-the-world flight* without refueling.

1988 CE Largest suspension bridge constructed in Japan; it has a span of 2 km.

1997 CE (Oct 15) **Andy Greene** (England) broke the sound barrier (1220 km/sec) with his *supersonic car* in the Nevada desert. The car was driven by two turbo-jet engines.

1765–1784 CE **Carl Wilhelm Scheele** (1742–1786, Sweden). Apothecary and chemist. First discoverer of *oxygen* (1772, ahead of Priestley), *chlorine*, *manganese*, and *barium* (1774). Claimed that air consists of oxygen and nitrogen (1777). Discovered and isolated various organic acids: [*prussic* (1765), *tartaric* (1770), *oxalic* and *uric* (1776), *lactic* (1780), *citric* (1784), *malic* (1785), *gallic* (1786)] and also *glycerine* (1783). Discovered action of light on silver salts (1777). Formed HCN (hydrocyanic acid) by the action of ammonia on a mixture of charcoal or graphite and potassium carbonate (1782).

Scheele was born at Stralsund, then the chief town of Swedish Pomerania. He was apprenticed in 1757 to an apothecary in Gothenburg, where he began to study chemistry. He occupied positions in pharmacies in Malmö, Stockholm, Uppsala and Köping, where he died at an early age.

Scheele was a man of great modesty and his circumstances were often poor. He worked with very simple apparatus and in periods of scanty leisure, in a cold and uncomfortable laboratory, yet he made a great number of discoveries of the very first rank.

1765–1785 CE **Lazzaro Spallanzani** (1729–1759, Italy). Physiologist, naturalist and 'microbe-hunter'. Known for his experiments in digestion, circulation of the blood, fertilization and regeneration of animals. Disproved the theory of spontaneous generation; pioneered in volcanology. The first to watch isolated bacterial cells divide. His main achievements:

- Suggested preserving the quality of food by sealing it in airtight containers (1765).
- Demonstrated (1767–8) that the experiments of John Needham (1713–1781, England), allegedly 'proving' spontaneous generation of microorganisms, were invalid since they derived from germs transported in the air.

- Discovered (1773) digestive action of saliva.
- Established importance of semen for fertilization.
- Showed that *digestion* was clearly a *chemical* process rather than a mechanical grinding of food (1780).
- Performed artificial semination of a dog (1785).
- Lay the foundations of modern *volcanology* and *meteorology*.

Born in Pavia. First educated by his father, who was a lawyer. At the age of 15 was sent to a Jesuit college at Reggio de Modena and took orders of the Roman Catholic Church. Studied natural history, languages and mathematics at the University of Bologna. Professor at the Universities of Reggio (1754–1760), Modena (1760–1769) and Pavia (1769–77). Made many journeys along the shores of the Mediterranean.

1766–1794 CE **Peter Simon Pallas** (1741–1811, Germany). Zoologist and botanist. Influenced the development of evolutionary theory.

Pallas was born in Berlin and attended the Universities of Halle, Göttingen and Leiden, where he earned his doctor's degree at the age of 19. His books *Miscellania Zoologica* (1766) and *Spicilegia Zoologica* included a new system of animal classification as well as a discovery of several vertebrates new to science.

In 1767 he was invited by Catherine II of Russia[295] to become a professor at the St. Petersburg Academy of Sciences, and during 1769–1774 he led an expedition to *Siberia* collecting natural history specimens on their behalf. He explored the upper *Amur*, the *Caspian Sea*, and the *Ural* and *Altai mountains*, reaching as far eastward as *Lake Baikal*. Between 1793 and 1794 he led a second expedition to southern Russia, visiting the *Crimea* and the *Black Sea*.

The first expedition resulted in his book: "*Journey through various provinces of the Russian Empire*" (1776–1778).

The stony-iron meteorite of *Krasnoyarsk* as well as a number of animals are named after him. His work provided great amounts of data on a variety of subjects, including botany, zoology, geology, geography, ethnography, philology, and medicine. Employing the comparative method, he thus laid the foundations of a new natural history that was influential in the development of evolutionary theory.

[295] During the reign of this empress, Russia became increasingly receptive to Western science, technology and culture. The German-born monarch invited scores of foreign scholars to take up residence in Russia in the hope of developing the material resources and intellectual life of her empire.

1766–1798 CE **Henry Cavendish** (1731–1810, England). Physicist and chemist. In 1766 he discovered the properties of hydrogen and identified it as an element. Later he showed that water is composed of hydrogen and oxygen.

In 1798, Cavendish performed a novel laboratory experiment to measure Newton's universal gravitation constant G [the apparatus he employed was devised by **John Michell** in 1784]. This constant remained unknown for over half a century after Newton. A rough estimate of G from guesses like Newton's of the average density of the earth, showed that the attractions between small objects in a laboratory must be almost hopelessly small. The common forces of gravity seem strong; but they are due to the huge mass of the earth[296]. The sun, with enormously greater mass still, controls the whole planetary system with its gravitational pull. But the gravitational tugs between human-sized objects are so small that we never notice them compared with earth-pulls and the forces between objects in "contact". It was clear that to measure G, very delicate and difficult experiments would be needed.

In a desperate attempt, several scientists at the end of the 18^{th} century tried to use a measured mountain as the attracting body. They estimated G by the pull of the mountain on a pendulum hung near it. They had to measure *astronomically*, the tiny deflection of the pendulum from the vertical caused by the sideways attraction of the mountain. They then had to *geologically* estimate the mass of the mountain and its "average distance" from the pendulum. Substituting these measurements in $F = G\frac{Mm}{d^2}$ gave the estimated value[297] $G \approx 7.5 \times 10^{-8}$ cgs and consequently $\rho = 4.5$ gm/cm^3 (1774).

Cavendish placed a pair of small metal balls on a light trapeze suspended by a long thin fiber. He brought large lead balls near the small ones in such positions that their attractions on the small balls pulled the trapeze about the fiber axis, twisting the fiber until its elastic forces balanced the effects of the tiny attractions.

He measured the masses and the distances between the small balls and the large attracting balls, but to calculate the value of G he also needed to know the attracting forces. Since the fiber was far too thin and delicate

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[296] For a mass m on the surface of a homogeneous spherical earth of radius a, mass $M = \frac{4\pi}{3}\rho a^3$ and average density ρ, Newton's law yields: $G\frac{Mm}{a^2} = mg$, where g is the earth's surface gravity. The two equations render the relation $g = \frac{4\pi}{3}G\rho a$. For known g and a, this relation enables one to calculate the mean density if G is measured independently.

[297] In 1887 **Thomas Preston** (1860–1900) obtained, in a similar experiment, $G = 6.6 \times 10^{-8}$ cgs.

for any direct measurement, Cavendish let the trapeze and its small balls twist to and fro freely with simple harmonic motion and timed the *period* of that isochronous motion. From that, with measurements of mass and dimensions of the trapeze, he could calculate the twisting strength of the fiber. He then proceeded to obtain a good estimate of G. To avoid convection currents, Cavendish placed his apparatus in a closed room and observed it with a telescope from outside the room. Cavendish's value for G was 6.75×10^{-8} cgs. The ensuing average density of the earth was $5.48\ \mathrm{gm/cm^3}$ (1798).

Cavendish was born in Nice, France, the elder son of Lord Charles Cavendish [brother of the 3^{rd} duke of Devonshire] and Lady Anne Gray, daughter of the duke of Kent. During 1749–1753 he studied in Cambridge without taking a degree. In the latter part of his life he inherited a fortune which made him one of the richest men of his time. He owned a huge private library, where he used to attend on appointed hours to lend the books to men who where properly vouched for. So methodical was he that he never took down a volume for his own use without entering it in the loan-book. He never married.

1768–1774 CE **William Hewson** (1739–1774, England). Surgeon, anatomist and physiologist. Sometimes referred to as the 'father of haematology.' Isolated *fibrin*, a key protein in the blood coagulation process. He also contributed work on the *lymphatic system* by showing the existence of lymph vessels in animals and explaining their function. Demonstrated that *red blood cells* were flat rather than spherical (as had been previously supposed by Leeuwenhoek).

In 1773 he produced evidence for the concept of a *cell membrane* in red blood cells — however, this last work was largely ignored.

Hewson was born in Hexham. He studied at Newcastle upon Tyne and Edinburgh, being the assistant of **William Hunter** (1761–1762). He died, at the age of 35, as a result of sepsis contracted whilst dissecting a cadaver.

An article "William Hewson (1739–74): the father of haematology" was published in May 2006 in the *British Journal of Haematology.*

Evolution of the Steam Engine[298]

A crude prototype of the first engine, in the form of an apparatus which employed the kinetic energy of jets of steam, is mentioned amongst the writings of **Hero of Alexandria** *(ca 150 BCE). In his book Pneumatica, he describes a primitive steam reaction turbine (Hero's engine was considered in his time to be mainly an interesting toy). Another apparatus described by Hero is a mechanism to close or open temple doors by a hidden mechanism: A hollow altar containing air is heated by a fire kindled on it. The air, in expanding, drives some of the water contained in a spherical vessel beneath the altar into a bucket. The descending bucket pulls ropes that are entwined on a pair of vertical posts, to which the doors are fixed, causing them to open. When the fire is extinguished, the air cools, the water leaves the bucket, the ascent of which closes the doors. In another device, a jet of water driven out by expanding air is turned to account as a fountain.*

Today, not only do jet propulsion *and* rocket motors *run airplanes — there are also* gasoline engines *for cars and planes;* diesel engines *for trucks, boats and trains;* steam turbines *to generate electricity and propel boats; and* steam engines *to run boats and locomotives.*

But all of these engines make use of the same basic principle which operated Hero's toy: a hot flame imparts increased motion to molecules and causes expansion of gases. When a substance is heated, its molecules move at great speeds but in random, haphazard motion. As many molecules go one way as another. The problem is to organize this chaos of movement so that the molecules act together, applying their energy in one direction. In all heat engines this collimation is accomplished by permitting the hot gas to

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[298] A steam engine is a machine for the conversion of heat into mechanical work, in which the working substance is water and water vapor. The working substance may be regarded from two points of view: *Thermodynamically* it is the vehicle by which heat is conveyed to and through the engine from the hot source (the furnace and boiler). Part of this heat undergoes a transformation into work as it passes through, and the remainder is emitted, still in the form of heat.

Mechanically, the working substance is a medium capable of exerting pressure, which effects this transformation in doing work by means of a change of volume which it undergoes in the operation of the machine.

Regarded as a thermodynamic device, the function of the engine is to extract as much work as possible from a given quantity of heat (or from the combustion of a given quantity of fuel). Accordingly, a question of primary importance is what is called the *efficiency* of the engine.

create pressure in a chamber which is completely enclosed except on one side. Sometimes there is a movable piston on this side and the bombardment of the molecules cause it to move. In a jet propulsion engine this side is left open; then the hot expanding gases rush out the back, at the same time reacting on the engine to push it forward. In the steam turbine, the motion of the gas rushing out of the open end pushes a wheel and makes it turn.

From the time of Hero to the 17^{th} *century no progress was recorded, though here and there we find evidence that appliances like those described by Hero were used for trivial purposes, such as organ blowing and the turning of spits.*

However, in 1601 ***Giovanni Battista della Porta*** *described in his treatise on pneumatics, an apparatus similar to Hero's fountain but with steam instead of air: Steam generated in a separate vessel passes into a closed chamber containing water, from which a pipe (open under the water) leads out. He also pointed out that the condensation of steam in the closed chamber may be used to produce a* vacuum *and suck up water from a lower level. In fact, his suggestions anticipated the machine which a century later became the steam engine.*

In 1629, ***Giovanni Branca*** *designed an engine shaped like a water-wheel, to be driven by the impact of a jet of steam on its vanes, and in its turn to drive another mechanism for various useful purposes.*

To **Edward Somerset**, 2^{nd} *marquis of Worcester, appears to be due the credit of proposing (1663), if not making, the first useful steam engine: Its object was to raise water, and it probably worked like della Porta's model — but with a pair of displacement-chambers, from which water was alternatively forced by steam from an independent boiler, while the other vessel was allowed to refill.*

The steam engine first became commercially successful in the hands of ***Thomas Savery*** *(1650–1715, England), who in* 1698 *obtained a patent for a water-raising engine. In the use of artificial means to condense the steam, and in the application of the vacuum so formed to raise water by suction from a level lower than that of the engine, Savery's engine was probably an improvement on Somerset's.*

Earlier, in 1678, *the use of piston and cylinder (long before known as applied to pumps) in a steam engine had been suggested by* ***Jean de Hautefeuille*** *(France), who proposed to use the explosion of gun-powder to raise a piston. In* 1680, ***Christiaan Huygens*** *described an engine in which the explosion of gun-powder in a cylinder expelled part of the gaseous contents, after which the cooling of the remainder caused a piston to descend under atmospheric pressure, doing work in the process by raising a weight.*

In 1690, ***Denis Papin*** *suggested that the condensation of steam should be employed to make a vacuum under a piston, previously raised by the expansion of the steam. Papin's was the earliest cylinder and piston engine.*

The first usable engine which made use of heat obtained by burning coal was invented by ***Thomas Newcomen*** *in* 1705. *In this engine, the pressure of the steam moving through a pipe controlled by a valve made the piston rise. Then the valve was shut manually and another valve on the opposite side was opened to let the steam out, condense it and thus make a vacuum under the piston. Air pressure above the piston then pushed it back to repeat the cycle (boys were hired at very low pay to turn the valves for* 14 *hours a day!).*

About half a century after Newcomen's engine first appeared, it was greatly improved by ***James Watt*** *(1769). He gets the credit for inventing the steam engine because he made the valve operation automatic and thus creating a practical engine. The non-condensing, high-pressure engine was the invention of* ***Richard Trevithick*** *(1800) and* ***Oliver Evans*** *(1755–1819, U.S.A.) in* 1805.

A steam engine is called an external-combustion engine because the fuel is burned outside the cylinder of the engine. There is a furnace to burn the coal and a boiler for the production of steam. In most steam engines, the furnace and the boiler are much larger than the cylinder itself. Watt's improvements and other improvements added since Watt's time, have made the modern steam engine 10 *times as efficient as Newcomen's original. Nevertheless, on the average, less than* 15 *percent of the total heat energy that is put into the engine is converted into mechanical energy. With expensive equipment and the greatest care of operation, this efficiency figure can be raised to* 27 *percent.*

1768–1777 CE **Jesse Ramsden** (1735–1800, England). Precision instrument-maker. One of the most skillful designers of mathematical, astronomical, surveying and navigational instruments in the 18th century. Introduced the first satisfactory *screw-cutting lath*[299] (1770), which had far reaching consequences.

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299 The availability of accurately cut screws, engaging with equally accurately cut gears, made it possible to effect the controlled movement of the scriber that cut the graduations of the scale. In this field **Duc de Chaulnes** (1714–1769, France) did much pioneer work, introducing the use of *microscopes*, with cross-

Using the ideas of Duc de Chaulnes (1768), he built the first dividing-engine suitable for work on an industrial scale. His machines excited great interest, and early in the 18^{th} century many of a similar type were built.

Ramsden was first to carry out *in practice* a method of reading off angles by measuring the distance of the index from the nearest division line by means of a *micrometer screw* which moves one or two fine threads placed in the focus of a *microscope*. His specialty was divided circles, which began to supersede *quadrants* in observatories toward the end of the 18^{th} century. He took out patents for improvements in the *sextant*, *theodolite*, *barometer* and *micrometer*. He also invented the *electrostatic machine* with glass plates (1768).

He was elected Fellow of the Royal Society (1786) and received the Copley Medal (1795).

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hair in the field of view, for the precise location of the graduations of the master plate; he also used the *tangent-screw drive*.

The development of the steam-engine by **Watt** (1769) and **Trevithick** (1800) made possible the creation of a civilization based on *power-driven machinery* but did not of itself create such a civilization. In fact, the steam-engine took almost 50 years to establish itself as the principal source of power for industry. One reason was the poor trade conditions existing during and after the great French wars; another reason was the purely technical difficulty of constructing steam-engines and the machinery they were to drive: the making by hand of parts of machinery to *precise standards* could prove not merely prohibitively expensive but even a practical impossibility. To this end, accurately threaded screws were important for a variety of purposes in the making of both precision machines and machine-tools [as, for example, the moving of the tool-holder — since each turn of the screw must correspond to a precisely determined linear movement forward].

The rate of which standards changed is illustrated in the fact that in 1776 the error in boring a meter-long cylinder was about 2 millimeters, whereas by 1856 workshop machines were capable of measuring 250 parts per million of a millimeter!

Ship of Doom

Principal causes of mortality among Royal Navy warship crews during the late 18^{th} century were: Enemy action 8.3%; Fire, sinking, wreck 10.2%; Accident 31.5%; Disease 50%.

Eye-patch, hook-hand and peg-leg, equipment of the pantomime pirate, are theatrical details with origins of historical accuracy. Throughout its history, the sailing ship was a death trap to the men who sailed in it, and mutilation often the lot of those who survived a lifetime at sea. Accidents could be of different origins:

- Falls *were an everyday hazard to man racing aloft to take in sail at the onset of a squall. They meant either broken bodies on deck or "man lost overboard". A fall into a billowing sail would catapult a sailor far into the sea.*
- Electrocution *(a storm hazard), if lightning struck a mast.*
- Rupture*, caused by hauling on ropes, or reefing heavy sails, was common among crew members.*
- Snapping cables *whiplashed, severing the limbs of bystanders if not killing them.*
- *On deck, a* gun breaking loose *from its mount would crush anyone in its path and had to be tipped on its side to stop it.*

After months at sea, the air below decks was rank and fetid and the bilges fouled. Respiratory diseases — tuberculosis, pneumonia — were common; stomach disorders, caused by bad food and bad water, were things with which every sailor had to live.

Headroom below decks was so limited that cracked heads were common. If a man found his way to the surgeon, he was in the hands of a man who supplied his own drugs and instruments and knew as little about his job as the man he treated.

To these everyday hazards, a passage in the tropics added many more. Yellow jack, or the black vomits, and malaria reduced crews to a point where, unless more hands could be pressed, a ship sailed undermanned and was more prone to foundering and running aground. Finally there was scurvy*, a horrifying deficiency disease leading to disfigurement and death; it remained common until steam came along to shorten passage times and reduce the crew's reliance on stored food. During the late 18^{th} century, when British sea power was approaching its peak, press-gangs roamed the seaports, clubbing*

unwary passers-by senseless and carrying them off to lives of hardship and likely death — not in action, but from accident or disease.

1768–1779 CE **James Cook** (1728–1779, England). Navigator, surveyor and explorer of the Pacific Ocean. Commanded three scientific expeditions around the globe.

The first (1768–1771), was launched under the joint sponsorship of the English Admiralty and the Royal Society, to observe the transit of Venus (June 03, 1769, Tahiti), produce a detailed survey of the coastline of the '*South Continent*'[300], and observe its flora, fauna and inhabitants.

In the *H.M.S. Endeavor* (368 tons, 31 m long, crew of 97 men), he circumnavigated the North and South islands of New Zealand and mapped its coasts. During this voyage, the naturalist **Joseph Banks**[301] (1743–1820) and his assistants collected specimens of 1400 previously unidentified species of plants.

Soon afterwards, Cook embarked on his 2^{nd} voyage (1772–1775) with the *Resolution* (462 tons), the *Adventure* (330 tons), and a crew of 193 men. He became the first man to sail across the Antarctic circle. His pioneering work led him to conclude that a frozen continent lay further south in the Antarctic (later explorers proved him right).

Cook's 3^{rd} and last voyage (1776–1779) was primarily to settle the question of the *northwest passage*. He proved that there was no direct water route from the Pacific Ocean to Hudson Bay. He was killed by the Hawaii islanders in a trivial incident.

"*In ten years, he explored more of the earth's surface than any other man in history*". This tribute was to James Cook, who by sheer competence as a navigator, sailor and leader of men rose in his life from obscure origins to

[300] Geographers have speculated for hundreds of years the existence of a continent that extended from the South Pacific to the South Pole. Like many explorers before him, he looked for it in vain.

[301] A wealthy young naturalist and member of the Society, who put up £10,000 to help the expedition and supplied some of the telescopes and other scientific instruments. With him were **Carl Solander**, a Swedish botanist and pupil of **Linnaeus**. There were also two artists: **Alexander Buchan**, a landscape painter, and **Sydney Parkinson**, who specialized in natural history.

a permanent place in history. In his youth and early twenties he served in collier brigs working out of Whitby (a busy port in his native Yorkshire), before joining the Royal Navy in 1755 as a seaman. In two years he was master — warrant officer in charge of handling a warship — aboard the 64-gun *Pembroke*, and by his navigational skills was instrumental in the successful assault on Quebec by General Wolfe in 1759. The talents that had gained him promotion made him a giant among explorers.

1769 CE **Richard Arkwright** (1732–1792, England). Inventor and manufacturer. Improved on earlier versions of *spinning machines* by adding mechanical details that made them work. The machine was powered by water. Sets of rollers turning at different speeds drew cotton from the carding machine, which straightened out the fibers. Spindles then twisted the cotton into thread.

Arkwright was born in Preston in Lancashire, the youngest of 13 children. After serving his apprenticeship in his native town, he established himself as a barber about 1750, and later amassed a little property from dealing in human hair, and dyeing it by a process of his own. He worked 16 hours a day and studied at night to make up for his lack of schooling. He was knighted in 1786.

1769–1781 CE **Pierre Sonnerat** (1748–1814, France). Naturalist and explorer. Made several voyages to southeast Asia, visiting the *Philippines and Molucces* (1769–1772), *India and China* (1774–1781) and *New Guinea* (1776). He was the first person to give a scientific description of the south Chinese fruit tree *lychee.*

In the latter half of the 18^{th} century, France made serious attempts to break the monopoly in the spice trade which the Dutch had long enjoyed. Having annexed the Seychelles islands in the Indian Ocean (1743), they built permanent settlements (1768) and spice plantations, later dispatching expeditions to India, the Malay archipelago, and elsewhere. Sonnerat was a naturalist accompanying one such voyage. He made extensive observations of primitive societies and exotic wildlife, which he subsequently reported.

1770 CE **Edward Waring** (1734–1798, England). Physician and mathematician. In his *Meditationes algebraicae* asserted without proof:

- Every positive integer is the sum of nine or fewer cubes (known as the '*Waring Conjecture*'; yet unproven).
- Every positive integer is a sum of a fixed number s of non-negative k^{th} powers (known as the '*Waring Problem*').

Clearly, the 'conjecture' is a special case of the 'problem' for which $s(3) = 9$. In general $s = s(k)$, such that for a given non-negative integer N

$$N = u_1^k + u_2^k + \cdots + u_{s(k)}^k, \qquad k \geq 2.$$

For each s there are two problems:

(1) Prove that $s(k)$ exists (done by **Hilbert** in 1908).

(2) Find the minimum value $s(k)$ for a given N.

Lagrange proved (1770) that $s(2) = 4$ and Waring himself claimed that $s(3) = 9, \quad s(4) = 19$.

G.H. Hardy had pointed out (1938) that the most fundamental and most difficult aspect of the problem is that of deciding not how many cubes are required for the representation of *all* numbers, but how many are required for the representation of *all large* numbers, i.e. of all numbers with some finite number of exceptions.

In 1986, **Ramachandran Balasubramanian**, **Jean-Marc Deshouillers** and **Fancois Dress** proved that $s(4) = 19$.

Waring also classified quartic curves and was first to set forth a method of approximating values of imaginary roots of polynomial equations.

Waring practiced medicine in various London Hospitals. From 1760 he was Lucasian professor of mathematics at Cambridge, although he did not give up practicing medicine until 1770.

1770 CE **John Wilson** (1741–1793, England). Discovered a theorem that bears his name:

$$(p-1)! + 1 \equiv 0 (\text{mod } p)$$

for p prime [or: $\frac{1 \cdot 2 \cdot 3 \cdots (p-1)+1}{p}$ is an integer]. This theorem was presumably discovered on numerical evidence alone, and reported without proof in Waring's book: '*Meditationes Arithmeticae*' (1770). Among the posthumous papers of Leibniz there were later found similar calculations on the remainders of $n!$, and he seems to have made, already in 1682, the same conjecture. The first proof of the theorem was given by **Lagrange** in 1770.

Wilson was a senior Wrangler at Cambridge and left the field of mathematics quite early to study law. Later he became a judge and was knighted.

1770 CE **Nicolas Joseph Cugnot**. French army captain. Operated successfully a three-wheeled steam-powered vehicle. It was used as a tractor for hauling cannon. It could travel 5 km/hour and had to stop every 10 or 15 minutes to build up steam.

1770–1789 CE **Antoine Laurent Lavoisier** (1743–1794, France). French chemist who gave the first accurate scientific explanation of the mystery of fire. In 1777, after a series of careful experiments, he stated that burning is the result of rapid union of the burning material with oxygen and that *respiration* is a form of combustion (1780). In 1789 he wrote the first modern textbook of chemistry[302] in which he formulated the principle of conservation of matter[303]. These ideas led him to write the first chemical equation.

Originated the modern concept of the chemical *element* through the definition: "*an element is a substance that cannot be decomposed into simpler substances*". On the basis of this definition, he drew up a list of 30 or so elements, most of which are still recognized as such.

Lavoisier was born in Paris of well-to-do parents, and attended the College Mazarin, where he studied mathematics, astronomy, chemistry, and botany. In 1768 he became a member of the Academy of Sciences. He established an agricultural experiments station, and tried to improve farming methods in France. Among his other varied interests were the increase of production of salt, improved manufacture of gunpowder, plans for improving the social and economic conditions of the community by means of saving banks, insurance societies, canals, and workhouses. He was further associated with committees on hygiene, coinage, the casting of cannon, etc., and was secretary of the treasurer of the commission appointed in 1790 to secure uniformity of weights and measures.

In 1787, Lavoisier, **Berthollet** and their associates introduced the first method of chemical nomenclature based on scientific principles ("*Method d'une nomenclature chimique*").

He was executed on the guillotine for his membership in a financial company that collected taxes for the government[304].

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302 *Traité élémentaire de chimie* (1789), a work sometimes described as marking the start of chemistry as a science, and classed with Darwin's *Origin of the Species* (1859) in biology and Newton's *Principia mathematica* (1687) in physics. In his book, Lavoisier describes *fermentation* as the splitting of sugar into alcohol and CO_2. He characterized the reaction as an oxidation-reduction process. He also made the first measurements on human metabolic rate.

303 He arrived at this conclusion after realizing that the *total weight* of all the products of a chemical reaction must be exactly equal to the total weight of the reacting substances. He was able to draw correct inferences from his weighings because, unlike many of the *phlogistonists*, he looked upon heat as imponderable.

304 A petition in his favor addressed to Coffinhal, the president of the tribunal, is said to have been met with the reply: "*La Republique n'a pas besoin de*

The name Lavoisier is indissolubly associated with the overthrow of the phlogistic doctrine that had dominated the development of chemistry for over a century, and with the establishment of the foundations upon which the modern science rests. Justus von Liebig said of him:

"*He discovered no new body, no new property, no natural phenomenon previously unknown; but all the facts established by him were the necessary consequences of the labors of those who preceded him. His merit, his immortal glory, consists of this — that he infused into the body of science a new spirit; but all the members of that body were already in existence, and rightly joined together*".

Founders of Modern Chemistry – from Lavoisier to Mendeleev (1778–1889)

If a specific date is to be set for the science of chemistry, it may be said to have begun with ***Robert Boyle****'s clear definition of a chemical element in his* Sceptical Chymist *of 1661. Nevertheless, it would be quite wrong to suppose that there was a sharply defined transition from empiricism to science. Chemical theory was built on a strong foundation of knowledge laboriously built up over the centuries by practical men and, on a level more detached from reality, by the alchemists with their fruitless preoccupation with the transmutation of base metals into gold and the preparation of an elixir of life. More realistic than the alchemists, though more limited in ambition, were the so-called iatrochemists of the* 16th *century Paracelsian school, who looked on chemistry as primarily the handmaiden of medicine.*

The discovery of gases[305] *during* 1620–1774*, and the investigation of their properties slowly undermined the* phlogiston *theory*[306]*, and after* 1785 *it*

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savants".

[305] *Hydrogen* (Boyle, 1670; **Cavendish**, 1766); *Ammonia* (**Kunckel**, 1677; **Berthollet**, 1787); *Oxygen* (**Scheele**, 1772; **Priestley**, 1774); *Nitrogen* (**Cavendish** and others, 1772); *Chlorine* (Scheele, 1772; Priestley, 1774); *Carbon dioxide* (**Van Helmont**, 1620; **Black**, 1754); *Carbon monoxide* (**Daniel Rutherford**, 1772).

[306] The name *phlogiston* was coined (1703) by **Georg Ernst Stahl** (1660–1734), a professor of medicine and chemistry at Halle.

rapidly disappeared except among a few very conservative chemists. [Although the theory had the advantage of coordinating a large number of facts into a system, it retarded the progress of chemistry, and prevented a number of the best investigators from seeing the correct explanation of the facts they brought to light.]

With the publication in 1789 *of the* Elements of chemistry *by* Lavoisier, *the science of chemistry severed its remaining connections with the alchemical past and assumed a modem form. Lavoisier stressed the importance of quantitative methods of investigation in chemistry, and in this connection, he introduced the principle of conservation of matter. Lavoisier's new viewpoint led to the elaboration of several empirical laws. The first was the law of equivalent proportions (1791), formulated by* **Jeremias Benjamin Richter** *(1762–1807, Germany). After this discovery, tables of equivalent weights were drawn up, showing the relative amounts of chemical elements that would combine with each other. Richter also introduced the name stoichiometry.*

A second law, that of constant proportions, was put forward (1797) by **Joseph Louis Proust** *(1754–1826, France). Finally, the revival of the atomic theory (1803) by* **John Dalton** *(1766–1844, England), opened the road to the quantitative analysis and synthesis of compounds and put chemistry on solid foundations upon which the scientific method could rest.*

Gay-Lussac *(1778–1850, France) then established the law of combining gaseous volumes (1807), followed by an hypothesis (1811) of* **Amadeo Avogadro** *(1776–1856, Italy) which reconciled Dalton's atomic theory with Gay-Lussac's law. But Avogadro's hypothesis was not accepted until the* 1860*'s, and chemists long continued to base atomic weights on arbitrary rules*[307].

William Prout *(1785–1850, England), a London physician, suggested (1815) that the atoms of all the elements were composed of a discrete number of hydrogen atoms, but* **Jöns Jacob Berzelius** *(1779–1848, Sweden), who*

[307] Dalton himself denied to the end the validity of Avogadro's hypothesis(!) because Avogadro pointed out that the molecules of elementary gases are not necessarily the atoms themselves, but usually consist of *groups* of atoms. Both kind of particles, atoms and molecules, had been called "atoms" by Dalton, but they are really different. Dalton held that like atoms must repel one another and could not combine. With his logic, the fact that one volume of oxygen combined with one volume of nitrogen to produce two volumes of nitric oxide meant that nitric oxide should contain only half as many particles in a given volume as nitrogen or oxygen. But the true reaction is $N_2+O_2=2NO$ in full accord with Avogadro's hypothesis. The hypothesis was also rejected by **Gay-Lussac** and **Berzelius**.

devised the modern chemical symbols (1813) and introduced the name Halogen (1825), showed that the atomic weights of the elements were not exact multiplies of the weight of the atom of hydrogen.

From about 1820 *to* 1860 *the atomic theory did not play a prominent role in chemistry. For the most part chemist preferred to use the directly determined equivalent weights of the elements, rather than the atomic weights which involved uncertain estimates as to the combining numbers of the atoms. The rejection of Avogadro's hypothesis left chemists without a general method of ascertaining the combining numbers of the elementary atoms.*

As early as 1824, *chemists discovered isomers* (compounds with the same *chemical formulas but different molecular structure). It all began with* ***Eilhard Mitscherlich*** *(1794–1863,* Germany), *one of Berzelius' pupils, who noticed (1819) that compounds with similar chemical formulae had the same crystalline form. This was the advent of isomorphism, through which Berzelius could determine the formulae of many salts and the atomic weight of their constituent elements.*

In the same year ***Pierre Louis Dulong*** *(1785–1838,* France) and ***Alexis Thérèse Petit*** *(1791–1820,* France) *found in Paris that in the case of a number of metals, the product of their atomic weight and specific heats was constant. This law enabled rough values of the atomic weights of the metals to be determined.*

The discoveries of ***Galvani*** *(1771) and* ***Volta*** *(1775) and the availability of the Voltaic cell, soon led to the development of the new branch of electrochemistry. It appeared that electricity could bring about chemical action;* ***William Nicholson*** *(1753–1815,* England) and ***Anthony Carlisle*** *(1768–1840,* England) *performed (1800) the first electrolysis of water. The experiments of* ***Humphry Davy*** *(1778–1829,* England) *during* 1801–1806 *on the electrolysis of salt solutions led him to the theory that the chemical reaction between the elements, was essentially of an electric character and paved the road to the electrical theory of chemical affinity developed further by Berzelius. The laws of electrolysis were introduced (1833) by* ***Michael Faraday*** *(1791–1867,* England).

Inorganic chemistry developed rapidly in the period 1790–1830, *as geologists discovered numerous minerals for the chemists to analyze. Berzelius himself described the preparation, purification, and analysis of over* 2000 *inorganic compounds in the decade* 1810–1820. *Hundreds of chemists, mainly in German, French, English and Swedish universities were discovering each year new elements and compounds and determining their properties: Uranium (1789) was discovered by* ***Martin Heinrich Klaproth*** *(1743–1817,* Germany); *Chromium (1798) by* ***Louis Nicolas Vauquelin*** *(1763–1829,* France);

Bromine (1826) by **Antoine Jérome Ballard** *(1802–1876,* France*); Palladium* and *Rhodium (1803) by* **William Hyde Whollaston** *(1766–1828); hydrogen peroxide* by **Louis Jacques Thenard** *(1777–1857,* France*).*

Early recognitions of the law of mass action were made by **Carl Friedrich Wenzel** *(1740–1793,* Germany*) in* 1777 *and by* **Claude Louis Berthollet** *(1748–1822) in 1799.* **Jean Antoine Chaptal** *(1756–1832,* France*) proposed the name* nitrogen *(1790) and* **William Whewell** *(1794–1866,* England*) coined the names* electrolysis*,* electrolyte*,* anode*,* cathode*,* anion *and* cation *at the request of* Michael Faraday *(1833).* **Robert Wilhelm Bunsen** *(1811–1899) studies the chemical action of light (1857) and applied the spectroscope to chemistry (1859).*

In 1807*,* **Berzelius** *named the class of solid substances that melt upon heating —* inorganic *and those that burn —* organic*. It was soon discovered that while minerals could be characterized by the relative amounts of the elements which were contained in them, organic compounds from the start were seen to be complex arrangement of few elements, notably of carbon (C), hydrogen (H), oxygen (O), and nitrogen (N); quantitative analysis did not go far toward the characterization of such compounds.*

Isolated studies of carbon compounds go back to the Middle Ages (e.g., alcohol, ether, acetone). The investigations of **Scheele** *during* 1770–1784*, resulted in the discovery of many organic acids in plants and fruits, glycerin, HCN and esters.*

The first satisfactory method of organic analysis was worked out by **Gay-Lussac** *and* **Thenard** *(1810).* **Michel Eugène Chevreul** *(1786–1889) investigated the composition of oils and fats (from 1813), explained the reaction of saponification and worked on the analysis of organic compounds. The first amino acid was isolated and studied by* **Henri Braconnot** *(1781–1855, France) in* 1820.

The development of organic chemistry was boosted considerably by the works of the German chemists **Friedrich Wöhler** *(1800–1882) and* **Justus von Liebig** *(1803–1873). Liebig went to Paris to study under Gay-Lussac at the Ecole Polytechnique, and Wöhler went to Stockholm to study under Berzelius. The later, as late as* 1819*, had thought that organic compounds did not obey the law of constant proportions and did not belong to chemistry proper, as they were the products of "vital forces". But in* 1828*, Wöhler broke down the hypothetical barrier dividing inorganic substances from organic substances by heating some ammonium cyanate, classed as inorganic, and got urea, an organic chemical*[308].

[308] He had mixed solutions of *silver cyanate* (AgCNO) and *ammonium chloride* (NH_4Cl), producing the isomer *ammonium cyanate* ($NH_4.CNO$) which, when

Now organic and inorganic chemistry were brought close together. Nevertheless, certain fundamental difference were emerging. It was found already in 1815 *by* ***Jean Baptiste Biot*** *(1774–1862, France) that tartaric acid produced by grapes [HOOC−(CHOH)$_2$−COOH] polarizes light, while seemingly the same acid produced in the laboratory, did not polarize light — both acids having the* same *chemical formula. Liebig and Wöhler found other similar situations in* 1824*. In* 1830*, Berzelius named such pairs of compounds* isomers.

Louis Pasteur *(1822–1895, France) investigated the optical activity of organic compounds (1848) and had worked out the mechanism by which two otherwise identical isomers behave differently in living organisms. He suggested that the shape of the molecule might be different between the isomers. He also provided experimental proofs for the vitalistic theory of* fermentation *(1857) and later carried out fundamental research in bacteriology, disproving 'spontaneous generation'.*

Jean Baptiste Dumas *(1800–1884, France), chemist and politician, suggested that the chemical properties of organic compounds were due to their particular structural arrangements, or* type*, and not to the electrical character of the elements which compose them (1838). His theory had a direct influence on the revival of the atomic theory, which meanwhile had receded into the background of chemical theory.*

The theory of types was further developed by ***August Laurent*** *(1808–1853) [who also discovered* anthracene *(1832)] and* ***Charles Frederic Gerhardt*** *(1816–1856, France) who revived the theory of acid radicals and contributed to the developing concept of atomic weights.* ***Charles Adolphe Wurz*** *(1817–1884, France) developed the method of synthesizing* long-chain hydrocarbons *using hydrocarbon iodides and sodium (1855).*

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warmed, gave crystals of urea identical to those obtained as a waste product in urine. He wrote: "*I must tell you that I can make urea without requiring a kidney or an animal, either man or dog*".

Previously, urea had been obtained synthetically by **John Davy** in 1811 by the action of ammonia gas on carbonyl chloride (the poisonous gas *phosgene*, obtained when a mixture of equal volumes of chlorine and carbon monoxide is exposed to bright sunlight). The reactions are: $Cl_2+CO \Rightarrow COCl_2$, $4NH_3+COCl_2 \Rightarrow CO(NH_2)_2+2NH_4Cl$. Urea is then separated from ammonium chloride by warming with alcohol in which urea is soluble.

Davy, however, was not aware that urea was formed in the reaction.

In 1859, ***Marcel Morren***, *the dean of the science faculty of Marseille, discovered acetylene* ($H-C \equiv C-H$) *by activating an electric spark in a glass containing carbon electrodes and hydrogen*[309] ($2C+H_2 \rightleftharpoons C_2H_2$).

Pierre-Eugéne-Marcellin Berthelot *(1827–1907) succeeded in synthesizing many organic compounds (1854–1868), among them alcohol (1857) and the first that do not occur naturally (1860). Through this he seriously undermined the remaining support for the theory of vitalism. Berthelot also carried out important work in physical chemistry on reaction velocity, thermochemistry and detonation waves.*

Christian Friedrich Schönbein *(1799–1868, Switzerland) discovered ozone (1840) and synthesized (1846) the new organic compound nitrocellulose*[310] *which heralded the age of high explosives.* **Thomas Graham** *(1805–1869, Scotland), one of the founders of physical chemistry, discovered his law of diffusion of gases (1829).* **Hermann Kolbe** *(1818–1884, Germany) foreshadowed modern structural formula and valence.*

However, it was **Edward Frankland** *(1825–1899, England) who introduced the concept of valence of elements into chemistry (1852) [*valenc = the *definite capacity of each atom to combine with other atoms] and recognized that the valency of an element could vary. He noted that the elements fell into groups which had the same valency.*

Friedrich August Kekulé *(1829–1896, Germany) recognized the* 4-*valency of carbon (1857) and began to use structural diagrams based on bonding in organic chemistry (1861). His diagrams showed that Pasteur was correct in assuming that the shape of the organic molecule determines its properties. Kekulé also put forward the hexagon formula for benzen (1865).*

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309 **Morren** reported that he obtained 'carbonized hydrogen', the nature of which he has not yet established. Three years later (1862) **Marcelin Berthelot** repeated the same feat. However, he *knew* that he had synthesized acetylene. There was one difference between the two experiments. Morren had made a *discovery* and Berthelot an *invention*. **Jean Baptiste Dumas**, who was president of the Academy, drew Berthelot's attention to what Morren had done before him. Berthelot replied that Morren had not been able to verify his production of a carbonized hydrogen. In fact, Morren had verified it by a *spectral analysis* of the gas obtained, but he was not up to contesting the matter with Berthelot; he stepped down and history has forgotten his name.

310 It was discovered when Schönbein's wife's apron, which he had used to wipe up a spilled mixture of acids, exploded and vanished in a puff of smoke. When others tried to manufacture guncotton in quantity, many were killed by premature explosions.

At the international Karlsruhe conference (1860) ***Stanislao Cannizzero*** *(1826–1910, Italy) revived the work of Amadeo Avogadro, particularly, Avogadro's hypothesis* and the important distinction between atoms and molecules (1811). Cannizzero employed the Avogadro hypothesis in the straightforward determination of molecular weights of gaseous compounds by comparing the weight of the volume of the gas to that of an equal volume of hydrogen at the same pressure and temperature. From molecular weights he proceeded to atomic weights. The valency could then be obtained by dividing the atomic weight by the equivalent weight of the element.*

With settled values for the valencies of the elements, structural models of their compounds were constructed. The reaction of those compounds provided tests for the validity of such structures, while, in turn, the proposed structures indicated possible new reactions.

The final addition to the classical theory of molecular structure came in 1874 *when* ***Joseph Achille Le Bell*** *(1847–1930,* France*), and* ***Jacobus Hendricus Van't Hoff*** *(1852–1911,* Holland*), suggested independently that the* 4 *valencies of carbon were directed in space toward the apices of a regular tetrahedron, in order to account for the two isomeric forms of tartaric acid isolated by Pasteur (1848), and other cases of optical isomerism discovered later.*

The acceptance of Avogadro's hypothesis, followed by the establishment of the definitive valencies and atomic weights of the elements, had its influence upon inorganic as well as organic chemistry. The works of ***Johann Dobereiner*** *(1780–1849,* Germany*) and* ***Antoine Jérome Ballard*** *(1802–1876, France) helped to classify the elements into equivalency groups.* ***Alexandre Émile Beguyer de Chancourtois*** *(1820–1886,* France*) was first to publish a list of elements in the order of their atomic weights (1862), but since he failed to furnish an accompanying diagram, the periodicity of the elements was far from clear.*

Finally, ***Dimitri Ivanovich Mendeleev*** *(1834–1907,* Russia*) in* 1869 *and* ***Julius Lothar Meyer*** *(1830–1895,* Germany*) in* 1870*, formulated the periodic law, stating that the properties of the elements varied in a periodic manner with their atomic weights. Both emphasized that there were gaps in the periodic table which elements as yet unknown should occupy. Mendeleev specified three gaps, all of which were filled by discoveries between* 1875 *and* 1885*. As a consequence, he got most of the credit for the periodic table.*

The periodic classification provided the first theoretical guide to the search for new elements: 23 elements known to Lavoisier had been discovered by the trial and error study of their specific chemical relations. Practical chemical analysis became more systematized, and, applied to the mineral specimens

provided by the geologists, it led to the discovery of 31 *new elements in the period* 1790–1830.

Between 1830 *and* 1860 *little was accomplished in regard to the isolation and identification of new elements, save the rare-earths lanthanum and erbium by* **Carl Gustaf Mosander** *(1797–1858,* Sweden*) in* 1839–1841. *With his new spectroscope,* **Bunsen** *discovered the new alkali metals cesium and rubidium in* 1860–1861.

In London, **William Crookes** *(1861–1919) found thallium (1861) and in the Freiberg School of Mines,* **Ferdinand Reich** *(1799–1882,* Germany*) discovered indium (1863). Then came the discoveries of gallium (1874), ytterbium (1878), scandium (1879), gadolinium (1880) and germanium (1885) by the respective chemists* **Paul Emile Lecoq de Boisbaudran** *(1838–1912, France; Ga),* **Jean Charles Galissard de Marignac** *(1817–1894,* Switzerland; *Gd),* **Lars Fredrik Nilson** *(1840–1859,* Sweden; *Sc) and* **Clemens Alexander Winkler** *(1838–1904,* Germany; *Ge).*

The development of chemistry in the second half of the 19^{th} *century was mainly due to the rapid growth of synthetic organic chemistry: attempts were made to prepare in the laboratory those compounds which built up the plant and animal organisms. In addition, numerous drugs and dyestuffs have been prepared which are not found in the storehouse of nature.*

Three of the leading organic chemists who contributed most of this trend were **Adolf Johann Friedrich Wilhelm von Baeyer** *(1835–1917),* **Emil Hermann Fischer** *(1852–1919), and* **Victor Meyer** *(1848–1898) — all from Germany. Baeyer synthesized the dye indigo blue (1878) Fischer won the Nobel Prize (1902) for sugar and parine synthesis (1891–1898), and Meyer discovered thiophene (1882).*

In the 19^{th} *century chemistry held sway as the leading science. Yet it was running ahead of its theories, emerging with little understanding of why certain rules worked; for example, the concept of valence was introduced in* 1852 *and the periodic table in* 1869*, but it was not until the discovery of the Pauli exclusion principle in* 1925 *that either of these could be understood from first principles. Likewise, with no understanding of how electron behave (indeed, without any suspicion of electrons!) the wonders of spectroscopy emerged without a theoretical basis.*

1771 CE Encyclopedia Britannica published.

1771–1772 CE **Alexandre-Théophile Vandermonde** (1735–1796, France). Mathematician and musician. Founded the general theory of determinants (1772).[311]

His role in this field is similar to the one played by Cayley with regard to matrices in 1857. In his "*Mémoire sur la resolution des équations*" (1771), he approached the general problem of solubility of algebraic equations through a study of functions invariant under *permutations* of the roots of the equation.

Kronecker (1888) claimed that the study of modern algebra began with this paper of Vandermonde. **Cauchy** stated that Vandermonde had priority over **Lagrange** (1771) for this remarkable idea which eventually led to the study of *group theory*. He thus discovered the first truly group-theoretic properties of permutations and the key to understanding of the solution of equations by radicals.

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[311] His name is best known today for the *Vandermonde determinant*. Yet nowhere in his four mathematical papers (1771–1772) does this determinant appear! Let

$$P(x) = x^n - s_1 x^{n-1} + \cdots + (-)^n s_n = (x - x_1)(x - x_2)\cdots(x - x_n),$$

let $\sigma_i = \sum_{j=1}^{n} x_j^i$ for $i = 1, 2, ...$ and let A and B be the $n \times n$ matrices

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \cdots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{bmatrix},$$

$$B = \begin{bmatrix} n & \sigma_1 & \sigma_2 & \cdots & \sigma_{n-1} \\ \sigma_1 & \sigma_2 & \sigma_3 & \cdots & \sigma_n \\ \sigma_2 & \sigma_3 & \sigma_4 & \cdots & \sigma_{n+1} \\ \vdots & \vdots & \cdots & \vdots & \\ \sigma_{n-1} & \sigma_n & \sigma_{n+1} & \cdots & \sigma_{2n-2} \end{bmatrix}.$$

It can be then shown that

- $\det A = \prod_{i>j} (x_i - x_j) =$ *Vandermonde's determinant*
- $B = AA^T$ (A^T = transpose of A)
- $D(s_1, s_2, ..., s_n)$ = discriminant of $P(x) = \det B$
- Discriminant of $P(x) = x^n + px + q$ is $(-)^{\frac{n(n-1)}{2}} \left[(-)^{n-1}(n-1)^{n-1}p^n + n^n q^{n-1}\right]$

Solved (1771) the irreducible cyclotomic equation

$$(z^{11}-1)/(z-1) = z^{10}+z^9+z^8+z^7+z^6+z^5+z^4+z^3+z^2+z+1=0$$

in radicals.

1772 CE *Nitrogen* was discovered independently by **Daniel Rutherford** (1749–1819, Scotland), **Carl Scheele** (1742–1786, Sweden), **Joseph Priestley** (1733–1804, England), and **Henry Cavendish** (1731–1810, England).

1772 CE **Johann Bode** (1747–1826, Germany) stated an empirical relation that gives the approximate mean distance of the known planets from the sun. It is known as *Bode's law* although it was stated earlier (1766) by **Johann Titius** (1729–1796, Germany) and it is not a law in the strict physical sense. If 4 is added to 0, 3, 6, 12, 24, 48, 96, and each sum is divided by 10, a sequence of number results, each of which is the approximate distance of a planet in astronomical units[312] [Mercury 0.4 (0.39); Venus 0.7 (0.72); Earth 1.0 (1.00); Mars 1.6 (1.52); Ceres 2.8 (2.77); Jupiter 5.2 (5.20); Saturn 10.0 (9.54); Uranus 19.6 (19.18); Neptune 38.8 (30.06); Pluto 77.2 (39.44).]

When Bode's law was stated, there was an apparent gap in the series between the distances of Mars and Jupiter and it was thought that there was a planet in the gap.

On Jan. 1, 1801, **Guiseppe Piazzi**[313] (1746–1826, Italy) discovered the first known asteroid. This "star" was found to move and was observed for about a month before it became "lost" owing to the illness of its discoverer.

Based on complicated calculations with meager evidence (41 days of data), **Carl Friedrich Gauss** (1771–1855, Germany), then only 23 years old, determined its orbit using a new method developed by him ad hoc. When its distance from the sun was found to fit almost exactly the mean distance computed by Bode's law, it was assumed to be the missing planet and named Ceres (diameter $\sim$ 1000 km), for the goddess of agriculture and protector of Sicily.

It is not clear today whether Bode's law is just a coincidence of numbers, or whether it describes some deeper interrelation among the planet's orbits. If this progression of numbers indeed has any meaning, it could provide some insight into the *early history* of the solar system. But it could also just be an

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[312] The number in brackets is the actual value.

[313] The astronomer **Piazzi** was born in Ponte di Valtellina, Italy. He became a Theatine monk, professor of theology in Rome (1779), and professor of mathematics at the Academy of Palermo (1780). He set up an observatory at Palermo (1789) and published a catalogue of fixed stars (1813), listing 7646 entries.

arrangement that would hold for *any* system of bodies orbiting about a center of mass, given enough time for those bodies to reach some state of dynamic equilibrium.

Gauss' method depends on accurate positions of the body on 3 dates, preferably separated by a few weeks. It is still used in modified form. Ceres was found again in the position predicted by Gauss. Other asteroids were discovered (often called minor planets): **Pallas** (1802), **Juno** (1804), **Vesta** (1807), **Astraea** (1845). Presently, nearly 2000 asteroids are known.

The fame earned by **Gauss** through his efforts on the problem eventually led, in 1807, to his appointment as director of the Göttingen Observatory, where he remained for the rest of his life.

1773–1778 CE **Otto Frederik Müller** (1730–1784, Denmark). Naturalist. Taxonomically separated bacteria[314] from protozoa and was able to distinguish two morphological types of bacteria: *bacillum* and *spirillum*. Bacteria were first *stained* (with indigo and carmine) by **Wilhelm Friedrich von Gleichen-Russworm** (1778).

1773–1825 CE **Pierre Simon de Laplace** (1749–1827, France). An eminent mathematician and astronomer. His most outstanding work was done in the fields of celestial mechanics, probability, differential equations and geodesy.

He was first to examine the conditions of stability of the system formed by Saturn's rings, pointed out the necessity for their rotation and assigned to it a period (10^h33^m) virtually identical with that established by the observations of **Herschel**. In 1773 he began his studies of the figure of equilibrium of a mass of rotating fluid. The related subject of the attraction of spheroids was also promoted by him, and in 1784 he generalized the results of **Legendre** and **Maclaurin** and treated exhaustively the general problem of the attraction of any spheroid form upon a particle situated outside or on its surface.

Laplace was born of poor parents at Beaumont-en-Auge in Normandy. His early mathematical ability won him a teaching post in the military school of Beaumont. He came to Paris and with the support of d'Alembert, became a professor of mathematics in the Ecole Militaire of Paris.[315]

At the age of 24 he rose to fame following his discovery (1773) concerning the mutual gravitational interactions of the constituents of the solar system

[314] First discovered by **Leeuwenhoek** (1683).

[315] **Napoleon Bonaparte** was a cadet at this school from October 1784 to October 1785.

(sun, planets and their satellites). He showed that while perturbations[316] introduced small changes into planetary orbits, these changes were *periodic*: that is, the orbit would alter its properties in one direction, then back in the other, and so on indefinitely. Over the long run, the *average* shape of the orbit would remain constant[317]. This is Laplace's celebrated conclusion of the invariability of the planetary mean motions, carrying the proof as far as the cubes of the eccentricities and inclinations. This was the first and most important step in the establishment of the stability of the solar system. It meant that the solar system was in dynamic equilibrium, and could continue indefinitely into the future and might already have existed through an indefinite past [assuming of course that there is no appreciable overriding influence by stars outside the system].

Laplace's results were followed by a series of investigations, in which **Lagrange** and Laplace alternatively surpassed and supplemented each other in assigning limits of variation to the several elements of the planetary orbits.

In his monumental five-volume treatise: "*Traité de Mécanique Céleste*" (1799–1825), Laplace summed up the work on gravitation of several generations of illustrious mathematicians [giving credit only to himself (!) and suppressing references to discoveries of his predecessors and contemporaries, including Lagrange]. The principal legacy of *Mécanique Céleste* to later generations lay in Laplace's wholesale development of *potential theory*[318], with its

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[316] To dig deeper, see:

- Nayfeh, A., *Perturbation Methods*, Wiley, 1973, 425 pp.
- Bellman, R., *Perturbation Techniques in Mathematics, Physics and Engineering*, Holt, Rinehart and Winston, 1964, 118 pp.
- Bush, A.W., *Perturbation Methods for Engineers and Scientists*, CRC Press, 1992, 303 pp.
- Hinch, E.J., *Perturbation Methods*, Cambridge University Press, 1991, 160 pp.
- Bender, C.M. and S.A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers*, McGraw-Hill, 1978, 593 pp.

[317] **Newton** himself was bewildered by the complexity of the solar system and was of the opinion that divine intervention would occasionally be needed to prevent this complex mechanism from degenerating into chaos. Laplace, apparently, decided to seek reassurance elsewhere.

[318] For further reading, see:

- Kellogg, O.D., *Potential Theory*, Dover, 1953, 384 pp.
- MacMillan, W.D., *The Theory of the Potential*, Dover, 1958, 469 pp.

far-reaching implications for different branches of physics ranging from gravitation and fluid dynamics to electromagnetism and atomic physics. Even though he lifted the idea of the potential from Lagrange without acknowledgements, he exploited it so extensively that ever since his time the fundamental partial differential equation of potential theory has been known as the *Laplace equation*[319].

The overall aim of Laplace in his treatise was to "offer a complete solution of the great mechanical problem presented by the solar system, and to bring the theory to coincide so closely with observation that empirical equations should no longer find a place in astronomical tables".

The first part of the work (2 volumes, 1799) contains methods for calculating the movements of translation and rotation of the heavenly bodies, for determining their figures and resolving *tidal* problems.

In his book '*Exposition du Systeme du Monde*', published in 1796, Laplace speculated on the subject of planetary origin in his famous *nebular hypothesis*.

Already in 1644, **Descartes** advanced the idea that the sun and its solar system formed from a gigantic whirlpool, or vortex, in a universal fluid, with the planets and their satellites forming from smaller eddies. This crude theory did not include any clearly specified idea of the nature of the cosmic substance from which the sun and planets arose, but it did account for the fact that all orbital motions are in the same direction.

The hypothesis of Descartes[320] was the first of a general type known as *evolutionary theories*, in which the formation of the solar system is posited to

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- Webster, A.G., *Partial Differential Equations of Mathematical Physics*, Dover, 1956, 440 pp.
- Webster, A.G., *Dynamics – Lectures on Mathematical Physics*, Hafner, 1949, 588 pp.
- Bateman, H., *Partial Differential Equations of Mathematical Physics*, Cambridge University Press, 1959, 522 pp.

[319] It is the equation $\nabla^2\phi = 0$ of the gravitational potential ϕ, outside the source region. It had actually been discovered by Euler in 1752, in connection with his studies of hydrodynamics.

[320] The Cartesian idea of a set of universal laws which control natural occurrences exercised a powerful appeal in the succeeding centuries. Laplace, even as he developed his theory of a naturally evolving cosmos, took the motion to its logical conclusion by endorsing the idea that, given the laws of gravitation and other forces, Newtonian mechanics, and the "initial conditions" of the universe, every subsequent event not only can be accurately predicted, but is *predetermined*. The whole history of the universe, and of earth, is but the inevitable unfolding

have occurred as a natural by-product of the sequence of events that produced the sun.

Immanuel Kant in 1755 further elaborated Descartes' idea by applying the recently discovered Newtonian mechanics to show that a rotating interstellar gas cloud would flatten into a disc as it contracted.

Laplace added to this model the notion that as the spinning cloud flattened into a disc, rotational inertial forces broke off concentric rings of matter, so that at one point the early solar system would have resembled the planet Saturn with its rings. Each ring was supposed to have condensed into a planet.

In the same book, Laplace raised another speculation about an object which he called "*corps obscurs*". He noted (1796) that a consequence of Newtonian gravity and Newtonian corpuscular theory of light was that light would not be able to escape from a sufficiently massive object, but would be bent around and stay trapped near the object[321]. In spite of this deduction, the idea that a '*black hole*' could actually exist in nature did not occur to astronomers for almost 2 centuries. Laplace's "*Corps obscurs*" were taken up in the mid 1960's by modern physicists, armed with the new General Relativity Theory.

His other masterpiece was the treatise "*Théorie Analytique des Probabilités*" (1812). Nowhere did Laplace display his genius more conspicuously than in the theory of probabilities. The science which **Pascal** and **Fermat** had initiated, was brought by him to near perfection. In this book he amalgamated his own discoveries with many ideas of others (unacknowledged!),

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of the consequences of a set of eternal laws. Laplace believed that mathematical physics is capable of explaining *everything*, although in practice he ignored physical phenomena not governed by the basic mathematical laws known in his day. For example, he did not take into account (because he could not) the electrical and magnetic interactions of bodies, their chemical reactions, their nuclear transformations, the processes by which they are heated and cooled — in short, all the phenomena now known to science but unknown to him. Nevertheless, the Laplacian idea of a deterministic, natural evolving universe – suitably modified by modern insight of quantum physics and chaos theory – is nowadays taken for granted by science – even in the realms of *biology* and *cosmology* (the study of the universe as a whole, including its very *creation*).

321 This was noted before (1784) by **John Michell**. Laplace calculated that no light could escape from a body with the *earth's density* and a radius 250 times that of the sun. Did Laplace read Michell's paper, published in the Philosophical Transactions of the Royal Society of London?

but even discounting this, his book is considered to be the greatest contribution to this branch of mathematics by any one man. Here he harnessed, for the first time, the powerful machine of the infinitesimal calculus to discrete mathematics.

To this end he invented the *Laplace Transform*, *generating functions* and many other highly nontrivial tools. He showed how the *Laplace transform* can be used to reduce the solutions of linear differential equations to definite integrals, and furnished an elegant method by which a linear partial differential equation of the second order might be solved.

Laplace died exactly 100 years after the death of **Isaac Newton**. His last words were: "*That which we know is a trifle — that which we are ignorant of is immense*".

Worldview XV: Pierre-Simon de Laplace

* * *

Nature laughs at the difficulties of integration.

* * *

Read Euler: he is our master in everything.

* * *

Such is the advantage of a well constructed language that its simplified notation often becomes the source of profound theories.

* * *

"An intelligence, which at a given moment knew all of the forces that animate nature, and the respective positions of the beings that compose it, and further possessing the scope to analyze these data, could condense into a single formula for the movement of the greater bodies of the universe and that of the least atom: for such an intelligence nothing could be uncertain, and past and future alike would be before its eyes."

(1812)

* * *

Napoleon and his scientists

(**Lagrange** 1736–1813; **Monge** 1746–1818; **Laplace** 1749–1827; **Fourier** 1768–1830)

In September 1785, ***Laplace*** *examined and passed a cadet by the name of Napoleon Bonaparte in a military school at Beaumont. With this education, Napoleon qualified himself for the artillery.*

Napoleon Bonaparte *(1769–1821) Emperor of the French, was an enthusiastic amateur mathematician, particularly fascinated by geometry, which of course had great military value. He was also a man with unbounded admiration for the creative French mathematicians of his day. Whatever Napoleon's ability as a geometer may have been, it is to his credit that he so revolutionarized the teaching of French mathematics that, according to several historians of mathematics, his reforms were responsible for the great upsurge of creative mathematics in the 19th century France.*

It is said that in 1797, while discussing geometry with ***Lagrange*** *and* ***Laplace****, Bonaparte surprised them by explaining some of* ***Mascheroni****'s solutions that were completely new to them. "General", Laplace reportedly remarked, "we expect everything of you, except lessons in geometry".*

Yet, a theorem named after Napoleon exists. It states that the centers of equilateral triangles constructed on the sides of an arbitrary triangle form another equilateral triangle (Napoleon's theorem). [The theorem provides a generalization in which the word 'equilateral' is replaced by 'similar'.] It is very doubtful that Napoleon was well enough versed in geometry to have discovered and proved it himself.

Monge *gained the close friendship and admiration of Napoleon and accompanied the latter, along with* ***J. Fourier****, on the Egyptian Expedition (1798).*

The publication of Mécanique Céleste gained ***Laplace*** *world-wide celebrity. Asked by Napoleon why in the entire work he had not once mentioned God, Laplace replied: 'Sire, je n'avais pas besoin de cette hypothèse' (Sir, I had no need for that hypothesis).*

But scientific distinctions by no means satisfied his ambition, and after the French revolution Laplace's political talents and greed for position came into full play. He set a memorable example of a genius degraded to servility for the sake of a riband and a title. He smoothly adapted himself by changing

his principles – back and forth between fervent republicanism and fawning royalism – and each time emerged with a better job and grander titles. The ardor of his republican principles gave place to devotion towards Napoleon, a sentiment promptly rewarded with the post of minister of the interior. His incapacity for affairs was however so flagrant that it became necessary to supersede him at the end of 6 weeks. "He brought into the administration", said Napoleon, "the spirit of the infinitesimals".

His failure was consoled by elevation to the senate, of which body he became chancellor in 1803. The title of Count he had acquired on the creation of the Empire. Nevertheless, he cheerfully gave his voice in 1814 for the dethronement of his patron, and his compliance merited a seat in the chamber of peers and, in 1817, the dignity of a marquisate.

Table 3.8: Greatest Mathematicians of the 18^{th} century

Name	Nat.	Life-span	Major Contribution
Jakob Bernoulli (Jacque, James)	SW	1654–1705	Theory of probability; isoperimetric problems (early calculus of variation); Bernoulli numbers and polynomials.
Antoine Parent	F	1666–1716	Solid analytical geometry (1700).
G. Saccheri	I	1667–1733	Forerunner of non-Euclidean geometry.
Johann Bernoulli (Jean, John)	SW	1667–1748	Principle of virtual work; L'Hospitale rule; partial differentiation; Brachistochrone; Applied calculus.
Abraham de Moivre	E	1667–1754	Normal and binomial distributions; Probability theory; De-Moivre formula; generating functions; Approximation for $n!$.
J.F. Riccati	I	1676–1754	Differential equations.
Roger Cotes	E	1682–1716	Algebraic equations; Trigonometry.
Fagnano dei Toschi	I	1682–1766	Addition theorems for elliptic integrals; Rectification of curves; $\pi = 2i \ln \frac{1-i}{1+i}$.
Brook Taylor	E	1685–1731	Polynomial approximation to analytic functions; Integration by parts; Calculus of finite differences.

Table 3.8: (Cont.)

Name	Nat.	Life-span	Major Contribution
Christian Goldbach	G	1690–1764	Goldbach conjecture (1742).
Colin Maclaurin	E	1698–1746	Applied calculus; Attraction of ellipsoids; determinants.
P.L.M. de Maupertuis	F	1698–1759	Principle of least action (optics and mechanics).
Daniel Bernoulli	SW	1700–1782	Hydrodynamic theory; First 'Fourier expansions'.
Thomas Bayes	F	1702–1761	Principle of Inverse probability (conditional probability).
Gabriel Cramer	SW	1704–1752	Determinants.
Leonhard Euler	SW	1707–1783	One of the last, and one of the greatest mathematical universalists. Contributed to all fields of pure and applied mathematics. Established analysis as an independent science.
Comte de Buffon	F	1707–1788	Geometrical probability. Forerunner of Monte-Carlo methods.
Alexis-Claude Clairaut	F	1713–1765	Differential geometry. Differential equations.
Jean Le Rond d'Alembert	F	1717–1783	Scalar wave-equation; Mathematical theory of gravitational perturbation; Foundations of calculus.

Table 3.8: (Cont.)

Name	Nat.	Life-span	Major Contribution
John Landen	E	1719–1790	Elliptic Integrals and functions.
Johann H. Lambert	G	1728–1777	Irrationality of π and e; Non-Euclidean geometry; map projections; Infinite Series; descriptive geometry.
Etienne Bezout	F	1730–1783	Algebraic equations; (Bezout eliminant).
A.T. Vandermonde	F	1735–1796	Theory of determinants. Notion of 'group'.
Erland S. Bring	S	1736–1798	The quintic equation ($x^5 + px + q = 0$, 1786).
Joseph Louis Lagrange	F	1736–1813	Calculus of variations; Theory of numbers; Interpolation formulae; Theory of equations; Continued fractions; 3-body problem; Theoretical dynamics.
John Wilson	E	1741–1793	Theory of numbers (1770).
Caspar Wessel	S	1745–1818	Geometry of complex numbers.
Gaspard Monge	F	1746–1818	Descriptive geometry; differential geometry of space curves and surfaces.
Jean-Baptiste-Joseph Delambre	F	1749–1822	Spherical trigonometry.

Table 3.8: (Cont.)

Name	Nat.	Life-span	Major Contribution
Pierre Simon de Laplace	F	1749–1827	Celestial mechanics; Theory of analytic probability; Partial differential equations.
Lorenzo Mascheroni	I	1750–1800	Geometry; infinite series.
Marie Adrien Legendre	F	1752–1833	Number theory; Elliptic Integrals; method of least squares; Law of quadratic reciprocity.
Lazare Carnot	F	1753–1823	Synthetic geometry.
J.B.M.C. Meusnier	F	1754–1793	Differential geometry; Minimal surface (1785).
Aimé Argand	SW	1755–1803	Geometry of complex numbers.
Marc-Antoine Parseval	F	1755–1836	Parseval Equality.
Paolo Ruffini	I	1765–1822	The quintic equation (1799).

Key: SW = Switzerland; F = France; I = Italy; E = England; G = Germany; S = Scandinavia.

1774–1784 CE **Marcus (Mordecai) Herz** (1747–1803, Germany). Physician, physicist and philosopher. One of the best physicians of his time. Was concerned with the ethical aspects of his profession and published (1783) "The Physician's Prayer".

Born to a poor Jewish copyist of scriptures. At the age of 15 left his parent's home and moved to Königsberg, where he began his philosophy studies, under **Kant**. Befriended **Moses Mendelssohn** in Berlin (1770) and studied medicine (1770–1774) at the University of Halle. Published books on philosophy and medicine. King Frederick William III (1744–1797) of Prussia appointed him Professor for life. His wife Henriette (1764–1847) was a famous beauty and society leader who conducted a brilliant salon frequented by **Boerne**, **Humboldt**, **Fichte** and **Schleiermacher**. She married Herz at age of 15 and after his death adopted the Christian faith (1817).

1774–1786 CE **Joseph Priestley** (1733–1804, England). Chemist. Shares the credit for the discovery of oxygen[322] with **Carl Wilhelm Scheele** of Sweden. His experiments are described in his 6 volume treatise "*Experiments and Observations of Different Kinds of Air*". Priestley prepared and examined oxygen, nitrous oxide, nitric oxide, nitrogen dioxide, hydrogen chloride, ammonia, silicon fluoride and sulphur dioxide. His work firmly established the fact that different gaseous forms of matter exist, each with definite properties.

Priestley was born near Leeds. He studied for the ministry and became a *dissenting* (nonconformist) minister in Leeds and Birmingham. Through his friendship with **Benjamin Franklin** he became interested in electricity, on which he performed many brilliant experiments. He turned to chemistry in 1772.

Priestley's sympathies for the cause of the French Revolution made him unpopular in England. In 1791 an angry mob burned his home and chapel in Birmingham. He then left England and moved to the United States in 1794.

1774–1800 CE **Alessandro Giuseppe Antonio Volta** (1745–1827, Italy). Physicist. Pioneer of electrical science. Invented the electric battery, the first electrochemical source of electric current. His discovery of the decomposition of water by electrical current laid the foundation of electrochemistry. He also invented the electric condenser.

Volta was born in Como, Italy, a member of a noble family. By 1774 he had established a reputation by his research work in electricity. In 1779, a

[322] Priestley called the gas "*dephlogisticated air*". The French chemist **Antoine Lavoisier** named it *oxygen*. At that time gases were called "*airs*".

chair of physics was founded in Pavia, and Volta was chosen to occupy it. In 1782 he journeyed through France, Germany, Holland and England, and became acquainted with many scientific celebrities. In 1801 Napoleon called him to Paris, to show his experiments on contact electricity, and a medal was struck in his honor. He was made a senator of the kingdom of Lombardy. In 1815, the emperor of Austria made him a director of the philosophical faculty of Pavia. The *volt*, a unit of electric potential, is named for him.

1774–1804 CE **Johann Heinrich Pestalozzi** (1746–1827, Switzerland). Educational reformer. Influenced strongly methods of instruction in elementary schools throughout Europe and America.

Pestalozzi believed a pupil learned best by using his own senses and by discovering things for himself. His emphasis, therefore, lay upon concrete approach in education, with objects used to develop powers of observation and reasoning.

He was born in Zurich. He first studied for the ministry, but later changed to law. Poor health forced him to abandon law, and Pestalozzi settled on his farm near Zurich. In 1774 he established a school for poor children on his estate and endeavored to put in practice educational theories of **Jean-Jacques Rousseau**; although the school failed (1780), he derived from his experience a knowledge of certain principles for effective education, which he explained in his influential book *Lienhard und Gertrude* (1787). His most famous educational experiments were carried on at an institute for training teachers which he established at Yverdon (1805–1825). His theories are also expounded in *Abendstunde eines Einsiedlers* (1780) and *Wie Gertrude ihre Kinder lehrt* (1801).

1775–1785 CE **William Withering** (1741–1799, England). Physician and botanist. Discovered the use of *digitalis*, the most important drug in the treatment of heart disease[323].

Withering was born in Wellington, England. In 1766 he received his medical degree from the University of Edinburgh and established a general practice in the town of Stafford in Shropshire. He listened with interest (1775) to the country folk in his native district as they described the benefit of foxglove-tea

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323 *Digitalis* was a medical herb for centuries. **Dioscorides** (ca 80 AD) praised it as a plant whose leaves, applied to the skin, could cure many diseases. Rural people made hot water infusions of leaves and drank *foxglove tea* to experience inexpensive but dangerous intoxication.

for 'dropsy'[324]. For several gold sovereigns he purchased the recipe from a local 'witch' and for the next ten years studied digitalis therapy in dropsy. Chemistry was not sufficiently advanced to permit the isolation of the active ingredient; biology in general and human physiology in particular were just in their infancy.

It was thus left to Withering to answer questions such as: which part of the plant was most active, could the leaves be dried; what was the best solvent for the active material (cold water); should one pick leaves in early or late summer and, most importantly, what was the optimum dose and how frequently should it be administered. In 1785, he published *An Account of the Foxglove and Some of its Medical Uses*, a clinical study so detailed and so accurate, so as to make the use of digitalis effective and safe.

The report was something of a bombshell in England where the standard treatment of dropsy was to puncture the water-logged tissues with an unsterilized scalpel and stretch the patient over bedsprings to allow the fluid to drip into buckets.

William Withering died of tuberculosis (1799) and was buried in a vault on which a *Digitalis* plant was engraved.

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[324] A cardiac-stimulating chemical, occurring naturally in the dried leaves of the common garden flower purple foxglove (*Digitalis purpurea*). It is a mixture of several cardiac *glucosides*. Doctors use digitalis in small doses to stabilize heartbeat when the action of the heart muscle is too weak to force blood out of the heart normally.

Dropsy (edema) is a condition in which watery fluid gathers in the body cavities and tissues caused by disorders in blood circulation in anemia, heart disease, kidney failure, etc. The word *Digitalis* was coined (1539) from the Latin *digitis* = little finger, and is directly derived from the German name for the plant *fingerhut*.

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“Behold, I make all things new”

(Revelations 21:5)

* * *

“The Revolution was effected before the War commenced.”

(John Adams)

* * *

1775–1783 CE *American War of Independence*. Thirteen of Britain’s North American Colonies broke away from rule by the mother-country and formed the *United States of America*; they were: Connecticut, Delaware, Georgia, Massachusetts, Maryland, North Carolina, New Hampshire, New Jersey, New York, Pennsylvania, Rhode Island, South Carolina, and Virginia. The colonies already made their own local laws, but the Britain Parliament kept control of financial matters, and particularly trade. The colonies had to use their own or British ships and to trade mainly with Britain or British colonies.

The Revolution could not have taken place without the religious background afforded by the *Great Awakening* — an American revivalist movement (1719–1775)[325]. It arose in response to the growing formalism of early 18^{th} century American Christianity, but was also influenced by the European

[325] Just as in France, rather late in the century, the combination of Voltairean rationalism and Rousseauesque emotionalism was to create a revolutionary explosion, so in America, but in a characteristically religious context. The thinking elements, and the fervid, personal elements were to combine to make Americans see the world with new eyes.

The essential difference between the *American Revolution* and the *French Revolution* is that the American Revolution, in its origin, was a religious event, whereas the French Revolution was an anti-religious event. That fact was to shape the American Revolution from start to finish and determine the nature of the independent state it brought into being. Indeed, in his *Farewell Address*

Enlightenment and the economic boom of middle-class people in Colonial America.

Revival began in New Jersey in 1719; Key figures were: **William Tennent** (1673–1745), a Presbyterian preacher, **Jonathan Edwards** (1703–1758), a puritan scholar, and **George Whitefield** (1714–1770), evangelist. Finally, Presbyterians, Baptists, Calvinists and Methodists all over America embraced the new movement. By questioning established authority, founding new colleges, and revivifying evangelical zeal, it helped to prepare the revolutionary generation in America.

The historian **Paul Johnson** in his book '*A History of the American People*' (1997) summarized the impact of the *Great Awakening*, in these words:

> *"The Revolution was in the mind and hearts of the people... It was the marriage between the rationalism of the American elites touched by the Enlightenment, with the spirit the Great Awakening among the masses which enabled the popular enthusiasm thus aroused to be channeled into the political aims of the Revolution — itself soon identified the coming eschatological event. Neither force could have succeeded without the other."*

After the *Seven Years' War* which brought Britain the French possessions in North America (1755–1763), Britain felt it necessary to keep a standing army there and taxed the colonies to pay for it. The colonists objected to 'taxation without representation' in Parliament. The British tried imposing taxes on newspapers, tea, paper, lead, and paint, but had to repeal all but the tea tax when the colonists refused to buy British goods as protest.

On December 16, 1773, a band of colonists disguised as Indians boarded British ships in Boston harbor and threw cargos of tea overboard. To this '*Boston tea party*' the British Parliament retorted with the so-called '*Intolerable Acts*', which included closing the port of Boston.

The *First Continental Congress* at Philadelphia (Sept. 1774) protested at the Acts, and the colonies decided not to buy British goods. British troops were sent from Boston to destroy an arms cache held by the colonists in

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(1796), **Washington** dispelled for good any notion that America was a secular state. He insisted: "*Religion and Morality are indispensable supports... There can be no security for property, for reputation, for life, if the sense of religious obligation desert the oaths which are the instruments of investigation in the Courts of Justice.*"

In fact, Washington was saying that America, being a free republic, dependent for its order on the good behavior of its citizens, cannot survive without religion.

nearby Concord. Just after dawn on April 19, 1775, at Lexington on the road to Concord, the troops were confronted by armed colonists and the war began.

The British retreated from Concord to Boston, and in June won the *Battle of Bunker Hill*, near Boston, despite heavy losses. The *Second Continental Congress* assembled in May 1775 and on July 4, 1776 issued the *Declaration of Independence*, largely drafted by Thomas Jefferson, claiming complete freedom from the British rule.

In 1777, the British army gained an important victory at Brandywine Creek (Pennsylvania), but a few weeks later, under General John Burgoyne, were forced to surrender at Saratoga (New York). France entered the war on the American side, followed later by Spain.

The end came when the British under Cornwallis surrendered to the American commander-in-chief, **George Washington** at Yorktown (Virginia) on October 19, 1781. The *Treaty of Paris* (September 03, 1783) formally recognized the independence of the United States. Washington was elected first president in 1789.

1776 CE The submarine is first used in combat, during the American Revolution. This 2-meter vessel, called the *Connecticut Turtle*, was designed (1775) by **David Bushnell** (1742–1824) of wood, iron and pitch. Driven by a hand-cranked propeller, it attempted unsuccessfully to sink British warships in New York harbor.

1776 CE **Adam Smith** (1723–1790, Scotland). Economist. Regarded the founder of modern economics. Worked out a theory of division of labor, money, prices, and wages in his book "Inquiry into the Nature and Causes of the Wealth of Nations". It laid foundations of the science of *political economy* and is the most influential economic treatise ever written, founding the classical school of economy. It contains the germ of nearly all economic ideas which have since appeared, even in rival systems.

The book dealt with the basic problem of how social order and human progress can be possible in a society where individuals follow their own self-interests, free from any government interference. This is the policy of '*laissez faire*' – leaving things done. Smith argued that this individualism led to order and progress. In order to make money, people produce things that other people are willing to buy. Buyers spend money for those things that they need or want most. When buyers and sellers meet in a market, a pattern of production develops that results in social harmony provided that all this would happen without any conscious control or direction. Smith also believed that:

- *Labor* (not land or money) was both the source and the final measure of value; *wages* developed on the basic needs of the workers and *rent* on the productivity of the land. *Profits* were the difference between selling prices and the cost of labor and rent.

- *Profits* should be used to expand *production*; this expansion would in turn create more jobs, and the *national income* would grow.

- *Free trade* and a self-regulating economy would result in *social progress.* Government need only preserve law and order, enforce justice, defend the nation, and provide for a few social needs that could not be met through the market.

Smith attacked the British mercantile system's limit on free trade and criticized the British government's tariffs and other limits on individual freedom in trade.

Smith was born in Kirkcaldy, Scotland and studied at the University of Glasgow and Oxford University. He became professor at Glasgow (1751). Appointed tutor of the young duke of Buccleuch (1764) and later received a regular income from that family. This enabled him to retire from teaching and devote the years 1766–1776 to the writing of his book.

1776–1784 CE **Jean Baptiste** (**Marie Charles**) **Meusnier** (de la Place) (1754–1793, France). Mathematician, engineer and army general. A pupil of Monge at the school in Mézières. Derived the *Meusnier theorem* (1776) on curvature at a point on a surface[326]. In 1783–1784, after Montgolfier's ascent in a balloon, he did fundamental research on aerostatics and designed (1784) a dirigible balloon. In this period he collaborated with **Lavoisier** in his work on decomposition of water into its elements. Meusnier joined the Jacobins

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[326] *Euler's theorem* (1760): $k_n = k_1 \cos^2 \alpha + k_2 \sin^2 \alpha$. It expresses the normal curvature (l.h.s.) to a surface in an arbitrary direction α in terms of the principal curvatures k_1, k_2.

Meusnier's theorem (1776): If a set of planes be drawn through a tangent to a surface in a nonasymptotic direction, then the osculating circles of the intersections with the surface (*normal sections*) lie upon a sphere ($k_n = k \cos \gamma$, where k_n = normal curvature, k = curvature of normal section, γ = angle between the principal normal to the curve and the normal to the surface).

The theorem was published in his *Mémoire sur la courbure des surfaces*, which he wrote after Monge had shown him Euler's paper. Together with Euler's theorem it gives full information concerning the curvature of any curve through a point on the surface.

(1790), became a field-marshal (1792) and was killed in defense of the fortress of Kassel, Mainz.

1777–1789 CE **Charles Augustin de Coulomb** (1736–1806, France). Scientist, inventor and military engineer. Made fundamental contributions in the fields of friction, electricity, and magnetism. In 1777 he invented the *torsion-balance* for measuring torsional elasticity, which he used to derive the laws of *torsion* of metal wires[327], strands of hair and thin silk (1784). In his memoir on the theory of simple machines (1779–1781) he discovered the fundamental law of *friction*; In 1785 he made precise measurements of the forces of attraction and repulsion between charged bodies and between magnetic poles, using his torsion balance. He then demonstrated conclusively that electric charges and magnetic poles obey the *inverse-square laws* like that of static gravity.

Coulomb was born at Angouleme. He chose the profession of military engineer. After spending nine years at Martinique in the West Indies he was stationed, in 1781, permanently at Paris. Upon the outbreak of the Revolution in 1789, he resigned his position and retired to a small estate at Blois. But he was recalled to Paris, to take part in the new determination of weights and measures, decreed by the revolutionary government. He was appointed inspector of public instruction in 1802 and died in Paris a few years later.

The practical unit of quantity of electricity, the *coulomb*, is named after him.

1778–1802 CE **Joseph Bramah** (1748–1814, England). Engineer and prolific inventor whose inventions introduced practical techniques that founded the engineering industry. Suggested the possibility of screw propulsion for ships (1785), and the hydraulic transmission of power (1802). Invented the hydraulic press (1795), a numerical printing machine (1806) and the ball-drive siphon system for a water closet (1778).

Bramah was born at Stainborogh in Yorkshire, the son of a farmer. He worked as a cabinet-maker in London, where he subsequently started a business of his own.

1779 CE **Jan Ingenhousz** (1730–1799, Holland). Physician and scientist. Discovered *photosynthesis*[328], and the *carbon cycle* in the earth-atmosphere system.

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[327] The torque τ in a thin cylinder with diameter d and length ℓ is $\tau = \mu d^4\theta/\ell$, where μ is the rigidity and θ is the angle of twist.

[328] The overall process by which plants absorb, store and use radiant energy: red light is absorbed by certain plant pigments (mainly *chlorophyll*), and converted into potential chemical energy. This energy is used to break the water molecule

He showed that in the presence of sunlight plants absorb water and carbon dioxide and give off oxygen through their green portions. In the dark the roots, flowers, and fruits give off carbon dioxide. In this process, plants obtain carbon *from the atmosphere* and not from the soil. On the other hand, he maintained, animals, by eating plants and breathing oxygen, recombine plant tissue and oxygen and re-form carbon dioxide and water. Thus, he concluded, plant and animal life on earth formed a balance.

Moreover — nothing material is used up; carbon, hydrogen, and oxygen shuttle between plants and animals, and from land to sea in a process that is most commonly called "*the carbon cycle*". Other elements, too, are engaged in cyclic processes: nitrogen, sulfur, phosphorus, and so on are absorbed *from the soil* by the plants and incorporated into their tissue. Animals eat plants and make use of the various elements, then finally restore them to the soil in their droppings, and in the form of their own bodies when death is followed by bacterial decomposition. Only one thing is permanently used up in these cyclical chemical changes, and that is the energy of the solar radiation.

Ingenhousz was born in Breda, Holland. During 1772–1779 he was court physician to empress Maria Theresa of Austria. He died in England.

Ingenhousz' discovery was not accidental, but rather a natural consequence of the Industrial Revolution: Following the ascent of the first hydrogen balloons, and the introduction of coal, gas and metallurgical innovations associated with the use of steam power, chemical science grew rapidly.

Black's work on CO_2 (1756), **Rutherford**'s isolation of nitrogen (1772), the researches of **Watt** (1776), **Cavendish** (1766–1781), and **Charles** (1787) on hydrogen, those of **Priestley** (1770), **Lavoisier** (1783–1789), and **Scheele** (1771) on oxygen — distinguished the principal constituents of

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into H and OH, thus reducing atmospheric CO_2 to glucose, according to the general scheme: $683\ \text{Kcal}+6CO_2+12H_2O \rightarrow C_6H_{12}O_6+6O_2+6H_2O$. The above equation glosses over the complexity of what is perhaps a 100-step molecular sequence(!), each step probably requires the action of specific catalytic enzymes, with monosaccharides, proteins, and fats (lipids) also being produced along the way. None of this, of course, was known to Ingenhousz in 1779.

The *reverse process*, that which occurs in the dark, is none other than *animal respiration*, through which energy is formed by oxidation of the stored glucose (or any other carbohydrate) and giving off CO_2. It thus became evident that CO_2 and H_2O are *essential for life*: both take an active part in carrying out the various life functions of living organisms, and in storing energy from the sun in the form of sugars and starches — through a process by which the chlorophyll of green plants catalyzes the formation of carbohydrates from carbon dioxide and water, through the action of sunlight.

air and the 4 elements which make up the bulk of plant tissues. They reawakened interest in the problem of breathing and made it possible to analyze the elementary constituents of the atmosphere and of plant body.

While prosecuting the researches which led to the modern view of metallurgical processes, Scheele, Priestley, and Lavoisier also devoted their efforts to the analogous problems of combustion, breathing, and animal heat. An important by-product of these subsidiary inquiries were two contradictory observations: One was made by Priestley who claimed that plants remove CO_2, and the other by Scheele who maintained that they produce it.

Lazzaro Spallanzani (1729–1799) then noticed (1768) that aquatic plants give off bubbles of oxygen in sunlight and do not do so in darkness. At this point, Ingenhousz took up the clues and showed that while plants remove carbon dioxide from the air and give up oxygen to it in sunlight, they evolve carbon dioxide and take up oxygen in the dark. He thus showed that two kinds of gaseous exchange between the green plant and the air occur: one is comparable to respiration in animals, the other is essentially different from it.

1780–1794 CE **Luigi Galvani** (1737–1798, Italy). Physician and Physicist. Made pioneering researches in electrophysiology, as causing muscular contraction in a frog's legs by application of static electricity. Galvani was professor of obstetrics at Instituto delle Scienze, Bologna (1782 to 1798).

1781 CE **Charles Messier** (1730–1817, France). Astronomer. First to compile a systematic catalog of nebulae, galaxies and star clusters. There are about 110 objects in the Messier catalog. Messier's objects (known by the prefix M and their catalog numbers) include the emission nebulae M8 (Lagoon) and M16 (Eagle) – two of the most famous hot clouds of interstellar matter (emission nebulae).

Massier was born in Badonville, Lorraine. He began professional career as an assistant to **Delisle** in Paris (1751). He observed the return of the Halley Comet (1759) and from that time onwards was an arid searcher of comets, discovering independently a total of 13 of them. His purpose in compiling his catalog was to make comet hunting easier by taking careful note of permanent deep-sky objects that might be mistaken for comets.

1781 CE **Johann Carl Wilcke** (1732–1796, Sweden). Physicist. Introduced the idea of *specific heat*[329], quantity of heat required to raise the temperature of a given substance by a given amount (usually 1 °C).

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[329] The Irish physician and chemist **Adair Crawford** (1747–1795) may also be credited with an independent introduction of the concept of specific heat in 1779 in connection with his treatment of heat generation in animals.

Wilcke was born in Wismar, Germany. Moved to Sweden with his parents (1739). Entered Uppsala university (1750) to study theology but concentrated instead on physics and mathematics. Received his doctorate at Rostock (1757). Prepared a comprehensive map of the earth's magnetic inclination (1768). Drew up a list of specific heats for different substances (1781), independently of **Joseph Black** (1728–1799).

Wilcke also discovered that in changing phase (e.g. from solid to liquid), a body absorbs (or releases) heat without changing temperature (*latent heat*). These were the first important discoveries about heat in modern times.

1781–1786 CE **Moses Mendelssohn of Dessau** (1729–1786, Germany). Distinguished scholar. The apostle of Jewish enlightenment in Germany. Catalyzed the merger of German and Jewish cultures, thus creating the proper climate and opportunity for the involvement of Jews in modern European science.

However, by asserting the pragmatic principle of the possible plurality of truths and his unfailing efforts to emancipate the Jews at the price of weakening the firm adhesion to their traditional values – he opened the floodgates of apostasy, secularization and assimilation on a scale that the Jewish people had never known before. Mendelssohn own descendants – a brilliant circle, of which the composer Felix[330] [331] was the most noted – left the synagogue for the Church.

Moses' father, Mendel Heyman was a poor scribe – a writer of Torah scrolls. The maternal grandfather, however, was a direct descendant from a

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[330] His father, Abraham Mendelssohn-Bartholdy (1776–1835) reflected on his status: "Before, I was known as the son of my father, and now I am the father of my son."

[331] Moses had 5 sons and 5 daughters. Of these: 4 died young, 4 were baptized [Dorothea 1764–1839; married the philosopher **Friedrich Schlegel** in 1804; Henriete 1775–1831; Avraham 1776–1835; Nathan] and 2 remained Jews [Rachel 1767–1831; Joseph 1770–1848]. The converts were baptized during 1814–1815. Abraham had 3 daughters and one son:

- Fanny 1805–1847, grandmother of the mathematician **Kurt Hensel**.
- Felix 1809–1847 (composer).
- Rebeca 1811–1858, married the mathematician **Dirichlet**.
- Ottilie 1819–1848, married the mathematician **Kummer**. Their daughter Marie-Elisabeth married the mathematician **Herman Amandus Schwartz**.

most illustrious rabbinic lineage[332]. His early education was cared for by his father and by the local rabbi. The latter, besides teaching him the Bible and Talmud, introduced him to the philosophy of Maimonides. When the rabbi received a call to Berlin (1743) the lad followed him there.

For the next seven years he embarked on a program of self-education, learning German, English, French, Latin, mathematics, philosophy and general history. His life at this period was a struggle against crushing poverty, but his scholarly ambition never relaxed. In 1750 he was appointed by a wealthy silk-merchant as teacher to his children. Mendelssohn soon won the confidence of his benefactor, who made the young student successively his book-keeper and his partner.

No stage director would have dared select an ugly ghetto hunchback as the central character in this drama. *But history dared.* It selected Moses Mendelssohn from the ghetto of Dessau, to reintroduce a knowledge of Judaism to the Christians and sell Christians cultural values to the ghetto dwellers. In a matter of a few years he befriended **Lessing** (then Germany's foremost dramatist and the great liberator of the German mind) and **Immanuel Kant**. His subsequent philosophical works earned him the sobriquet "German Socrates"; his reviews on literature made him the leading German stylist, while his critical essays on art made him the founder of modern aesthetic criticism.

He was soon challenged publicly to quit straddling the religions issue and either refute Christianity or be baptized (1781). In wrestling with his conscience, Mendelssohn became reinfected with the spirit of Judaism. From then on he dedicated the rest of his life to the emancipation of the Jews. To this end he saw his task as twofold: first, to give the Jews a tool for their own emancipation; second, to prepare a new basis for Judaic values once the old religious norms were rejected.

The German language was to be the tool whereby the Jews would lift themselves out of the ghetto. It was with this in mind that Mendelssohn translated the Pentateuch into lucid German, written in Hebrew letters (1783). His book *Jerusalem* (1793) was a forcible plea for freedom of conscience and noninterference of the state with the religion of its citizens.

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[332] Moses' grandfather, Saul Whal of Dessau, was the 6^{th} generation of Rabbi Meir Katzenellenbogen, known as the MAHARAM OF PADUA (1482–1565). **Karl Marx** was the 12^{th} generation of this very ancestor along another route.

Mendelssohn had 10 children: 4 died young, 4 were baptized and 2 assimilated. All his grandchildren but one were apostates, and the last Jewish Mendelssohn died in 1871, thus bringing this rabbinic line into final extinction.

Whether his influence was for good or evil in the next generation has been a subject of much debate. But the real difficulty was not at all of his doing. It was due to the fact that the Jews caught his spirit of eagerness to re-enter European society much more quickly than the Christians were willing to permit them to enter.

1781–1800 CE **Frederick William (Friedrich Wilhelm) Herschel** (1738–1822, England). One of the greatest observational astronomers in history and founder of the present day system of stellar astronomy. Made a series of astronomical discoveries that established the universality of the law of gravitation — its not being confined to the solar system alone.

In 1781 he discovered the planet *Uranus* and the phenomenon of *binary stars*. In 1782 he discovered that *the entire solar system is moving* relative to the fixed stars. Finally in 1783 cosmology received an enormous boost when Herschel observed diffuse patches of light, or *nebulae*, through his telescope. He considered them to be 'island universes'.

Thomas Wright and **Immanuel Kant** had previously speculated about such nebulae, but Herschel's observations established extragalactic astronomy as an independent branch of astronomy. He realized that the 'Milky Way' might be similar in structure and scale to other faint nebulae. He was also able to resolve the globular star clusters in our own galaxy into stars.

In so doing Herschel took a major step toward placing the earth in its proper perspective with respect to the rest of the universe.

In 1800, he discovered infrared radiation by moving a thermometer along the color spectrum produced by a prism.[333]

Herschel was born in Hanover. His father, Isaac, was a Jewish musician employed in the Hanover guard. His grandfather's family had left Moravia for Saxony in the early part of the 17^{th} century on account of religious persecutions. He started his career as an oboist in the Prussian army, but the hardships of the Seven Years War caused his parents to send him to England, where he became organist and teacher of music.

During 1766–1772 he was director of all public musical entertainment at Bath, and in his free time taught himself mathematics and astronomy. Moreover, in 1772 he brought his sister **Caroline Lucretia** (1750–1848) to England and both built, after toiling at it for few years, a 7 ft. Newtonian reflecting

[333] Although William Herschel is best remembered for his hand-built telescopes and his discovery of Uranus (1781), the simple experiments he performed with glass prisms and thermometer (1800), with which he detected what is now known as *infrared light*, were far more momentous: They gave science its first evidence that an entire world lay hidden beyond the limit of our visual perceptions.

telescope — having an aperture of $6\frac{1}{2}$ inch (16.25 cm). His observations were communicated by him to the Royal Society in a series of memoirs from 1781 to 1797. In 1782 he accepted the offer of King George III to become his private astronomer.

In a series of papers from 1784 up until 1818 (when he was 80 years of age) he demonstrated that our sun is an ordinary star of the Milky Way, and that all the stars visible to us lie more or less in clusters scattered throughout a comparatively thin, but immensely extended disc. In 1789 he finished the construction of his large 4 ft (122 cm) aperture reflecting telescope, through which he could observe the Saturnian system with its 7 satellites, two of which he discovered himself (*Enceladus* and *Mimas*). The 8^{th}, *Hyperion*, escaped his notice.

His son **John Frederick William** (1792–1871) continued his father's studies on double stars and nebulae and contributed further to the knowledge of the *Milky Way*. His sons: **Alexander Stewart** (1836–1907) and **John** (1837–1921) were also astronomers.

1781–1828 CE **Caroline Lucretia Herschel** (1750–1848, England). Astronomer. She was born at Hanover, Germany, while this territory was still part of the British crown. Her father, Isaac Herschel, a musician in the Hanoverian guard, encouraged the development of her musical talents and she learned to play the violin competently enough to perform in concerts.

After her father's death, she was brought by her brother William to England to keep his house for him. During her stay at Bath she established herself as a popular vocalist and also an as assistant to her brother in his astronomical observations. When William became the Astronomer Royal, Caroline became his official assistant at a stipend of 50 pounds annually. Never before or since has any government purchased such a dedicated servant for such a relatively low cost of hire.

To this end she taught herself mathematics and took care of all the laborious numerical calculations and reductions, all the record keeping and the other tedious minutiae. She also did her own observations and during 1783–1797 she discovered 8 comets, 3 nebulae and issued a comprehensive star catalogue. In 1828 she completed the cataloging of 1500 nebulae. For this immense and valuable labor, the Royal Astronomical Society presented her with a gold medal and in 1835 elected her an honorary member of the society.

Synthetic vs. Analytic Geometry

Projective geometry investigates those properties of geometrical figures that are unaltered by projection. The impetus for these investigations was provided by the study of perspective in painting and architecture.

The first beginnings of this synthetic approach *(in contradistinction to the* analytic geometry *of Fermat and Descartes) are to be found in the work of* ***Pappos*** *(ca 300* CE*) who introduced the cross-ratio, referring to the lost work of* ***Apollonios of Perga*** *(262–200 BCE).*

The first projective geometer of modern times is ***Girard Desargues*** *(1593–1662). In a highly original treatise on conic sections (1639), he went beyond the Greek geometers and presented a systematic foundation to projective geometry, and in addition — a number of beautiful theorems unknown to Apollonios.*

Following the development of descriptive geometry, principally by ***Gaspard Monge*** *(1746–1818), the first outline of projective geometry was given by* ***Victor Poncelet*** *(1788–1867). Analytical methods in projective geometry were introduced mainly by* ***August Ferdinand Möbius*** *(1790–1868) and* ***Julius Plücker*** *(1801–1868), while* ***Jacob Steiner*** *(1796–1863) and* ***Christian Von Staudt*** *(1798–1867) perfected a development of projective geometry without these methods.*

The connection between projective and Euclidean geometry was clarified by ***Felix Klein*** *(1849–1925). He also introduced the idea of a geometry as the invariant theory of certain groups of mappings.*

The essence of analytical geometry of space consists in setting up a correspondence between the points of the space and real numbers: Curves (1-dimensional manifolds) and surfaces (2-dimensional manifolds) then correspond to solution of sets of equations, and geometrical constructions can be replaced by algebraic and analytic methods. Since these methods form the basis of analytic geometry, the subject did not arise until progress was made in algebra and analysis.

1781 CE **Gaspard Monge** (1746–1818, France). Mathematician. The inventor of descriptive geometry and the father of differential geometry of space curves and surfaces.

Monge was born at Beaune. He started his career as a teacher at the military school in Nézieres (on the Meuse in N. France), where he discovered a clever representation of 3-dimensional objects by appropriate projections on a 2-dimensional plane. His method was adopted by the military and classified as top-secret(!) It later became widely taught as *descriptive geometry*.

In 1778 Monge married Mme Horbon, a young widow whom he had previously defended in a very spirited manner from an unfounded charge, and in 1780 he was appointed to a chair of hydraulics at the Lyceum in Paris. Unlike the three L's (**Lagrange**, **Laplace**, **Legendre**) who remained aloof from the French Revolution, Monge was an active Jacobine and occupied leading scientific positions.

As temporary head of the government on the day of the King's execution, he incurred lasting royalist resentment as the *chief regicide*. He served as a Minister of Marine and engaged in the manufacture of arms and gunpowder for the army.

After 1795 he was the principal organizer of the Polytechnical School in Paris [the prototype of all technical institutes in Europe and the U.S., even West Point] and became a professor of mathematics there.

Monge gained the close friendship and admiration of Napoleon and accompanied the latter, along with **J. Fourier** on the Egyptian expedition in 1798. Monge was a great teacher and his lectures in algebraic and differential geometry inspired many young men. Among them were: **E.L. Malus** (1775–1812), **J. Dupin** (1784–1873, geometry of surfaces), **J.V. Poncelet** (1788–1867, projective geometry), **A.L. Cauchy** (1789–1857), **O. Rodrigues** (1794–1851), **A.J.C. Barré de Saint-Venant** (1796–1886, theory of curves through his work in elasticity), **M.A. Lancret** and **J.B. Meusnier**, all of whom have theorems in differential geometry named after them. Others, whose principal papers in differential geometry were written in the period 1840–1850, are: **F. Frenet** (1816–1888), **J.A. Seret** (1819–1885), **V. Puiseax** (1820–1883) and **J. Bertrand** (1822–1900).

The school of Monge contributed greatly to the geometry of surfaces, introducing the concept of *developable surfaces*[334].

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[334] A surface that may be unrolled or developed *into a plane* without stretching or tearing, e.g.: cylinder, cone. In order to find geodesics on such surfaces, we "unwrap" the surface, flattening it to a plane, draw the relevant straight lines in the plane, and then wrap the plane up again. Using this idea, it is not too

Monge himself contributed to differential geometry in the topics of space evolutes and lines of curvature on 3-dimensional surfaces.

In 1816 he was discharged as director of the Polytechnical School after the fall of the Emperor. He was also purged with his friend, the geometer **Lazare Carnot** (1753–1823) from the Académia, while **Cauchy** was assigned to it by royal decree — although he had not been elected by the members of the Académia. This created an enormous scandal in scientific circles and Cauchy became very unpopular. [The tables were turned in 1830: Louis-Philippe came to power and Cauchy, as a loyal Burbon, refused to swear allegiance to the new government. He then went into voluntary exile.] Monge died soon afterwards.

1781–1793 CE **Jean Pierre Francois Blanchard** (1753–1809, France). Aviation pioneer and inventor. Proposed heavier-than-air machines in 1781. But as soon as the Montgolfiers made successful balloon flights, Blanchard became an ardent balloonist; he made his first balloon ascent in England (1784) and on Jan. 07, 1785 made the first aerial crossing of the English channel with Dr. **John Jeffries** (1745–1819, U.S.A.), an American physician. Invented the *parachute* and survived the first jump (1784). In 1793 he made the first balloon ascent over North America (Philadelphia, 1793). He was born in Les Audelys, France.

1783 CE The brothers **Jacque Étienne** (1745–1799) and **Joseph Michel** (1740–1810) **Montgolfier** (France) invented and built the first balloon to carry men into the air. The balloon was made of cloth and paper, filled with *hot air*. On its first public trial (June 05, at Annonay, France), their balloon (unmanned) rose about 1800 meters. Five months later (Nov 21), two men: **Francois Pilatre de Rozier** and **Marquis d'Arlandes** rose to height of 24 meters and flew across Paris for 25 minutes in a Montgolfier balloon – the first human beings to fly. Ten days later, the physicist **Jacques Charles** and a member of his team made the first *hydrogen balloon* flight.

1783 CE, June 08 The eruption of the *Laki volcano* in Iceland, started. Fluid basalt lavas flooded out of the fissure for a period of 2 months, spreading out over tens of square kilometers. Large volumes of sulphurous fumes were emitted throughout the eruption, forming a ground-hugging layer extending many kilometers down-wind from the fissure. A heavy fall of ash also rained

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difficult to show that *geodesics* on a cylinder or a cone are curves that make constant angle with the elements of the cylinder or cone. In the case of the cylinder, the geodesics are *helices*.

A *sphere*, for example, is not developable. It can be shown that the *Gaussian curvature* of a developable surface is zero.

on the countryside. 10,000 people, about $\frac{1}{5}$ of Iceland's population in the 18^{th} century, died of the eruption. Most died a lingering death of the resulting environmental damage: ash-fall destroyed growing crops and carpeted grazing lands, so that cattle either died of starvation, or were forced to eat ash-covered grass. The loss of livestock produced a severe famine which resulted in starvation. The Laki event was the largest eruption of historic times in terms of the volume of material erupted.

Benjamin Franklin, while serving in Paris in 1783 as the first diplomatic representative of the newly-formed United States of America, related the severe Northern Hemisphere winter of 1783–1784 to the Laki eruption and speculated that the injection of ash, dust and gases from the volcano into the atmosphere could result in lower temperatures by screening out some of the solar radiation.

1783–1790 CE Advent of the *steam boat*; Thomas Newcomen's first practical *steam-engine*, employing piston and cylinder, was first used to power a 45 m paddle-boat by the **Marquis de Jouffroy** (France) in 1783. Two years earlier, **James Watt** patented a way to change the power produced by a steam engine from a back-and-forth motion to rotary motion. This was used by **John Fitch** (England) in 1787 to successfully test his steam boat on the Delaware river. By 1790, one of Fitch's steamboats was in regular service for several weeks during the summer, but it was a commercial failure.

1783–1801 CE **Jacques-Alexandre-César Charles** (1746–1823, France). Physicist and mathematician. First, in 1783, to employ hydrogen for the inflation of balloons (he made the first ascent in that year to an altitude of 3.2 km). In 1787 he discovered (ahead of **Joseph Gay-Lussac**, 1802) that a gas expands, under constant pressure, such that its volume is proportional to the absolute temperature (*Charles law*)[335].

Charles was born at Beaugency, Loiret. After spending some years as a clerk in the ministry of finance, he turned to scientific pursuits, and attracted

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335 Charles did not publish his findings, but explained his experiments to the French chemist **Joseph Louis Gay-Lussac** (1778–1850). The latter performed similar experiments and published his results in 1802.

As a result, Charles' law is sometimes called Gay-Lussac's law. The *ideal gas law* $PV = nRT$ combines Boyle's law $[(PV)_T = \text{constant}]$, Charles' law $[(V/T)_P = \text{constant}]$, and Avogadro's law into a single statement. [P = pressure; V = volume; T = absolute temperature in degrees Kelvin (K); n = number of moles of gas; R = universal gas constant.] For one mole ($n = 1$), $R = \frac{PV}{T}$.

Taking a gas at $p = 1$ atm, $T = 0°\text{C} = 273.15°\text{K}$ and molar volume of 22.4 liter, one obtains $R = 8.3144\ \frac{\text{J}}{\text{mole}\times°\text{K}}$. In general $PV = nRT = NkT$, $N = nN_A$ = total number of molecules in the sample, N_A = Avogadro's

considerable attention by his skillful and elaborate demonstrations of physical experiments. In 1785 he was elected to the Academy of Sciences, and subsequently became professor of physics at the Conservatoire des Arts et Metiers.

1784 CE **George Atwood** (1746–1807, England). Applied mathematician. Graduated from Trinity College, Cambridge in 1769 and remained there until 1784 as a fellow and tutor. In 1776 he was elected a fellow of the Royal Society of London. In 1784 he was appointed to the office of a patent searcher of the customs. In the same year he published his work "*Treatise on the Rectilinear Motion and Rotation of Bodies*" in which he invented a machine for the demonstration of the laws of free fall (*Atwood machine*)[336]. With this machine he was able to improve the accuracy of the measurement of the acceleration of a body in free fall.

1784–1794 CE **William Jones** (1746–1794, England). Orientalist, linguist and jurist. Recognized and demonstrated that six groups of kindered languages (known today as Indo-European) — Sanskrit, Greek, Latin, Gothic, Celtic and Persian originated from a *common source* (proto Indian-European) which no longer exists[337].

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number, k = Boltzmann's constant. Therefore

$$k = \frac{R}{N_A} = \frac{8.3144\ \text{J/mole} \times {}^\circ\text{K}}{6.02205 \times 10^{23}/\text{mole}} = 1.38066 \times 10^{-23}\ \frac{\text{J}}{{}^\circ\text{K}}.$$

336 The apparatus "dilutes" the effect of gravity so that acceleration could be accurately measured to determine the value of g. It consists of a light frictionless pulley on which two nearly equal masses $m_2 > m_1$, are hung vertically. Neglecting the mass and rotational effect of the pulley, the energy conservation can be expressed as:

$$(m_2 - m_1)gs = \frac{1}{2}(m_1 + m_2)v^2,$$

where s is the vertical displacement of the blocks and v their linear velocity. But since $v^2 = 2as$, the acceleration a of the system is

$$a = \frac{m_2 - m_1}{m_2 + m_1}g.$$

337 Two hundred years earlier, an Italian, **Filippo Sassetti**, had already noticed the similarity between Sanskrit and Italian. Sassetti lived in India (1581–1588).

Jones was born in London and distinguished himself at Harrow and Oxford in the study of Oriental languages. By 1766 he mastered Arabic, Hebrew, Persian, Chinese, French, Italian, Spanish and Portuguese. During 1768–1774 he occupied himself with translations from Asiatic to European languages. To enhance his income he studied law and gained high reputation in this field both in England and America. In 1783 he was appointed judge of the supreme court at Calcutta. In this capacity (1783–1794) he compiled a digest of Hindu and Muhammadan law and translated from the ancient Hindu literature into English.

An extraordinary linguist knowing 13 languages well, and having a moderate acquaintance with 28 others, his range of knowledge was enormous. As a pioneer in Sanskrit learning he rendered the language and literature of the ancient Hindus accessible to European scholars, and thus became the indirect cause of later achievements in the field of Sanskrit and comparative philology.

1784–1809 CE **René Just Haüy** (1743–1822, France). Mineralogist and the founder of the science of crystallography. Elucidated geometrical properties of various crystals and laid theoretical basis for further work in his *Traité de mineralogie* (1801) and *Traité de cristallographie* (1822). Also studied pyroelectricity.

Haüy was born at St. Just, Oise. He studied at the college of Navarre and afterwards at that of Lemoine. Becoming one of the teachers (Abbé Haüy) at the latter (1770 to 1784), he began to devote his leisure hours to the study of botany; but an accident directed his attention to another field of natural history: while looking at a particular collection of minerals, he supposedly dropped a group of calcite crystals that crystallized as hexagonal prisms. As he went down to examine the shattered fragments, he found they were all perfect rhombohedra, in every detail the identical shape of Iceland spar, a different crystalline form of calcite (= calcium carbonate, or limestone in one form).

Thus he found that all crystals of calcite, whatever their external form, could be reduced by cleavage to a rhombohedron with interfacial angle of 75°. Further, by stacking together a number of small rhombohedra of uniform size he was able to reconstruct the various forms of calcite crystals[338].

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338 In the same manner a regular octahedron is built of cubic elements, such as given by the cleavage of rock-salt. By making the steps one, two or three bricks in width and one, two or three bricks in height the various secondary faces on the crystal are related to the primitive form or "cleavage nucleus" by the law of whole numbers, and the angles between them can be arrived at by mathematical calculation. By measuring, with a goniometer, the inclination of the secondary faces to those of the primitive form, Haüy found that the secondary forms are

When the revolution broke out he was thrown into prison (1792), and his life was even in danger, when he was saved by the intercession of the naturalist **Étienne Geoffray Saint-Hilaire**[339] (1772–1844, France). He later taught at École des Mines (1795 to 1802). In 1802, under Napoleon, he became a professor of mineralogy at the museum of natural history and the Sorbonne (1809), but after 1814 he was deprived of his appointment by the government of the Restoration. His latter days were consequently clouded by poverty, though he lived cheerful and respected till his death in Paris.

1784–1809 CE **Adrien Marie Legendre** (1752–1833, France). An outstanding mathematician whose works have placed him at the forefront of achievement in widely distinct fields of pure and applied mathematics. He had the misfortune of seeing most of his best work — in elliptic integrals, number theory and the method of least squares — superseded by the achievements of younger and abler men.

For 40 years he slaved over *elliptic integrals* (his 2-volume treatise appeared in 1827) without noticing what both **Abel** and **Jacobi** saw almost at once (1828) that by considering the *inverse functions* the whole subject drastically simplified.

The readiness with which Legendre, who was then 76 years of age, welcomed these important researches, that quite overshadowed his own, and included them in successive supplements to his work, eloquently testify to his integrity.

Legendre was born in Paris. In 1775 he was appointed professor of mathematics in the École Normale. In 1783 he was elected a member of the French Academy in succession of **J. Le Rond d'Alembert**. During the revolution, he was one of the three members of the council established to introduce the metric system.

In 1767–1769 Legendre used continued fractions to find approximations to the irrational roots of algebraic equations, and approximate solutions of ordinary differential equations.

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always related to the primitive form (on crystals of numerous substances) in the manner indicated, and that the width and the height of a step are always in a simple ratio, rarely exceeding that of 1 : 6. This laid the foundation of the important *law of rational indices* of the faces of crystals.

339 Geoffray was a student of medicine in Paris when Haüy, his former teacher, and other professors of the colleges of Lemoine and Navarre were arrested by the revolutionists as priests. Through the influence of Daubenton, Geoffray obtained an order for the release of Haüy in the name of the Academy.

In 1784, Legendre encountered his polynomials in his research on the gravitational attraction of ellipsoids. He introduced the celebrated expressions, which though frequently called 'Laplace coefficients', are more correctly named after Legendre. The definition of the coefficients is that if $(1 - 2h\cos\phi + h^2)^{-1/2}$ be expanded in ascending powers of h, and if the general term be denoted by $P_n h^n$, then P_n is the Legendre coefficient of the n^{th} order.

In 1805 Legendre issued the first published account of the *method of least squares* in connection with his work on the orbits of comets. It had, however, been applied earlier (1795) by **Gauss**, and was independently used by **Laplace** (1810).

To Legendre is due the theorem known as *law of quadratic reciprocity* in the theory of numbers, the most important general result in the science of numbers which has been discovered since the time of **Fermat** and which was called by Gauss "the gem of arithmetic". The symbol $\left(\frac{a}{p}\right)$ for odd prime p is known as *Legendre's symbol*[340]. *Legendre's formula* $\{x/(\log_e x - 1.08366)\}$ for the approximate number of primes less or equal to a number x, was first given by him in 1801 in his famous treatise '*Théorie des nombres*'.

In 1825 Legendre provided a complete proof, for the case $n = 5$, of the Fermat conjecture.

In 1794 he published a popular textbook on Euclidean geometry called '*Eléments de géométrie*', in which he attempted a pedagogical improvement of Euclid's '*Elements*' by rearranging and simplifying many of the propositions. This book was translated by **Thomas Carlyle** (1795–1881), who early in his life was a teacher of mathematics, and ran through 33 American editions for 100 years.

In one respect, Legendre's life resemble that of Lagrange: in 1792, at the age of 40, he married a girl 22 years his junior. His young wife helped him put his affairs in order and also brought the tranquility to his life which greatly aided him in his work.

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340 Equal to zero when p divides a; equal to $+1$, when p is prime to a but p divides $N^2 - a$, for some N; equal to -1, when p is prime to a but there is no N such that p divides $N^2 - a$.

The Remarkable Legendre

Legendre' works have placed him at the very forefront in the widely distinct subjects of elliptic functions, theory of numbers, potential theory, geodesy and analysis. His versatility is demonstrated in the following relations discovered by him.

I. Elliptic Integrals (1811)

The relation

$$EK' + E'K - KK' = \frac{\pi}{2}$$

relates the complete elliptic integrals

$$\begin{aligned} K(k) &= \int_0^{\pi/2} d\theta (1 - k^2 \sin^2 \theta)^{-1/2}, \qquad\qquad k^2 + {k'}^2 = 1 \\ K'(k) &= \int_0^{\pi/2} d\theta (1 - {k'}^2 \sin^2 \theta)^{-1/2} \equiv K(k') \\ E(k) &= \int_0^{\pi/2} d\theta (1 - k^2 \sin^2 \theta)^{1/2} \\ E'(k) &= \int_0^{\pi/2} d\theta (1 - {k'}^2 \sin^2 \theta)^{1/2}. \end{aligned}$$

It is known as the Legendre relation. To prove it, one must show that

$$\frac{dE}{dk} = \frac{1}{k}(E - K), \qquad \frac{dK}{dk} = \frac{1}{k{k'}^2}(E - {k'}^2 K),$$

$$\frac{dE'}{dk} = \frac{k}{{k'}^2}(K' - E'), \qquad \frac{dK'}{dk} = \frac{1}{k{k'}^2}(k^2 K' - E').$$

It then follows that $\frac{d}{dk}(EK' + E'K - KK') = 0$. *To find the value of the constant, we let* $k \to 0$.

With the aid of the Legendre relation and Gauss' arithmetic-geometric mean, **Eugene Salamin** *discovered (1976) a new formula that is currently used as a computer algorithm for the fast computation of* π *with the property of doubling the number of digits at each step.*
Thus, **Y. Tamura** *and* **Y. Kanada** *computed* π *in 1982 to 4,194,293 digits with a* CPU *time of 2 hours and 53 minutes!!.*

II. Prime Factorization of $m!$

Legendre enriched mathematics, both pure and applied, with many important and beautiful results. But one can recognize the signature of the master even in one of his lesser known theorems.

Suppose we wish to find how many zeros there are at the end of the number

$$1000! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot 999 \cdot 1000$$

Clearly, the number of terminal zeros of a number depends on how often the factor $10 = 2 \cdot 5$ *occurs in its factorization. We must therefore find the exponents of the factors* 2 *and* 5 *in the prime factorization of* $1000!$. *The smaller of these exponents will yield the largest exponents, say* α, *for which* $10^\alpha = (2 \cdot 5)^\alpha$ *divides* $1000!$, *and* α *will be the number of terminal zeros in* $1000!$.

Now, every fifth one of the numbers

$$1,\ 2,\ 3,\ \ldots,\ 1000$$

is a multiple of 5. *Since*

$$1000 = 5 \cdot 200 + 0,$$

there are 200 *factors in* $1000!$ *which are divisible by* 5. *Of these* 200, *i.e., of*

$$5,\ 10,\ 15,\ 20,\ 25,\ \ldots,\ 1000,$$

every fifth is a multiple of 5^2. *Since*

$$200 = 5 \cdot 40 + 0,$$

there are 40 factors divisible by 5^2. *Moreover, since*

$$40 = 5 \cdot 8 + 0, \qquad 8 = 5 \cdot 1 + 3, \qquad 1 = 5 \cdot 0 + 1,$$

there are 8 *numbers divisible by* 5^3, *one by* 5^4, *and none by any higher power of* 5.

Thus, the prime factorization of $1000!$ *contains* $5^{200} \cdot 5^{40} \cdot 5^8 \cdot 5^1$; *the exponent of* 5 *in the prime factorization is*

$$200 + 40 + 8 + 1 = 249.$$

On the other hand, the prime 2 *occurs to a higher power in the prime factorization of* $1000!$, *since* 500 *of the factors are even,* 250 *are divisible by* 2^2, *etc. Hence there are* 249 *zeros at the end of* $1000!$.

Legendre attacked the general problem of the exponents in the prime factorization of $m!$. He asked: What is the highest power, say p^α, of a prime p such that p^α divides $m!$, where m is any given natural number? In order to answer this question he generalized the procedure used in the above solution for the case $m = 1000,\ p = 5$.

First, divide the given number m by the prime p:

$$m = pq_1 + r_1 \qquad (0 \le r_1 < p);$$

Next, divide the quotient q_1 by p:

$$q_1 = pq_2 + r_2 \qquad (0 \le r_2 < p).$$

One continues this process until one obtains a quotient which is zero, i.e.,

$$q_{k-1} < p, \qquad q_k = 0.$$

By examining the solution of the problem, it is seen that the exponent α (which was 249 in the case $m = 1000,\ p = 5$) is the sum of the quotients obtained in the above algorithm:

$$\alpha = q_1 + q_2 + \cdots + q_{k-1}.$$

It is now claimed that the remainders $r_1, r_2, \ldots, r_k$ obtained in this algorithm are the digits in the representation of the number m to the base p. To see this, just substitute repeatedly for the quotients; thus,

$$m = pq_1 + r_1 = p(pq_2 + r_2) + r_1 = \cdots = p^{k-1}r_k + p^{k-2}r_{k-1} + \cdots + pr_2 + r_1.$$

The last expression proves the claim. In the solution of the problem, the remainders in the divisions were $0, 0, 0, 3, 1$ and the representation of 1000 to the base 5 is, indeed, 13000.

Next it is shown how to express the exponent α in terms of the remainders. One adds the expressions

$$\begin{array}{rcl} m &=& p\,q_1 + r_1 \\ q_1 &=& p\,q_2 + r_2 \\ \vdots && \vdots \qquad \vdots \\ q_{k-1} &=& p \cdot 0 + r_k \end{array}$$

and obtains

$$m + q_1 + q_2 + \cdots + q_{k-1} = p\,(q_1 + q_2 + \cdots + q_{k-1}) + r_1 + r_2 + \cdots + r_k$$

or

$$m + \alpha = p\,\alpha + s,$$

where $s = r_1 + r_2 + \cdots + r_k$. *Solved for* α, *this gives*

$$\alpha = \frac{m-s}{p-1}\ .$$

This proves the following theorem of Legendre:

> *If* m *is a positive number and* p *a prime, then the exponent of* p *in the prime factorization of* $m!$ *is*
>
> $$\frac{m-s}{p-1}$$
>
> *where* s *is the sum of the digits of the representation of* m *to the base* p.

The prime power in a factorial, i.e. the highest power of p *that divides* $m!$ *is alternatively given by the expression*

$$\left[\frac{m}{p}\right] + \left[\frac{m}{p^2}\right] + \left[\frac{m}{p^3}\right] + \cdots$$

which include only finitely many non-zero terms. Here $[A]$ *indicates the greatest integer not exceeding* A.

To see this we write

$$m! = 1 \cdot 2 \cdots (p-1) \cdot p \cdot (p+1) \cdots (2p) \cdots (p-1)p \cdots p^2 \cdots .$$

It is obvious that there are $\left[\frac{m}{p}\right]$ *multiples of* p, $\left[\frac{m}{p^2}\right]$ *multiples of* p^2, *and so on. Similarly, the number*

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

is an integer because the power of p *in* $\binom{n}{r}$ *is*

$$\sum_m \left(\left[\frac{n}{p^m}\right] - \left[\frac{r}{p^m}\right] - \left[\frac{n-r}{p^m}\right] \right) .$$

Since $[A] - [B]$ *is either* $[A-B]$ *or* $[A-B]+1$, *the sum is a non-negative integer.*

Examples:

- *If* $m = 1000$, $p = 3$ *then*

$$\left[\tfrac{1000}{3}\right] = 333; \qquad \left[\tfrac{1000}{3^2}\right] = 111; \qquad \left[\tfrac{1000}{3^3}\right] = 37; \qquad \left[\tfrac{1000}{3^4}\right] = 12;$$

$$\left[\tfrac{1000}{3^5}\right] = 4; \qquad \left[\tfrac{1000}{3^6}\right] = 1$$

Therefore the exact power of 3 *which divides* 1000! *is* 498.

- *Defining* $3!!! = [(3!)!]! = 720!$, *prove that it has more than* 1000 *digits, and find the number of zeros at the end of the expansion.*

Since $(720)! > 99!100^{621} > 10^{1242}$, $3!!!$ *has more than* 1000 *digits. The largest power of* 5 *which divides* $3!!! = 720!$ *is*

$$\left[\frac{720}{5}\right] + \left[\frac{720}{25}\right] + \left[\frac{720}{125}\right] + \left[\frac{720}{625}\right] = 144 + 28 + 5 + 1 = 178,$$

while the largest power of 2 *dividing* 720! *is still greater (since already* $\left[\frac{720}{2}\right] = 360$*). It follows that the number* 3!!! *has* 178 *zeros at the end of its decimal expansion.*

1784–1810 CE **Johann Wolfgang von Goethe** (1749–1832). Poet-philosopher and scientist. The last of the great universal men. In the course of a long life he engaged in a wealth of activities: poet, lawyer, politician, civil servant, physicist, botanist, zoologist, painter, theater manager and literary critic. Yet there is nothing fragmentary about him, and his mature writings are the expression of the harmony he created by conscious effort out of the manifold experiences of a richly varied life.

Goethe made important contributions in *anatomy*, *botany* and *optics*. He tried to introduce an *evolutionary perspective* into every one of these disciplines. He advocated a holistic approach toward science, emphasizing intuition and a concern for the whole rather than a separation into parts (1791).

As a student he studied in Leipzig and Strasbourg where his thinking was strongly influenced by the works of **Bacon**, **Spinoza** and **Kant**.

Goethe discovered the intermaxilary bone, a feature of the human upper jaw that is missing in most other mammals.

In the two-volume *Zur Farbenlehre* (1810), he attacked Newton's theory of light (1704) and presented a psychologically-oriented examination of color. However, Goethe would not recognize the distinction between *physical* and *physiological*[341] optics; this was the reason for his fruitless fight against Newton: reviving the old Aristotelian view, Goethe abhorred the theory that white light is a mixture of the seven colors of the rainbow.

He was certainly correct in regard to the white[342]-*sensation* which he had primarily in mind, but the rainbow should have convinced him that white light is decomposed into colors by a spectral apparatus (in this case, water droplets).

In attempting to explain the metamorphosis of plants (1789) he claimed incorrectly that all plant structures are modified leaves, but clearly espoused *evolution*.

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341 Today we understand without difficulty that the *sensation yellow* which is caused by the D-lines of sodium is a phenomenon which is entirely different from the wavelengths $\lambda = 5890\text{Å}$ and $\lambda = 5896\text{Å}$ by which we must describe these lines physically. For we know that the psychological response to an event is something entirely different from the physical event itself; the two are different in nature, though related.

342 We perceive the sun's natural light as white (i.e., as lacking all spectral colors) because the eye is *adapted* to see sun; that is to say, because our eye and the associated physiological-psychological vision apparatus has in its evolutionary development adapted itself to the spectrum of the sun. If we lived in the vicinity of a *red giant*, we would presumably perceive its red color as the normal white.

Worldview XVI: Johann Wolfgang von Goethe

* * *

"In the inside of Nature — Nature has neither kernel nor shell".

* * *

"The history of science is science itself; the history of the individual, the individual".

* * *

"As for what I may have done as a poet, I take no pride in it whatever... Excellent poets have lived at the same time with me, poets more excellent lived before me, and others will come after me. But that in my country I am the only person who knows the truth in the difficult science of colors — of that, I say, I am rather proud, and here I have a consciousness of superiority to many".

* * *

"In the eternal silence within a crystal they may see the happenings of the world outside".

* * *

"Gray and ashen, my friend, is every science. And only the golden tree of life is green".

* * *

"Duration is change".

* * *

"Man is naturally disposed to consider himself as the center and end of creation, and to regard all the beings that surround him as bound to subserve his personal profit... He cannot imagine that the least blade of grass is not there for him".

* * *

1785 CE *The Times* (London) was founded.

1785–1788 CE **James Hutton** (1726–1797, Scotland). Geologist, chemist, physician and farmer. The father of modern geology. Concluded (1785), on the basis of geological observations, that vast periods of time were required for the earth to have reached its present state and form. He saw the earth as a living machine, immensely old, continuously changing and powerful[343].

Noticing how little the *Hadrian wall* was affected by erosion during the 1600 years of its existence, he concluded that there was simply no time for mountains and valleys to be carved during the meager 6000 years allocated by **Ussher** (1650)[344].

Hutton was born in Edinburgh and educated at the high school and university of his native city. He completed his medical education in Paris and Leyden (1749) but later abandoned the medical profession, and after extensive travels in the Low Countries and France (1750–1754) he settled on his own farm in Berwickshire. In 1768 he established himself in Edinburgh for the rest of his life, living unmarried with his 3 sisters. Surrounded by congenial literary and scientific friends he devoted himself to research.

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[343] Hutton observed the layering of rock in Scotland's provinces, and deduced that the layers of limestone, sandstone, and shales had been laid down in distant times as soft sediments that settled to the bottom of the sea. He then conjectured that these sediments had been compacted and slowly turned to stone by the pressure of sediments settling above them; at last the sea had withdrawn, or the sea bed had risen, exposing some of the rock to the open air. The wearing action of wind and weather (erosion) had broken up the topmost layers of rock into fine bits, helping to make the rich mix we call soil. Rains *continually* wash the soils into streams and rivers and hence to the sea, where the sediment is compacted once again into solid rock. Heat within the earth heaves this bedrock up above the sea to form new mountains. This cycle, Hutton argued, from rock to soil to rock, from sea to air and again to sea, had endured for an extraordinary length of time, and will go on indefinitely. In his own words: "*We find no vestige of beginning — no prospect of an end*". Hutton did not venture to estimate the time-scale involved in these processes. His ideas were further elaborated by **Wegener** (1912).

[344] Ussher's age for the earth proves much too small for a totally different reason: we know the Sumerians invented writing in ca 3000 BCE. We cannot expect that in 900 years, a civilization complex enough to require a writing system can be developed.

Edinburgh was at that time the center of Scottish Enlightenment, which included **James Watt**, **Adam Smith**, **David Hume**, **John Clerk** (1728–1812) and **Joseph Black**. They met one evening a week at the *Oyster Club*, which Hutton helped to found. These men brought into the world the engines of the *Industrial Revolution* and some of the ideas and attitudes that made the revolution possible.

Hutton communicated his views in 1785 to the recently established *Royal Society of Edinburgh*, in a paper entitled *Theory of the Earth*.

1785–1787 CE **Edmund Cartwright** (1743–1823, England). Inventor and clergyman. Developed a steam-powered loom for wearing cotton that led to the invention of more effective power looms and to the development of modern wearing industry.

Cartwright was born at Marnham, in Nottinghamshire. He studied literature at Oxford, but had no scientific education. He became pastor of a rural parish in Goadby Marwood in Leicestershire. In 1784 Cartwright learned of the need for a wearing machine that could make cloth faster than the hand loom. Even though he had never seen a loom in operation, he hired a carpenter and a blacksmith to help him build a power loom. In 1787 he used it in a spinning and wearing factory he opened at Doncaster.

In 1791, a mill at Manchester ordered 400 of Cartwright's looms. But the factory was burned down by workmen who feared the new power machinery would eliminate their jobs. Although Cartwright's looms were never fully practical, Parliament recognized his pioneering work (1809) by awarding him the equivalent of $ 50,000.

1785–1803 CE **Claude Louis Berthollet** (1748–1822, France). Chemist. Made the first systematic attempt to grapple with the problems of *chemical physics* (the physics of chemistry), such as chemical affinity and rate of chemical reactions.

Berthollet was born at Talloire, near Annecy in Savoy. He graduated in medicine at Turin. In 1722 he moved to Paris and became the private physician of Phillip, duke of Orleans. In 1785 he declared himself an adherent of the Lavoisierian school, and in 1787 participated in the revision of chemical nomenclature with **Lavoisier** himself. He determined the composition of ammonia (NH_3) and hydrogen sulphide (H_2S), and introduced the process of chemical *bleaching*[345] (1785). After 1794 he became a professor of chemistry at the École Polythechnique. He accompanied Napoleon to Egypt in 1798.

[345] Berthollet found that a solution of chlorine in water, when exposed to *light*, gave of bubbles of *oxygen*, which causes the bleaching action

$$Cl_2 + H_2O \rightleftarrows HCl + HOCl \rightleftarrows H + Cl + HOCl;$$

During 1799–1803 he determined the factors influencing chemical reaction: *affinity* (tendency to combine) and *concentration*. In 1803 Berthollet anticipated the *law of mass-action*[346] which was formulated in a general form by **Guldberg** and **Waage** (1864 to 1867).

Under the empire Berthollet became a Count, and after the restoration of the Bourbons he was made a peer. In the later years of his life he had established, at Arcueil, a well-equipped laboratory which became a center frequented by some of the most distinguished scientific men of the time.

1785–1802 CE **William Paley** (1743–1805, England). Philosopher and churchman. Author of "*The Principles of Moral and Political Philosophy*" (1785) and "*Natural Theology*" (1802).

Argued that living things are far too complicated to have arisen by chance, and that the existence of creatures as beautifully fitted for their way of life as ourselves, reveal the presence of a designer. Paley's arguments come both from biology and astronomy: he realized, for example, that the inverse-square law of the gravitational attraction force is unique in giving rise to stable orbits. If the law of gravity had, for example, been an inverse *cube*, then planetary orbits would be unstable, and a planet that moved a little closer to the sun would immediately begin to fall inwards permanently, while one that moved slightly outwards in its orbit would continue receding forevermore. Tiny changes, such as those caused by the impact of a meteorite, would be disastrous. In our universe, if the earth's orbit, say, shifts slightly inwards or outwards because it is hit by a piece of rock from space, the natural tendency is for the planet to return close to its old, regular path. Paley saw this

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$$HOCl \rightleftarrows HCl + O.$$

The photon energy decomposes the chlorine molecule into two chlorine atoms, and the oxygen, through the oxidizing OH group, helps to convert pigments and strains from cotton goods.

346 A *chemical reaction*, e.g. $A + BC \rightarrow AB + C$, is countered by the inverse reaction: $AB + C \rightarrow A + BC$ which can often take place under the *same* conditions and simultaneously with the direct reaction. A state of *equilibrium* is then reached when the two opposing reactions balance each other, i.e., proceed with equal rates. This is denoted $A+BC \rightleftarrows AB+C$ [example: reducing steam by heated iron $3Fe+4H_2O \rightleftarrows Fe_3O_4+4H_2$]. Such reactions are called *reversible reactions*.

According to Berthollet, the *activity* of a substance is proportional to the affinity and the concentration. The product of these he called active *mass*. His *law of mass-action* then states: In the equilibrium state, the extent of chemical change is proportional to the active masses of the interacting substances.

"choice" of the universe square law of gravity as another example of the work of intelligent design in a universe suitable for human life. He did not elaborate, however, on the fact that the inverse square law is a byproduct of the fact that the universe has three spatial dimensions — although this had been noticed by Immanuel Kant earlier in the eighteenth century.

Paley's arguments go back to authors such as **John Ray**, and have had a long intellectual history, surviving to the present day in the critique of Darwinian evolution. Yet Charles Darwin, while himself a student at Christ's College of Cambridge University, not only had to read Paley, but was deeply impressed with Paley's arguments. Even though Paley's concept of God as a designer is very different from Darwin's theory of natural selection, Darwin took from his reading of Paley a belief in adaptation — that organisms are somehow fit for the environments in which they live, that their structure reflects the functions they perform throughout their lives.

Where natural theology ran into trouble was in explaining the many cases of apparent pain, waste, and cruelty in the living world: why would a benevolent Designer have made cats play with mice before killing them, or parasites that eat their hosts from the inside?

Paley struggled to reconcile the apparent cruelty and indifference of nature with his belief in a good God, and finally concluded that the joys of life simply outweighed its sorrows. Where Darwin departed from Paley was in his concept of natural selection as a process that could produce adaptation and design without the all-encompassing intervention of a benevolent Designer.

Paley was born in Peterborough and graduated from Christ's College, Cambridge (1763). In 1782 he became Archdeacon of Carlisle.

1785–1794 CE **Antoine-Nicolas Caritat Marquis de Condorcet** (1743–1794, France). Mathematician. Tried to apply probability theory to situations of human judgment, such as the probability of election of a candidate by a given number of voters or the probability of a tribunal arriving at a true verdict in a trial if to each juror a number can be assigned that measures the chances he will speak or understand the truth. This *probabilité des judgments*, with its overtones of the Enlightenment philosophy, was prominent in the work of Condorcet[347]. Though an advocate of the French Revolution (and a believer in the necessary progress of the human race towards happiness and perfection), he himself became ironically, a victim of the revolutionary tribunal.

[347] His *Voteŕs Paradox* (1785) plays a central role in both the theory of group choice (voting) and the theory of group preference (utility aggregation), finding application in modern *game theory*

1787–1794 CE **Nicolas Leblanc** (1742–1806, France). Chemist and Surgeon. Father of modern industrial chemistry. Invented the first industrial chemical process[348] to be worked on a really large scale.

The great expansion to textile manufacturers in Britain which began with the industrial revolution, together with the expansion of glass-making and soap manufacture, greatly increased the demand for alkali, and so great a strain was put on natural resources, that before long the synthesis of alkali became essential. In France the general shortage of alkali has been made acute by the difficulties arising from the wars, and in 1775 the Academy offered a prize of 2400 livres for the satisfactory method of making soda from salt.

Leblanc, then a physician to the Duke of Orleans, took the challenge and established his process in a works by means of a loan from the Duke (1791). He opened factories at St. Denis, Rouen, and Lille, but reaped no lasting benefit from what was to remain for a century one of the most fundamentally important of all industrial processes. In 1793 the Duke was guillotined by the friends of liberty and fraternity, and Leblanc's factory was confiscated and dismantled. Moreover, Leblanc was forced to reveal the secrets of his process.

By way of compensation for the loss of his rights, the works were handed back to him in 1800, but all his efforts to obtain money enough to restore them and resume manufacturing were vain. Worn out with disappointment, the unfortunate inventor died by his own hand in a shelter for the poor in St. Denis, near Paris and buried in an unknown pauper's grave.

Four years after his death, Michel Jean Jacques Dizê published a paper in the *Journal de Physique* claiming that it was he himself who had first suggested the addition of chalk; but a committee of the French Academy came to the conclusion that the merit was entirely Leblanc's (1856).

Although Leblanc's process was announced in 1787, it was not worked in Britain until nearly 40 years later, an important factor in the delay being the excise duty on salt (1702–1823).

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[348] *The* '*Lablanc process*': the preparation of Soda (Na_2CO_3) from common salt. In this process, sulphuric acid made from pyrites is heated with salt: $2NaCl+H_2SO_4 = Na_2SO_4+2HCl$, the hydrochlorin acid being absorbed and converted into chlorine, used in the manufacturing of bleaching powder. The 'salt-cake' (Na_2SO_4) is now heated with carbon in the presence of limestone (chalk, $CaCO_3$). The reaction occurs in two stages, sodium sulphate being first reduced to sulphide: $Na_2SO_4+2C = Na_2S+2CO_2$, and the sulphide then reacting with chalk to form a mixture of sodium carbonate and calcium sulphide, together with unchanged carbon and impurities, called *black ash*: $Na_2S+CaCO_3 = Na_2CO_3+CaS$.

1788 CE **Charles Bladgen** (1748–1820, England). Scientist and physician. Discovered that the lowering of the freezing point of a solvent by a substance in dilute solution is proportional to the concentration of the solute.

Blagden was born in Gloucestershire. Assistant of **Cavendish**. Visited **Lavoisier** (1783) and told him of Cavendishe's experiments. Established the importance of sweating in maintaining constant body temperature of animals.

1789 CE **Jeremy Bentham**[349] (1748–1832, England). Social philosopher. Remembered as a legal and political critic and reformer.

Founded the philosophy known as *Utilitarianism*. It defines virtue in terms of utility (the enhancement of the happiness of many, expressed in the formula, "*the greatest happiness of the greatest number*", as the proper goal of society, and the function of a good government. Utilitarians advocated the intellectual and social independence of individuals, defended civil liberties, and declared their belief in democratic ideals. All such values, however, were regarded merely as steps toward the fundamental goal of universal happiness, *not as absolute truths*.

Bentham avowed his faith in the democratic rights of man, but he considered them to be only fictions necessary for the successful conduct of life. He proposed to organize a country's laws and institutions so that they placed the general good above each person's pleasure, effecting harmony between public and private interests. His criticisms brought about many needed *reforms*.

Bentham dabbled in various scientific and intellectual pursuits; for example, he propounded schemes for cutting canals through the isthmus of *Suez* and the isthmus of *Panama*.

As a teacher of the principles of legislation, Bentham inquires of all institutions whether their utility justifies their existence. His writings have been and remain a storehouse of instruction for statesmen and legal reformers, and the great legal revolution (1873) which in England accomplished the fusion of law and equity can be traced to him.

Bentham identified the useful and the good. This outlook influenced English thought, which took its spirit from a life of industry and trade, and looked up to *matters of fact* with a certain reverence.

The Baconian tradition had turned *thought* in the direction of *things*, *mind* is the direction of *matter*; The *materialism* of Hobbes, the *sensationalism*

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349 The stuffed and clothed skeleton of Jeremy Bentham are preserved according to his instructions in his will by the University of London. He left his entire estate to the university with the provision that his remains be present at all meetings of the board.

of Locke, the *skepticism* of Hume, and the *utilitarianism* of Bentham were variations on the theme of a practical and busy life.

1786–1789 CE **Abraham Bennet** (1750–1799, England). Physicist. Early pioneer of electricity. Tried to relate atmospheric electricity to the weather. In the course of these investigation invented (1786) the *gold-leaf electroscope* [based on the "portable electrometer" of **Tiberius Cavallo** (1740–1809), an Italian, who settled in London]. He also invented a simple *electric induction machine* (1789).

Bennet was born Taxal, Cheshire, the son of the schoolmaster.He was ordained in London (1785) and appointed to curacy in Wirksworth, Derbyshire. He held several other posts at the same time, including librarian to the Duke of Bedford.

1789 CE Bad weather and disastrous harvests infected with *ergot* resulted in mass hallucinations in Brittany – leading to widespread panic and irrational fears of food being stolen; it is not known how many people died of ergot outbreaks, but it had an impact on events leading up to the French Revolution.

1789–1795 CE *The French revolution*[350] transformed the government of France, shook the Establishment throughout Europe and led to many changes in ideas of government.

By 1789 France was deeply in debt because of expensive wars, and badly governed by an elite of the nobility, who lived in luxury while many poor people starved.

Faced with national bankruptcy, the King, Louis 16^{st}, decided to summon the *Estates General*, a national parliament which has not met since 1614. It consisted of 300 noblemen, 300 clergy and 600 commoners. Each estate had one vote, which meant that the nobility and clergy could outvote the commoners. So the commoners formed a national *Constituent Assembly*, pledged to make a new constitution for France.

Louis planned to dismiss the Assembly. This aroused the fury of the Paris mob, which stormed the fortress-prison of the Bastille on July 14, 1789. Louis had to give way, and the Assembly proceeded to bring many reforms. Louis then conspired with his allies in Austria and Prussia, and in June 1791 tried to flee the country. He was captured and taken back to Paris. War with Austria and Prussia followed in April 1792.

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[350] For further details, see:

- Hobsbawm, E.J., *The Age of Revolution 1789–1848*, Mentor Books: New York, 1962, 416 pp.

In August the Paris mob attacked the King in the Palace of the Tuileries, butchering his guards and imprisoning him. French victory against the Prussians in the *Battle of Valmy* (1792) encouraged the revolutionaries. A new assembly, the *National Assembly*, declared the monarchy abolished and set up a republic on Sept. 21, 1792. Power in the Convention passed to a political group called the *Girondists*, who had Louis tried for treason and executed (1793).

During 1793, a more extremist group, the *Jacobians*, gained power. The Girondists were executed, and a *Committee of Public Safety* ruled the country, headed by Maximilien Robespierre. Under his influence anyone suspected of opposing the new regime was executed, in a blood-bath known as the '*Reign of Terror*'. In July 1794, Robespierre himself was accused and guillotined, and the Terror gradually died away. In 1795, a new two-chamber assembly was elected, and order returned to France.

The French Revolution marked a turning point in European history. It unleashed forces that altered not only the political and social structures of states but also the map of Europe. Europe entered a world of class conflict, middle-class ascendancy, acute national consciousness, and popular democracy. Together with industrialization, the Revolution reshaped the institutions, the societies, and even the mentality of European men[351].

Industrial Chemistry I

No beginning can be set to chemical industry, for at the earliest times for which we have either archaeological evidence or written records, a considerable number of both chemicals and of chemical processes were in use.

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[351] The revolution did not, however, bring immediate equality for *women*; the French took the last word of the revolutionary slogan *liberté*, *égalité*, *fraternité* so literally that French women were not given the right to vote until 1945.
In the United States, women had to struggle for more than half a century before the 19^{th} Amendment to the constitution gave them full voting rights (1928); Great Britain reluctantly made the same concession in 1928.

One of the first chemicals in demand was salt[352]*, an early consequence of cooking food on fire; many cooking processes remove salt from the raw food. In the hot climate of the early civilizations much salt was also lost in sweat, and the need of replenishment was correspondingly great. For these reasons salt trade was one of the most ancient in the world. In addition of its use for seasoning food, salt was also used at an early date to preserve both meat and fish.*

Natron (an impure form of soda) was preferred for the preservation of the body after death. The production of natron, derived from three main natural sources in Egypt, and especially from the Wadi Natrun, was a state monopoly in Ptolemaic times.

While the roasting and grilling of food became possible as soon as mastery of fire had been won, boiling had to await the availability of vessels that would withstand the heat of the fire. Thus, cooking begat pottery vessels, the glazing of which demanded chemical skill; Egyptian potters used naturally occurring iron oxide to form red and black glazes.

Fermentation processes, too, had their origin in the preparation of alcoholic beverages by the conversion of sugar by yeast. Although Alexandrian alchemists were familiar with the processes of distillation, it is doubtful whether apparatus was sufficiently advanced for pure alcohol to have been available before the 12^{th} *century.*

In ancient Egypt it was known that the fermentation could proceed further, resulting in the formation of vinegar: chemically, this involves oxidation of alcohol to acetic acid.

The art of painting produced a need for natural pigments: blacks were produced with manganese dioxide, red with iron oxide and yellow with iron carbonate. By the time of the ancient empires, the paint for the decoration of houses, temples and tombs was made viscous by addition of such substances as egg-white, gum, or honey. Pigments used included red lead, yellow lead oxide, malachite and green copper silicate — the preparation of which needed considerable chemical skill.

The plastering of walls with lime, made by roasting limestone or chalk in kilns to expel carbon dioxide was introduced already around 2500 BCE. Likewise, roasted gypsum (hydrated calcium sulphate) was used for decoration of walls.

352 The Bible is abundant with such evidence (e.g., *Job* **6**, 6; *Lev* **2**, 13; *Zep* **2**, 9). Wars were fought for control over salt and asphalt sources (*Gen* **14**; *Chron II* **25**, 11; *Kings II* **4**, 7).

With increasing sophistication there came the increasing demand for artificial illumination; lamps of metal and pottery were modeled on sea-shells, using oil mixed with little salt to give a yellower and more luminous flame. Oil was made either from olives or the seed of the sesame plant.

The demands of clothing *provided the most powerful stimulus for the development of chemical processes: To this day, the chemical and the textile industries are very closely related. The origin of* soap *(in the chemical sense of saponified fats and oils) is probably in the* 4^{th} *century* CE*. Long before that, however, various cleansing agents of a different chemical character were in use. The basic process in soap-making is to boil fats or vegetables oils with strong alkali. From the* 12^{th} *century on, soft soap for the use of the textile industry was prepared using caustic alkali.*

The practice of dyeing *goes back to remote times, and the earliest records show that it was already a complex craft relying heavily on chemical processes. Until the* 19^{th} *century, virtually all dyes were of vegetable or animal origin. From very early times it was known that cloth would take up colors much more intensely and permanently if it was first treated with what we now know to be salts of aluminum (alums). The Greeks and Romans used potassium alum, obtained from certain volcanic regions, but by the* 13^{th} *century a method of purifying natural aluminum sulphate was described by Arabic writers.*

Of the origins of the three principal acids of modern chemical industry[353]

[353] By far the most important industrial chemical is sulfuric acid. It is used as a solvent and reactant in the preparation of a large number of other chemicals. Thus, it is involved in the manufacture of phosphate fertilizers, of inorganic pigments, of iron and steel, of ammonium sulfate [$(NH_4)_2SO_4$] and aluminum sulfate [$Al_2(SO_4)_3$], and of *rayon*. Sulfuric acid is also involved in the processing of nonferrous metals and in the manufacture of a variety of petroleum products. After H_2SO_4, ammonia (NH_3) is the industrial chemical produced in the greatest quantity. Its largest use is in the preparation of fertilizers. It is also used in the manufacture of soda ash, nylon, dyes, rubber and various plastics. HCl is used in the petroleum, food and metal industries. HNO_3 has its largest use in the manufacture of nitrates, explosives and as an oxidizing agent.

Some idea of the variety of compounds and uses can be obtained from the following list of the uses of the salts of sodium:

$NaCl$ (sodium chloride): table salt, manufacture of NaOH, Cl_2 and in the paper industry.

Na_2SO_4 (sodium sulfate): preparation of paper pulp and in the manufacture of glass.

$NaHSO_4$ (sodium bisulfate): dye industry.

$NaHSO_3$ (sodium bisulfite): tanning and bleaching of textiles.

$Na_2S_2O_3$ (sodium hyposulfate): photography.

— sulfuric (H_2SO_4), hydrochloric (HCl), and nitric (HNO_3), sulfuric acid seems to have been unknown until the early 16^{th} century, when it was made in Germany by dry distillation of green or blue vitriol (iron or copper sulfate). It was of virtually no industrial importance until the 17^{th} century, which is also when HCl was first clearly distinguished. Nitric acid, commonly obtained by distilling nitre (potassium nitrate) with vitriol, was described by the 8^{th} century Arabic alchemist, ***Jabir*** *(****Geber****). It was industrially important for separating large quantities of silver, which dissolves in it, from gold.*

Far more important than nitric acid was nitre (KNO_3), which with sulfur and charcoal, is an essential ingredient of gunpowder. By 1300 this mixture was prepared for use in artillery and, later, in small arms. The common source of nitre was from stables, pig-sites etc., in which it resulted from bacterial action on manure. At first, mixing of the three ingredients was done by artillerymen in the filed, but power-mills were soon established; the earliest were manually operated, but water-power had been introduced by the 17^{th} century.

Up to the 18^{th} century, the main specifically chemical trades were those of the apothecary, who prepared compounds on a small scale for use in medicine, and of the alum makers, who prepared alum on a comparatively large scale for the treatment and coloring of skins, paper, and textiles.

The new spinning and weaving machines introduced during the 18^{th} century increased the output of textiles to such a degree that the chemical problems of bleaching, and later of dyeing cloth, became considerable. Traditionally, textiles had been bleached by dipping them alternately in acid solutions of sour milk and alkaline solutions of plant ashes, and exposing them to the sun on 'bleach fields', a process that occupied all of the summer months in a given year. A shortage was soon experienced in sour milk and then also in natural alkali.

At the end of the 18^{th} century, the discoveries of ***Antoine Lavoisier*** *(1789) and* **Nicolas Leblanc** *(1791) in France had propelled a small chemical industry. But it was in Germany, which became the leading country in theoretical chemistry, that chemical research had the biggest impact on industry. By the end of the 19^{th} century, the country had developed into the*

$NaBO_3$ (sodium perborate): oxidizing and bleaching agent.

Na_2CO_3 (sodium carbonate; soda ash): manufacture of glass, soap and detergents, paper.

$NaOCl$ (sodium hypochlorite): disinfectants, deodorants, bleaches.

$NaClO_3$ (sodium chlorate): manufacture of rocket propellant and explosives.

Na_2S (sodium sulfide): preparation of dyes.

largest manufacturer of such chemicals as dyes, fertilizers, and acids used in industrial processes[354].

In England, **William Henry Perkin** *(1838–1907) was trying to synthesize quinine when he accidentally produced the first synthetic dye, which we know as mauve. Perkin got rich on mauve and then went on (1875) to create the first synthetic perfume ingredient coumarin.*

Perkin's chemistry teacher, **August Wilhelm von Hoffmann** *(1818–1892, Germany) was a German chemist teaching in England. He synthesized his first dye, magenta in* 1858. *After he returned to Germany he discovered many chemicals and aniline dyes. Other chemists in Germany worked on producing natural dyes from easily available chemicals, obtaining a red called alzarin (1869) and indigo (1880). All these dyes became the basis of an immense German chemical industry. They also had an impact on biology, for biologists discovered that coloring bacteria or cells with dyes made previously invisible structures apparent.*

1789–1842 CE **Martin Klaproth** (1743–1817, Germany). Chemist. Discovered an unknown metal in *pitchblende* (1789). Although he was unable to isolate the new metal, he named it *uranium* after the recently discovered plant *Uranus* (1781, William Herschel). In 1842 the chemist **Eugène Melchior Peligot** (1811–1890, France) isolated the element.

1790–1800 CE **Salomon ben Joshua Maimon** (1753–1800, Germany). Philosopher, historian of philosophy and logician. Attempted to expound an algebraic *symbolic system of logic* (1794). Developed a form of *monism* (1797) (i.e. there is but one fundamental reality) that pervaded not only philosophy, but all sciences, and by which **Fichte**, **Schelling** and **Hegel** were influenced. **Goethe**, **Schiller**, **Kant** and **Mendelssohn** payed him tributes of praise.

354 The relationship between scientific education and technological progress became fully understood during the 19^{th} century. Following the example of the *Ecole Polytechnique* in France, Germany (and later the United States) also founded technical schools with the idea of applying science to technology. At the end of the century, these technical universities played an essential role in the rapid expansion of Germany's industry. They developed the various kind of engineers who used science to solve technological problems rather than to advance knowledge.

Key works:

- *Versuch über die Transzendentalphilosophie* (Essay on the Transcendental Philosophy, 1790). Criticism of Kantian philosophy. Kant acknowledged Maimon as the most acute of his critics.
- *Versuch einer Neuen Logik* (Essay on the New logic, 1794)
- *Kritische Untersuchungen über den Menschlichen Geist* (Critical elaborations on the human spirit, 1797).
- *Lebengeschichte* (Autobiography, 1793). An important source for the study of Judaism and Hasidim in Eastern Europe in that period.

In his later writings he achieved synthesis of rationalism and Judaism. In his *Kritische Untersuchungen*, the great question at issue is Kant's question: *"Has man any ideas which are absolutely and objectively true?"*. The answer to this question depends on another question: *"Has man any ideas independent of experience?"*, for if all ideas depend on experience, there can be no question of objective ideas, experience being essentially subjective.

Kant answered the second question in the affirmative, and the first in the negative. He showed that in consciousness certain elements are given which are not derived from experience, but which are necessarily true. However these given elements or "things in themselves" man knows only as they appear to him, but not as they are "per se". This concept of "things in themselves" is rejected by Maimon, who holds that the matter of exterior objects which produce impression on man's sensibility is absolutely intelligible.[355] He also

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355 Maimon seized upon the fundamental incompatibility of a consciousness which can apprehend, yet is separated from, the "thing-in-itself". That which is object of thought cannot be outside consciousness; just as in mathematics $\sqrt{-1}$ is an unreal quantity, so things-in-themselves are *ex-hypothesis* outside consciousness, that is to say, unthinkable.

The Kantian paradox he explains as the result of an attempt to explain the origin of the "given" in consciousness. The form of things is admittedly subjective; the mind endeavors to explain the material of the given in the same terms, an attempt which is not only impossible but involves a denial of the elementary laws of thought. Knowledge of the given is, therefore, essentially incomplete. Complete or perfect knowledge is confined to the domain of pure thought, to logic or mathematics. Thus the problem of the thing-in-itself is dismissed form the inquiry, and philosophy is limited to the sphere of pure thought. The Kantian categories are, indeed, demonstrable and true, but their application to the given is meaningless and unthinkable.

contested the Kantian distinction between sensibility and understanding as well as the subjectivity of the intuitions of time and space. For him, sense is imperfect understanding, and time and space are sensuous impressions of diversity, or diversity presented as externality.

In practical philosophy he criticized Kant for having substituted an unpractical principle for the only motive for action – pleasure. The highest pleasure is in knowing, not in physical sensation, and because it recognizes this fact the "Ethics" of Aristotle is much more useful than the Kantian.

Maimon's autobiography was published by K. Ph. Moritz (Berlin, 1793). In this work he gives a résumé of his views on the Kabbalah, which he had expounded in a work written while he was still in Lithuania. According to him the Kabbalah is practically a modified Spinozism, in which not only is the world in general explained as having proceeded from the concentration of the divine essence, but every species of being is derived from a special divine attribute. God, being the ultimate substance and the ultimate cause, is called "En Sof," (infinity) because He can not be predicated by Himself. However, in relation to the infinite beings, positive attributes were applied to Him, and these attributes were reduced by the Kabbalists to ten – the *ten sefirot*. The ten "circles" correspond to the ten Aristotelian categories, without which nothing can be conceived.

In the same work Maimon expresses his views on Judaism. He divides Jewish history into five main periods:

(1) the period of natural religion, extending from the Patriarchs to Moses;
(2) the period of revealed or positive religion, from Moses to the Great Sanhedrin;
(3) the Mishnaic period;
(4) the Talmudic period;
(5) the post-Talmudic period.

Maimon censures the Rabbis for having burdened the people with minute prescriptions and ceremonies, but praises their hight moral standard.

Maimon was born at Nieswiez, Polish Lithuania. He was a child prodigy in the study of rabbinic literature. Married off by his father at the age of 11, he became a father himself at 14. He supported his family by working as a tutor

By this critical skepticism Maimon takes up a position intermediate between Kant and Hume. Hume's attitude to the empirical is entirely supported by Maimon. The causal concept, as given by experience, expresses not a necessary objective order of things, but an ordered scheme of perception; it is subjective and cannot be postulated as a concrete law apart form consciousness.

in neighboring towns. In his spare time he educated himself in philosophy, secular sciences and foreign languages. He adopted the name **Maimon** in honor of Maimonides, as a token of reverence for that great master.

Harassed both by his implacable mother-in-law and by his correligionists (who regarded him as a heretic) he left home and family (1770) to begin a life of material insecurity and wandering over Northern Europe which terminated only in 1790; he was then offered a retreat on the estate of Count Adolph Kalkereuth of Nieder Siegersdorf (Silesia), where he died.

1790–1820 CE **John Rennie** (1761–1821, England). Civil engineer. Constructed many canals, bridges, docks, breakwaters and harbors in Scotland and England. The most conspicuous are: Waterloo bridge, Southwark bridge and London bridge over the Thames.

Born at Phantassie, Haddingtonshire, and educated (1780–1784) at Edinburgh University.

A feature of his work was the use of iron for many portions of the machines which had formerly been made of wood.

1790–1850 CE Leading Western poets and novelists in the *Age of Romanticism* an *Naturalism*.

- Friedrich Hölderlin 1770–1843
- William Wordsworth 1770–1850
- Samuel Taylor Coleridge 1772–1834
- Heinrich von Kleist 1777–1811
- Adelbert von Chamisso 1781–1838
- Stendhal 1783–1842
- Jakob Grimm 1785–1863
- Wilhelm Grimm 1786–1859
- Lord Byron 1788–1824
- Percy Bysshe Shelley 1792–1822
- John Keats 1795–1821
- Heinrich Heine 1797–1856
- Adam Mickiewicz 1798–1855
- Honore de Balzac 1799–1850
- Alexander Pushkin 1799–1837
- Victor Hugo 1802–1885
- Prosper Merimeé 1803–1870
- Hans Christian Andersen 1805–1875
- Henry W. Longfellow 1807–1882
- Edgar Allan Poe 1809–1849
- Nikolai Gogol 1809–1852

•	Alfred Tennyson	1809–1892
•	Berthold Auerbach	1812–1882
•	Ivan Goncharov	1812–1891
•	Mikhail Lermontov	1814–1841
•	Herman Melville	1819–1892

1791 CE **Jeremias Benjamin Richter** (1762–1807, Germany). Chemist. Formulated the *Law of Equivalent Proportions* which states that if an amount x of substance A combines chemically with amount y of substance B and also with amount z of substance C, then amount y of substance B will combine with amount z of substance C. After this discovery, tables of equivalent weights were drawn up, showing the relative amounts of the chemical elements that would combine with one another.

Richter was born at Hirschberg in Silesia, and became a chemist at Breslau mines and the Berlin porcelain factory. Richter was a pupil of the philosopher **Immanuel Kant**, and he held, with his master, that the physical sciences were all branches of applied mathematics.

Emancipation – The second Exodus (1791–1917)

In antiquity the Jews were the great innovators in religion and morals. In the Dark Ages and early medieval Europe they were still an advanced people transmitting scarce knowledge and technology. Gradually they were pushed from the van and left behind until, by the end of the 18^{th} century, they were seen as bedraggled and obscurantist rearguard in the march of civilized humanity.

But then came an astonishing second burst of creativity. Breaking out of their ghettos, they once more transformed human thinking, this time on the secular scientific sphere.

With the decline of religious faith in post-medieval European society, the traditional theological hostility towards the 'deicide' people became less relevant, especially to intellectuals, who identified with the skeptical temper of the Age of Enlightenment. The rise of rational thinking in the 17^{th} and 18^{th}

centuries appeared to be positive development for the Jew, for it attacked the foundations of Christian religion and the unified Christian state which had excluded or oppressed Jews for reasons of creed.

It was partly from rationalist assumptions of the German Enlightenment that the Habsburg ***Emperor Joseph II*** *derived his Toleration edicts of the 1780s, that* ***Moses Mendelssohn*** *felt empowered to build a bridge between traditional Jewish and modern German cultures and that his friend* ***Gotthold Lessing*** *immortalized a more positive image of the Jew in his famous play, Nathan the Wise. Without the philosophy of the Enlightenment, the Prussian bureaucrat* ***Christian Wilhelm Döhm*** *would never have written his tract "Concerning the Civic Amelioration of the Jews" (1781), an indictment of the responsibility of the Christian world for the degradation of the Jews.*

Other forces were also in action; with the disappearance of the last vestiges of feudalism, the Court Jews went also. Their financial manipulations, which often saved European monarchs from bankruptcy, made way for the public banking system which we know today. Their financial services and counsel to sovereigns, states and private enterprises were vastly important in the commercial development and industrial growth of Europe. Thus, at the turn of the 19^{th} *century, the rulers of Europe were not blind to the economical potential of the Jews.*

The Emperor, as well as his neighbor, ***Frederick the Great*** *of Prussia, have certainly not overlooked the substantial revenue the Jew brought into their realms by stimulating industry. Indeed, the industrial revolution in Germany found its most enterprising pioneers among Jews. The first iron industry (1840), coke industry, and railroad industry were all founded and built by Jewish investors and entrepreneurs. The electrical, chemical, shipping and dye industries also owe much to Jews.*

With those factors combined, the ghetto dwellers faced, at the end of the 19^{th} *century, the most momentous political event in Jewish history since the loss of their state in 70 CE – the Emancipation:*

- *Sept. 28, 1791: Jews were declared to be equal (on paper) with all men and free citizens of the Republic of France.*
- *1798: The ghetto gates in Bonn, Germany, were broken down by Christians*

Consequently, the Jews of Germany and Austria emerged from the mental and physical isolation of ghetto life and rushed with burning enthusiasm into

the arts and sciences[356]*. They may have been newcomers to the German universities, but it seemed as though they had been preparing for the entrance examination for a thousand years.*

Their bibliophile tradition as "people of the Book" took them almost as a matter of course, into medicine, biology and mathematics as well as literature and music. Their grandfathers had studied the Talmud. They, no less attentively, read ***Kant****,* ***Goethe*** *and* ***Hegel****.* ***Karl Marx*** *(grandson of two orthodox rabbis) unconsciously reverted to an old Kabbalist technique when he "turned Hegel upside down" in order to formulate his own dialectical materialism. In the sciences, though most professorships still remained closed to them*[357]*, new doors were constantly opening.*

There was, however, a price to pay for joining the modern world; the "ticket of admission to European culture" was the baptismal certificate[358]*, common mostly in the first half of the 19*th *century. After 1848, apostasy declined while other forms of assimilation (intermarriage and renegades) were more fashionable.*

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356 **Moses Mendelssohn** translated the Pentateuch into German (1783): **Leopold Zunz** and his friends established (1819) the *Society for promotion of Jewish Culture and Science*

357 From 1818, Jews in Germany were excluded from state academic posts, by decree of King Frederick William III. Jews were also dismissed from state positions and conversion to Christianity was actively encouraged.

358 During the 19th century, at least 250,000 Jews converted to Christianity in Europe alone. [Germany 22,500; Britain 23,500; Russia 84,500; Poland 21,500; Austro-Hungary 45,000]. This amounted to about 5 percent of the total Jewish population during that century. Most apostates belonged to the wealthy and intellectual circles in major cities, who constituted about 15 percent of the Jewish population. The rest were poor religious people who would not change their old living style, thus being immune to assimilation of any form. Total Jewish population in Europe reached 1,430,000 (1800) [Russia and Poland 800,000; Austria, Hungary and Galicia 300,000; Germany 200,000; France 80,000; Holland 50,000] and 8,690,500 (1900); 9,462,000 (1939). In *Berlin*, the Jewish population never exceeded 4 percent [2000 (1743); 3300 (1812); 12,000 (1852); 24,280 (1864); 45,500 (1876); 106,000 (1900); 172,600 (1925); 160,500 (1933); 82,780 (1939); In *Vienna*: 178,000 (1933); 91,500 (1939)]. The number of Jews in Germany grew to 420,500 (1871) and 564,400 (1925), where the 1925 figure included some 80,000 immigrants from the east.

Intermarriages during 1906–1930 amounted to 27% of total Jewish marriages. In general, Jewry lost 80% of descendants via intermarriages. The Jewish intellectual elite amounted to about 3 percent of their total number at any given time during 1830–1930.

Apostasy was usually Protestant in Northern Germany (for ***Heine****,* ***Marx****,* ***Felix Mendelssohn*** *and many others) but Catholic in Bavaria and the Habsburg empire. Conversion tended to be a matter of good form rather than an act of faith. Most converted Jews blended entirely into the social background. A generation or two later, no one remembered that* ***Johann Strauss, Sr.*** *("the demon of Viennás innate musical spirit", as Richard Wagner described him), was the son of a baptized Jewish tavern-keeper from Budapest. Many German Jews bore the names of localities which, with the addition of a* von *or* zu*, furnished a pure Aryan flavor.*

Even those who stopped short of conversion, however, tended to abandon the more visible aspects of the Jewish faith. Since German life was becoming increasingly secular, baptism as such ceased to be the touchstone of social assimilation. Most of the emancipated German Jews did their best to become outwardly indistinguishable from their neighbors.

But despite all the efforts of the liberal elements and the strenuous fight of the Jews themselves to remove the barriers to their full acceptance by assimilation, anti-Semitic forces remained powerful, especially in the lower middle class and nobility. They frequently brought about violent outbreaks and riots differing in strength and extent in the cities and states where they took place. Whenever adverse events (economic, political or social) occurred, these feelings of hostility erupted.

The attitude of Bismarck to this problem during his reign (1862–1890) was ambiguous; on one hand he had great respect for the high qualities, great talents and competence of many Jews and had not the slightest hesitation to use their services for Germany or for his own interests.[359] *But Bismarck also used antisemitism as a convenient political weapon in his fight against liberals and Social Democrats, many of whose leaders were Jewish. Nevertheless, he would not tolerate any violent actions against Jews and would not permit their civil rights violated, fully recognizing their great value for the strength of Germany.*

The attitude of Kaiser Wilhelm II toward Jews was more complex and sometimes more emotional. During his reign (1888–1914), Jews increasingly

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[359] An example is his close association to the Jewish banker **Gerson von Bleichröder** (1822–1893) who helped to finance two of Bismarck's wars and was instrumental in the building of the empire. He was also Bismarck's personal financial adviser, as well as that of Benjamin Disraeli. Bleichröder's contributions to the greatness of Germany earned him only envy scorn and hatred, becoming the target of strong antisemitic reactions, especially from the economically declining ruling Junker class.

penetrated academic professions and many became recognized leaders in science and many other fields. But, owing to the newly accumulated wealth, Jews began to play a prominent role in society, and many Junkers were unable to compete with them. These Junkers, whose wealth was essentially based on their large estates were impoverished by the rise of capitalism and industrialization. Attributing their misfortunes to the Jews and not to the economic trends prevailing in Western Europe, they succeeded (with the full support of Wilhelm) to block the Jews from both government and the army.

Unfortunately, the Kaiser fell under the influence of the antisemitic ideology of Cosima Wagner[360] *and her son-in-law, Houston Stewart Chamberlain. Consequently he began to consider Jews in general as the deadly enemy of the "Aryan" Germans. This did not prevent many of his entourage from having Jews as their personal physicians, bankers, or advisers, despite their gross antisemitism. As for Wilhelm himself, he was fully aware of the major contribution of German Jews to German science and industry and like Bismarck would not tolerate any violent eruption of antisemitism; Law and order were untouchable!*

What were the factors permitting the Jews in Germany to become instrumental in the rapid rise of science in the Wilhelmian era (1888–1914) and later in the Weimar Republic until Hitler (1919–1933)?

[360] In December 1914, **Lord Balfour** (Britain's foreign secretary 1916–1919) told the scientist and Zionist leader, **Chaim Weizmann** that on his previous visit to Cosima Wagner in Bayreuth she had expressed the opinion that "...the Jews in Germany have captured Stage, Press, Commerce and the Universities. They are putting in their pockets, after only a hundred years of emancipation, everything for which the Germans have worked for centuries. We resent very much having to receive all the moral and material culture at the hands of the Jews ..."

To this, Weizmann had commented to Balfour: "The essential point which most non-Jews overlook and which forms the crux of the Jewish tragedy is that those Jews who are giving their energies and their brains to the Germans are doing it in their capacity as Germans and are enriching Germany and not Jewry, which they are abandoning .... The tragedy of it all is that whereas *we do not recognize them as Jews, Madame Wagner does not recognize them as Germans*, and so we stand there as the most exploited and misunderstood people."

At the turn of the century these extreme views of Cosima Wagner were shared by a small number of people, but they became widespread in the era of the Weimar Republic. It is remarkable how unaware Jews, as well as many non-Jews, were of these deep-rooted feelings, almost until the era when the Nazis came to power.

One factor was the rapid growth of cities which offered talented, intelligent Jews, with their willpower and drive, an unexpected chance.

A second factor was the rapid development of the capitalistic and industrial economy around 1850. New elite and new ideologies were formed; a new upper middle class began to emerge at the expense of the previously dominant nobility, the craftsmen, the lower middle class, the peasants, and the landowners, who lost many of their privileges. Jews seized upon these new opportunities and played an important role in the new economy.

The third factor was the vast expansion of the universities and the technical institutions. About 10% of the students were Jews while Jews constituted not quite 1% of the total population! In 1907, they amounted to 6% of all German physicians and dentists, 14% of all lawyers, and 8% of private scholars, journalists and writers.[361]

The acceptance of Jews in the universities was greatly facilitated by their assimilation to German civilization. At the turn of the century many Jews had lost almost all connections with Jewish tradition and the Jewish community. Many of them considered themselves as German citizens of Jewish faith. But the Jewish religion actually had little meaning for many of them. Even in the relatively liberal Wilhelmian era, a period of great prosperity and relative affluence, the freedom that the Jews enjoyed in the academic and free professions was not extended to all fields. Even baptized Jews were admitted to public office and civil service in very limited and insignificant numbers. They were virtually excluded from the government and the army. In these fields, the situation began to change only in WWI, when many Jews fought in the army.

Full emancipation (the right of religious freedom and the right to be chosen to governing bodies) was declared in The Netherlands (1796); Italy (1798); Belgium (1831); Canada (1832); Germany (1871); England (1878); U.S.A. (1785–1877); Russia[362] *(1917).*

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361 When the Nazis came to power (1933) almost half of the 6000 physicians in Berlin were Jews.

362 There was no other state on the European continent which officially pursued such repressive anti-Jewish policies in the 19^{th} century as the Tzarist Russian Empire. By 1897, more than 5 million Jews (about half of the world Jewry) lived under the totalitarian rule of the Tzars in poverty and deprivation, subjected to endless humiliating decrees. What the Russians did was to engage in the first modern exercise of *social engineering*, treating Jews as earth or concrete, to be shoveled around. Firstly they confined the Jews to what was called the "Pale of the Settlement" (1812) which consisted of one million square kilometers stretching from the Baltic to the Black Sea. A series of statutes (1804) forbade

The Jewish intellectual tradition, nurtured continuously for so many centuries by Talmudic studies and enriched at various periods by fusion with other cultures, received a new impetus with the emancipation.

Although severely handicapped by the numerus clausus *(an anti-Semitic device for limiting Jewish students in universities on a percentage quota), Jews nonetheless entered into all fields of study. They distinguished themselves especially in the sciences. While science does not know any racial or national boundaries, since it aims to serve all mankind, in Germany, however, sharp distinctions were often made between Jewish and non-Jewish scientists — to the detriment of science and of Germany.*

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them to live and work in the villages, thus destroying the livelihood of a third of the Jewish population, without allowing them to do any labor on the land. The real aim was to drive Jews into accepting baptism. The next turn of the screw came in 1827, when Nikolai I issued the 'Cantonist Decrees' which conscripted all male Jews, from 12 to 25, to 25 years of military service, the object again being to promote baptism. During 1827–1856, some 60,000 kids were forcibly kidnapped and conscripted, half of which were eventually baptized. The government destroyed Jewish education. Jewish books were censored or burned. Movements outside the pale were banned, and inside it – restricted.

Russian antisemitism was in its origins a combination of simple primitive hatred for the Jews as 'aliens' and of Christian orthodox religious prejudice which regarded Jewish people as deicides. Such prejudice remained alive and virulent both at the state level and among the millions of superstitious and illiterate Russian peasants. In fact, Russia was the only country in Europe, at this time, where antisemitism was the official policy of the government. It took innumerable forms, from organizing massacres (pogroms, 1871–1906) to forging and publishing the *Protocols of the Elders of Zion*.

History of Theories of Light II

B. Waves Versus Corpuscles, A *(1608–1800)*

The 17^{th} *century opened with a shower of new inventions and ideas. It seemed as if all the latent optical lore of* 2000 *years, suddenly burst the floodgates of human consciousness and materialized in the form of telescopes, microscopes, prisms, the new phenomena of dispersion, polarization, diffraction and aberration and the principles of least-time and Huygens'.*

Above all, however, hovered the great controversy on the nature of light: was it a stream of particles as maintained by ***Democritos*** *(420 BCE),* ***Descartes*** *(1637) and* ***Newton*** *(1672) or a rapid undulation of ethereal matter as argued by* **da Vinci** *(1490),* ***Grimaldi*** *(1665),* ***Hooke*** *(1665) and* ***Huygens*** *(1678)?*

Until about the middle of the 17^{th} *century, it was generally believed that light consisted of a stream of corpuscles, emitted by light sources, such as the sun or a candle flame, and traveled outwards from the source in straight lines. This theory provided simple explanations to the simple laws of reflection and refraction from smooth surfaces. With the discovery of the phenomenon of light diffraction by* **Grimaldi** *(1665) and* **Römer's** *proof (1676) that light travels with a definite velocity,* **Huygens** *(1690) showed that the laws of reflection and refraction could be explained on the basis of a wave theory (through the wavelets and secondary wavefronts) and that such a theory could furnish a simple explanation to the recently discovered phenomenon of double refraction by* **Erasmus Bartholinus** *(1670).*

*The great moments of this epoch were undoubtedly the invention of the telescope (1608), the mathematical statement of the law of refraction (***Snell***, 1621),* **Fermat***'s principle of least-time (1657), the advent of wave theory [***Grimaldi***,* **Hooke***, 1665;* **Huygens***, 1678], the determination of the velocity of light (***Römer***, 1676) and* **Newton***'s theory of corpuscles, dispersion and color (1672). The authority of Newton led to the rejection of the wave-theory and the abeyance of optics for nearly a century.*[363] *But it still found an occasional supporter, such as the great mathematician* **Leonhard Euler**

363 For one thing, it was objected that if light were a wave motion one should be able to see around corners, since waves can bend around obstacles in their paths. We know now that the wavelengths of light are so short that the bending, while it does actually take place, is so small that it is not ordinarily observed. The significance of Grimaldi's results was not realized at the time.

(1746). It was not until the beginning of the 19^{th} *century that the decisive discoveries were made which led to general acceptance of the wave theory.*

1791–1819 CE **William Smith** (1769–1839, England). Geologist, engineer and surveyor. Founder of stratigraphical geology. Discovered a method to assign relative ages to individual rock strata (formations) by means of their fossilized content and thus was first to point out the relationship between fossils and geologic data (1791).

Units of similar lithology are not continuous even in one region and the stratigraphic sequences differ significantly among widespread localities. The need for detailed mapping of rock formations required a new unifying principle — a new tool by which units could be categorized and recognized widely. [Detailed geologic mapping in a humid region like Europe is difficult. Almost everywhere the rocks are superficially covered by soil, vegetation, or alluvium.]

Smith found fossils to furnish just such a tool. His investigations of roads, quarries, mines, and canals acquainted him intimately with much of England's countryside. During his travels he recognized and traced out numerous sedimentary rocks, and he soon noticed that each successive unit contained its own diagnostic assemblage of fossils by which it could be distinguished from other units of different ages. Utilizing this principle he produced, in 1815, the first geological map of England, and correlation between distant localities now became feasible. The way was prepared to erect a stratigraphic classification based on *time relations* of strata rather then on rock type.

1792–1808 CE **Jean-Baptiste-Joseph Delambre** (1749–1822, France). Astronomer, erudite and historian of astronomy. Discovered new formulas in *spherical trigonometry* (1808). Published tables of the location of planets and their satellites (1792); with **Méchain**, measured an arc of the meridian between Dunkirk and Barcelona (1792–1799); wrote histories of ancient, medieval and modern astronomy. A large crater is named for him on the moon.

Delambre was born at Amiens. Despite extreme penury he studied indefatigably ancient and modern languages, history and literature, and it was not until he was 36 years of age that he begun a serious study of astronomy and mathematics. He was 40 before he published anything on the subject, and it was some years later that he was awarded a prize by the Academy for

his tables of Uranus. He succeeded **Lalande** (1807) as professor of astronomy at the College de France.

1792 CE Ca 800,000 died of the plague in Egypt. By 1799 the disease reached North Africa with ca 300,000 additional casualties.

1784–1830 CE **William Murdock** (1754–1839, England). Inventor. Invented *coal-gas lighting* (1792). First to construct a model of steam powered carriage (1784). Made important improvements of the steam engine.

Murdock was born near the village of Auchinleck in Ayrshire. He was first to realize that coal gas might be used for light. In 1807, London streets began to be illuminated by coal-gas lighting.

At the celebration of the centenary of gas lighting (1892), the bust of Murdock was unveiled by Lord Kelvin.

1793 CE **Christian Konrad Sprengel** (1750–1816, Germany). Botanist. Discovered the part played by nectaries, insects and the wind in the *pollination* of flowers (plant fertilization).

1793–1798 CE **Eli Whitney** (1765–1825, U.S.A.). Inventor. The father of *mass production*. His *cotton gin* (1793) made cotton-growing profitable, and helped make the United States the largest cotton producer in the world. His method of making guns by machinery (1798) marked the beginning of *mass production* in the world's industry.

Whitney was born in Westborough, Mass., the son of a farmer. Times were hard after the Revolutionary War, and Whitney did not have the money to go to college. He taught school for five years and with his saving financed his studied at Yale during 1788–1792. By 1793 he had built the cotton gin, which could clean cotton as fast as 50 men working by hand.

In 1798, he built a factory near New Haven and began to make muskets by a new method. Until then, each gun had been handmade by a skilled craftsman, and no two guns were alike. Whitney invented tools and machines that enabled unskilled workmen to turn out absolutely uniform parts.

1793–1814 CE **Thomas Young** (1773–1829, England). Distinguished physicist, physician and philologist. Discovered and explained the phenomenon of light interference and consolidated the wave theory of light on a firm experimental basis. Opposed Newton's particle theory of light in favor of light as a wave in the cosmic aether (1801). In 1807 he anticipated the nature of infrared radiation from hot bodies, claiming that heat, like light, is a wave vibration rather than a material substance. In 1809 he applied the wave

theory of light to the phenomena of light refraction and dispersion, although he believed light vibration to be mainly longitudinal.

Young is also the founder of physiological optics: in 1793 he explained the mode in which the eye accommodates itself to vision at various distances depending on the change of the curvature of the lens. In 1801 he described the defect known as *astigmatism*. In 1802 he put forward the theory that color perception depends on the presence in the retina of 3 kinds of nerve fibers which respond respectively to red, green and violet light.

In another field of research, he was one of the first successful workers at the decipherment of Egyptian hieroglyphic inscriptions: by 1814 he had completely translated the enchorial (demotic) text of the *Rosetta stone*.

Young, like Leonardo da Vinci, was a remarkably versatile scholar. His epitaph reports that "he first penetrated the obscurity which had veiled for ages the hieroglyphics of Egypt"; but his work shows only one facet of a brilliant career. His work in medicine and science later led the physiologist and physicist **Helmholtz** to say of him: "*He was one of the most profound minds that the world has ever seen*".

Young was born to a Quaker family in Somerset, England, the youngest of 10 children. At age 14 he was acquainted with Latin, Greek, French, Italian, Hebrew, Persian and Arabic. In 1796 he obtained his M.D. degree at Göttingen, Germany. Upon the death of his grand-uncle in 1797, he became financially independent and in 1799 established himself as a physician in London. In 1801 he was appointed professor of physics, but resigned his professorship in 1803, fearing that its duties would interfere with his medical practice.

1793–1828 CE **Thomas Telford** (1757–1834, Scotland). Civil engineer. Devised and improved methods of road construction. The Telford method of using large flat stones for road foundations is named after him. Telford engineered roads, bridges, harbors, docks, canals and waterways. He built the Menai Strait suspension bridge in Wales, the Ellesmere Canal in England, the Caledonian Canal in Scotland and the Göta Canal in Sweden.

Telford was born in Eskdale, Scotland and died in London (buried in Westminster Abbey). He was a son of a shepherd. From early childhood he was employed as a herd, occasionally attending the parish school of Westerkirk. He was mostly self-educated, learning architectural drawing in his spare time. He never married, living most of his adult life in hotels. He was a fellow of the Royal Societies of London and of Edinburgh.

Science Progress Report No. 7

"The Revolution has No Need for Savants"

With the rise to power of the Jacobines in 1793, the French revolution took a more radical turn, and many of the old institutions were closed down, including the Paris Academy of Sciences; scientists associated with the regime of the Girondists were executed, notably Lavoisier. The vice-president of the tribunal that tried Lavoisier declared that France "Already had too many scholars".

However, after 40,000 people were killed by the government and its agents, the National Convention was sobered by the terror and with the fall of the Jacobines in the summer of 1794, the revolution fell back into the hands of the bourgeoisie, which was the class that in the end gained the most from it.

1794–1835 CE **Carl Friedrich Gauss**[364] (1777–1855) Germany)[365]. Physicist, astronomer, and one of the greatest mathematicians of all times. Published the treatise '*Disquisitiones Arithmeticae*', which includes his discoveries in number theory and is one of the most important works in the history of mathematics.

Gauss was born in Brunswick, Germany, into a poor family. He was a child prodigy. At the age of 14 he discovered the prime number theorem, and completed his first original work at 19, when he showed how to construct a regular 17-sided polygon with a ruler and compass, the first 'new' n-gon for

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[364] For further reading, see:

- Bühler, W.K., *Gauss*, Springer-Verlag, 1981, 208 pp.
- Hall, T., *Carl Friedrich Gauss*, Massachusetts Institute of Technology Press: Cambridge, 1970, 176 pp.
- Dunnington, G.W., *C.F. Gauss Titan of Science*, New York, 1955.
- Gauss, C.F., *Disquisitiones Arithmeticae* (*1801*), Yale University Press: New Haven, CT, 1966, 472 pp.

[365] During 1777–1783, *both* Euler and Gauss were alive.

2000 years[366]. His doctoral thesis (1799) contained the first acceptable proof of the fundamental theorem of algebra, a result whose proof had defeated such giants as **Euler** and **Lagrange**. He did fundamental work in probability, geodesy, mechanics, optics, actuarial science and electromagnetism [with **W.E. Weber**, he build in 1833 the first operating electric telegraph]. Like Euler, he was a prodigious calculator.

In 1795, Gauss developed the method of least squares[367], thus founding the field of *mathematical statistics*. He applied it to problems as diverse as astronomy and prime number counting.

In 1811 Gauss opened the modern period of research on infinite series with his memoir on the *hypergeometric* series [name given by **Johann Friedrich Pfaff** (1765–1825, Germany)]. **Euler** had studied it and introduced its defining differential equation, but Gauss was the first to master it. He made the first adequate study of its convergence and associated functional relations. The hypergeometric function played a central role in Gauss' thinking, be-

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366 In 1801 Gauss took up the ancient problem of finding *all* regular polygons that can be constructed by means of compass and ruler. The construction of regular polygons of 2^n, $3\cdot 2^n$, $5\cdot 2^n$ and $15\cdot 2^n$ have been known since the time of the Greeks, but no one suspected before Gauss that polygons of any other number of sides could be constructed by ruler and compass. The way had to be paved by numerous theorems in algebra. This Gauss did, showing eventually that a circle can be divided by ruler and compass into n equal parts if and only if n is of the form

$$2^{\alpha_0}\left[2^{(2^{\alpha_1})}+1\right]\left[2^{(2^{\alpha_2})}+1\right]\cdots\left[2^{2^{\alpha_n})}+1\right],$$

where each of the quantities in parenthesis is a prime and where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are all different positive integers. The only known primes of the form $F_n = 2^{2^n}+1$ (*Fermat numbers*) are 3, 5, 17, 257 and 65537 corresponding to n values of 0, 1, 2, 3 and 4. Euler showed (1732) that F_5 is *not* prime. No prime F_n has yet been found for $n \geq 4$. In fact, F_n is known to be composite for all n such that $5 \leq n \leq 21$, as well as for some larger n.

Gauss was so proud of his discovery showing the relation between prime Fermat numbers and inscribed polygons that he wished to have a 17–gon inscribed on his tombstone (emulating the tombstone of Archimedes, which was decorated by a figure of a sphere and circumscribed cylinder, suggesting his formula for the area of a sphere).

For some reason his request was not granted and on Gauss' grave in Göttingen there is no such polygon. It does, however, appear on the side of a monument in his native town of Brunswick.

367 *Formal* priority belongs to Legendre (1805). Gauss *published* his results only in 1821.

cause he encountered many special cases of its defining series in the theory of elliptic integrals.

In 1819 Gauss obtained an explicit formula for the combination of two finite rotations (never published). This led him to the discovery of the quaternions 24 years in advance of Hamilton.

In 1827 Gauss made the first systematic study of quadratic differential forms in his *Disquisitiones generales circa superficies curves*, thus laying the foundation of differential geometry. He was led to this by his geodetic work, which concerned the precise measurement of large triangles on the earth's surface. This provided the stimulus that led him to found the intrinsic differential geometry of general curved surfaces. For this work he introduced curvilinear coordinates u and v on a surface. He obtained the fundamental quadratic differential form $ds^2 = Edu^2 + 2Fdudv + Gdv^2$ for the element of arclength ds, which makes it possible to determine geodesic curves.

He formulated the concepts of *metric coefficients*, *Gaussian curvature* and *total curvature*.

His main specific results were the famous *theorema egregium*, which states that the Gaussian curvature depends only on E, F, G, and the *Gauss-Bonnet theorem* on total curvature for the case of a *geodesic triangle*[368], which in its general form is the central fact of modern differential geometry. It is

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[368] Gauss was able to find a formula for the sum of the angles of a *geodetic triangle* on any surface. If s is the sum of the angles, measured in degrees, then $s = 180\big[1 + \frac{1}{\pi}\int KdA\big]$, where K is the Gauss' curvature and the integral is taken over the interior of the triangle. In the plane, $K = 0$ and we have $s = 180$ for any triangle. On a sphere of radius r, we have $K = \frac{1}{r^2}$ and $\int KdA = \frac{A}{r^2}$, where A is the area of the triangle. Then $s = 180\big(1 + \frac{A}{\pi r^2}\big)$ or $A = \pi r^2\big(\frac{s}{180} - 1\big)$. (This formula was first published by the Flemish mathematician **Albert Girard** (1629).) By measuring A and the sum of angles s, this equation can be used to determine the radius r of a sphere. In general, on a surface with positive curvature we have $K > 0$, $\int KdA > 0$, and the sum of the angles s is greater than 180 degrees. Negative curvature implies $K < 0$ and s is then less than 180 degrees.

Thus, an inhabitant of a 2-dimensional spherical surface can discover the radius of his spherical world by simply measuring the area and the sum of angles of an arbitrary spherical triangle, using the formula

$$r = \sqrt{\frac{(A/\pi)}{\frac{s}{180^\circ} - 1}}$$

e.g. a quarter-hemisphere spherical triangle has: $s = 270^\circ$, $A = \frac{1}{2}\pi r^2$.

the generalization of these concepts that opened the door to Riemannian geometry, tensor analysis and the ideas of Einstein.

In 1831, Gauss turned his attention again to number theory, where he broadened the ideas of number into the complex domain. He defined complex integers (now called '*Gaussian*' *integers*) as complex numbers $a + ib$ with a, b as ordinary integers. This led him to introduce a new concept of prime numbers in which 3 remains prime but $5 = (1 + 2i)(1 - 2i)$ does not.

He then proved the unique factorization theorem for these integers and primes. The ideas of this paper inaugurated algebraic number theory. [He used these concepts to prove Fermat's conjecture for $n = 3$.]

In 1839 Gauss published his fundamental paper on the general theory of inverse square forces, which established *potential theory* as a coherent branch of mathematics. Among his discoveries were the *divergence theorem*[369], the basic mean value theorems for harmonic functions and the very powerful statement which later became known as "*Dirichlet's Principle*" and was finally proved by **Hilbert** in 1899.

Unlike **Euler**, he restricted the amount of his research that he made public and in his publications, obliterated any description of how his ideas had been generated. He stuck to his motto "Few, but ripe". Like **Newton** before him, he ascribed his success in solving problems where others failed to 'always thinking about them'.

In his unpublished notes it was discovered, after his death, that Gauss had considered non-Euclidean geometry before **Lobachevsky**, quaternions before **Hamilton**, elliptic functions[370] before **Abel** and **Jacobi** as well as much of **Cauchy**'s complex variable theory. In a letter written to his friend **Bessel** in 1811, Gauss explicitly stated *Cauchy's integral theorem* (1827) and

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369 *Gauss' divergence theorem* (1839): The flux of a vector field out of a closed oriented surface equals the integral of the divergence of that vector field over the *volume* enclosed by the surface. The results parallel Stokes' theorem in that it relates an integral over a closed geometrical object (curve or surface) to an integral over a contained region (surface or volume).
Let $\mathbf{F}$ be a smooth vector field defined on Ω. Then

$$\int_\Omega \operatorname{div} \mathbf{F}\, dV = \int_{\partial\Omega} \mathbf{F} \cdot d\mathbf{S} = \int_{\partial\Omega} (\mathbf{F} \cdot \mathbf{n}) dS,$$

where $\partial\Omega$ is an oriented closed surface that bounds Ω and $\mathbf{n}$ is the outward unit normal to Ω.
This theorem arose in connection with electrostatic problems.

370 In 1799, Gauss defined the *sinus lemniscaticus* function (sin lem) $x = \text{sl}\, u$ via the inverse relation $u = \int_0^x \frac{dt}{\sqrt{1-t^4}}$. He then defined the lemniscate cosine by

remarked that he had found a fairly simple proof, and also that he knew how to make series expansion of functions of complex variable [**Laurent**, 1843]. For some reason, however, the suitable occasion for the publication of these theorems did not arise.

By 1820 he was in full possession of the main theorems of *non-Euclidean geometry* (the name is due to him), but he did not reveal his conclusions. The reason for this may be sought in the dominance of the ideas of **Kant** at that time in Germany, namely, the idea that Euclidean geometry is the only possible way of thinking about space. Gauss knew that this idea was totally false and that the Kantian system was a structure built on sand. However, he valued his privacy and quiet life and held his peace in order to avoid wasting his time on disputes with the philosophers.

The same thing happened again in the theory of elliptic functions, a very rich field of analysis that was launched primarily by **Abel** in 1827 and by **Jacobi** in 1828–1829. Gauss had published nothing on this subject and claimed nothing, so the mathematical world was filled with astonishment when it gradually became known that he had found many of the results of Abel and Jacobi before these men were born![371] Abel was spared this devastating knowledge

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cl $u = \mathrm{sl}\left(\frac{\varpi}{2} - u\right)$, where $\varpi = 2\int_0^1 \frac{dt}{\sqrt{1-t^4}}$, and showed that

$$\mathrm{sl}^2 u + \mathrm{cl}^2 u + (\mathrm{sl}^2 u)(\mathrm{cl}^2 u) = 1.$$

The equation of the John Bernoulli lemniscate (1694) is

$$(x^2+y^2)^2 - 2a^2xy = 0, \qquad \text{or} \qquad r^2 = a^2 \sin 2\theta.$$

Clearly, $\mathrm{sl}\ 0 = 0$, $\mathrm{cl}\ 0 = 1$, $\mathrm{sl}\left(\frac{\varpi}{2}\right) = 1$, $\mathrm{sl}(\varpi - u) = \mathrm{sl}\ u$, $\mathrm{cl}(\varpi - u) = -\mathrm{cl}\ u$. The graphs of sl u and cl u resemble in their appearance those of the circular functions $\sin u$ and $\cos u$. The lemniscate is the locus of points P, the product of whose distances from *two* fixed points F_1, F_2 (the foci) $2a$ units apart is constant and equals a^2.

371 **Gauss** (1818) devised a remarkable method for the numerical calculation of elliptic integrals: To begin with, Gauss (and independently, **Lagrange**) introduced (ca 1785) the concept of the *arithmetico-geometric mean* in the following way: Let two numbers $\{a_0, b_0\}$ be given. Then the arithmetic mean a_1 is defined by $a_1 = \frac{1}{2}(a_0 + b_0)$ and the geometric mean by $b_1 = \sqrt{a_0 b_0}$. He then formed the new means $a_2 = \frac{1}{2}(a_1 + b_1)$, $b_2 = \sqrt{a_1 b_1}$. By continuing this process, one obtains two series of numbers obeying the coupled recursions

$$a_{n+1} = \frac{1}{2}(a_n + b_n), \qquad b_{n+1} = \sqrt{a_n b_n}.$$

As $n \to \infty$, a_n steadily decreases and b_n steadily increases toward the *same* limit $M(a_0, b_0)$ known as the *arithmetico-geometric* mean. This follows from

by his early death in 1829, at the age of 26, but Jacobi was compelled to swallow his disappointment and go on with his work. His attention was caught by a cryptic passage in the *Disquisitiones*, whose meaning can only be understood if one knows something about elliptic functions. He visited Gauss on several occasions to verify his suspicion and tell him about his own most recent discoveries, and each time Gauss pulled a 30-year-old manuscript out of his desk and showed Jacobi what Jacobi has just shown him. The depth of Jacobi's chagrin can readily be imagined. At this point in his life Gauss was indifferent to fame, and was actually pleased to be relieved of the burden of preparing a treatise on the subject which he had long planned.

In 1832 Gauss established with **Wilhelm Eduard Weber** (1804–1891, Germany) the now-standard CGS system of units.

In 1835 he discovered that a moving charge exerts a different electric force from a charge at rest. His (unpublished) result was rediscovered by Weber in 1846. Earlier, in 1833, Gauss and his friend Weber built the first experimental electromagnetic telegraph which transmitted signals across a wire, 2000 meters long, connecting Gauss' house with his observatory.

Gauss married twice (1805, 1811) and had altogether 2 daughters and 3 sons, two of which emigrated to the U.S.A. after an extended conflict with

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the inequalities:

$$a_{n+1} - b_{n+1} = \frac{1}{2}(\sqrt{a_n} - \sqrt{b_n})^2 > 0$$

$$a_n - a_{n+1} = \frac{1}{2}(a_n - b_n) > 0$$

$$b_{n+1} - b_n = \sqrt{b_n}(\sqrt{a_n} - \sqrt{b_n}) > 0.$$

Next, Gauss established the relation

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{a_0^2 \cos^2\theta + b_0^2 \sin^2\theta}} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{a_1^2 \cos^2\theta + b_1^2 \sin^2\theta}}.$$

By applying this transformation repeatedly, he obtained

$$\lim_{n\to\infty} \int_0^{\pi/2} \frac{d\theta}{\sqrt{a_n^2 \cos^2\theta + b_n^2 \sin^2\theta}} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{M^2 \cos^2\theta + M^2 \sin^2\theta}} = \frac{\pi}{2M}.$$

Choosing $a_0 = 1,\ b_0 = k' = \sqrt{1-k^2}$, Gauss clinched the final beautiful result

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{\cos^2\theta + k'^2 \sin^2\theta}} = \frac{\pi}{2M(1,k')}$$

which provides a powerful means of constructing a table of values of $K(k)$.

him. His second wife died in 1831 and his youngest daughter kept the house for him until his death in 1855. Gauss did not seem to relish travel: In the last 27 years of his life he slept away from his observatory only once.

Gauss lived in a period of extraordinary political and social upheavals, even when measured by the standards of our fast-moving and eventful age. He was 12 years old when the French Revolution broke out (1789), 29 when the 1000-year old Holy Roman Empire was dissolved (1806), 38 when Napoleon was defeated (1815), and over 70 when Germany had its own Liberal Revolution (1848). During the same period, the so-called first Industrial Revolution took place, with its lasting and incisive effects on everyday life and on the political and social order. All this affected Gauss' life in an explicit and tangible way.

Gauss' contemporaries in Germany were: **Ludwig van Beethoven** (1770–1827), **Franz Schubert** (1797–1828), **Arthur Schopenhauer** (1788–1860), **Georg Wilhelm Friedrich Hegel** (1770–1831), **Johann Wolfgang von Goethe** (1749–1832), **Heinrich von Kleist**[372] (1777–1811) and **Caspar David Friedrich** (1774–1840).

[372] His father's uncle, **Ewald Georg Christian Johann von Kleist** (1700–1748, Germany) invented the *Leyden jar* (1745). It is a glass jar, partially filled with water and containing a nail projecting from its cork stopper, basically an early version of an electrical capacitor.

Worldview XVII: Carl F. Gauss

* * *

"Few, but ripe" (his motto)

* * *

If others would but reflect on mathematical truths as deeply and as continuously as I have,, they would make my discoveries.

* * *

I confess that Fermat's Theorem as an isolated proposition has very little interest for me, because I could easily lay down a multitude of such propositions, which one could neither prove nor dispose of.

[A reply to Olbers' attempt in 1816 to entice him to work on Fermat's Theorem.]

* * *

There are problems to whose solution I would attach an infinitely greater importance than to those of mathematics, for example touching ethics, or our relation to God, or concerning our destiny and our future; but their solution lies wholly beyond us and completely outside the province of science.

* * *

I have had my results for a long time: but I do not yet know how I am to arrive at them.

* * *

You know that I write slowly. This is chiefly because I am never satisfied until I have said as much as possible in a few words, and writing briefly takes far more time than writing at length.

* * *

God does arithmetic.

* * *

We must admit that, while number is purely a product of our minds, space has a reality outside our minds, so that we cannot completely prescribe its properties a priori.

Letter to Bessel, 1830

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I mean the word proof not in the sense of the lawyers, who set two half proofs equal to a whole one, but in the sense of a mathematician, where half proof = 0, and it is demanded for proof that every doubt becomes impossible.

* * *

Gauss' Class-Number Conjecture[373] (1801–1983)

Consider Gaussian integers of the form $a + b\sqrt{-N}$, *where* N *is some positive integer other than* 1 *and* (a, b) *are integers or half-integers. The question may arise as which values of* N *result in* unique factorization. *For* $N = 1, 2, 3$ *we do get it but for* $N = 5$ *we do not, since, for example* $6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$. *In Gauss' time,* 9 *values of* N *were known for which the* system of numbers $a+b\sqrt{-N}$ *has a unique factorization. They are* $N_1 = 1, 2, 3, 7, 11, 19, 43, 67, \mathit{163}$.

Despite considerable efforts by Gauss (1801) and others in the decades that followed, no one was able to find higher values of N_1. *However, in* 1952, *the Swiss mathematician* **Kurt Heegner** *proved that the special number 163 is the largest value of* N *for which the number system* $a + b\sqrt{-N}$ *allows unique factorization! [An independent and different proof was given in* 1967 *by* **Harold Stark** *(U.S.A.) and* **Alan Baker** *(England).]*

We turn to number systems that fail *the unique factorization. One can still group them into different classes according to the number of ways there are for factoring numbers in the system into* primes *in that system. We assign to each such class a figure of merit, called the* class number *(Gauss, 1801), and denote it by* $h(N)$. *Thus* $h(N_1) = 1$ *is given to the above class of values* N_1 *for which unique factorization holds.*

The class number $h(N_2) = 2$ *is assigned to the class where unique factorization just fails:* $N_2 = 5, 6, 10, 13, \ldots$. *To* $h(N_3) = 3$ *corresponds the series* $N_3 = 23, 31, 59, \ldots$ *and to* $h(N_4) = 4$ *holds for* $N_4 = 14, 17, 41, \ldots$ *and so on.*

In Article 303 *of his* Disquisitiones Arithmeticae, *Gauss described some extensive computations of class numbers, and observed that for each class number* k, *there seemed to be a largest value for* N_k. *Thus,* $N_1 = 163$, $N_2 = 427$, $N_3 = 907$ *are maximal in their respective classes. But Gauss was able neither to confirm that any of these values really was maximal, nor to prove that there always was a largest* N, *though he conjectured that this was, nevertheless, the case.*

The class number problem, which assumes the truth of Gauss' conjecture, is to determine for each class number k *the largest* N *for which* $h(N) = k$. *In*

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373 To dig deeper, see:

- Stark, N.M., *An Introduction to Number Theory*, The MIT Press, 1987, 344 pp.

1983, **Don Zaiger** *(U.S.A.) and* **Benedict Gross** *(U.S.A.) announced that they had proven Gauss' conjecture, and the ' class-number conjecture' was finally put to rest. Yet this subject still has a number of interesting sidelines.*

Euler *discovered in 1772 that the formula* $f(n) = n^2 + n + 41$ *yields primes for all values of* n *from zero to 39. No other quadratic formula has been discovered which produces as many prime numbers. Of the first 10 million values, the proportion of primes is about one in three — far greater than for any other quadratic formula.*

It turns out that the roots of $f(n) = 0$ *are* $-\frac{1}{2} \pm \frac{1}{2}\sqrt{-163}$*. Is it a coincidence that the number 163 arises again in connection with primes? Yet this is not all: It was found that the number* $e^{\pi\sqrt{163}}$ *differs from an integer by less than* 10^{-12}*, namely*

$$e^{\pi\sqrt{163}} = 262\ 537\ 412\ 640\ 768\ 743\ .\ 999\ 999\ 999\ 999\ 250\ \cdots .$$

Incidentally[374]*, the very simple* $2n^2 + 29$ *found by* ***Legendre*** *in 1798 generates 29 primes for* $n = 0, 1, 2, \ldots, 28$*.*

Statistics Comes of Age – The Normal Distribution and the Method of Least Squares (1795–1827)

Statistics is the system of computation that deals with the collection, classification, analysis and interpretation of numerical data. By using the theory of probability it aims at discovering laws that govern complex physical

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374 In 1752, **Christian Goldbach** (1690–1764, Germany) proved that no polynomial with integer coefficients can yield a prime for all integer values of x. The proof of this statement is elementary: Let $g(x) = a_0 + a_1x + a_2x^2 + \cdots + a_kx^k$, with a_i integers and $g(m)$ prime for any integer m. Then, if p is a prime derived from this formula when $x = m$, we have $p = g(m)$. Likewise, let q be another prime such that $q = g(m + np)$. Clearly $q = p + pf(m, n)$ with f integer, and thus q is composite for some integer m. This shows that no such polynomial $g(x)$ can exist.

as well as biological and social systems. As a science, statistics began in Germany in the 18^{th} and 19^{th} centuries when governments used statistics to count their citizens and to collect taxes. Today, statistics helps all sciences to deal with masses of facts and it makes vital contributions to business and industry; advertising, finance, insurance, manufacturing, retailing, and many other fields depend on statistics. It helps politicians plan their campaigns, and the use of statistics forms the basis of public-opinion polls.

I. History[375]

Over the two centuries, from 1700 to 1900, statistics underwent a simultaneous horizontal and vertical development: horizontal in that the method spread among disciplines, from astronomy and geodesy, to psychology, to biology, and to social sciences, being transformed in the process; vertical in that the understanding of the role of probability advanced as the analogy of games of chance *gave way to probability models for* measurements*, leading finally to* statistical inference.

The roots of modern statistics, since 1650, encompassed the following disciplines and scientists:

- *The works of mathematicians on* probability: ***P. Fermat*** *(1601–1665),* ***B. Pascal*** *(1623–1662),* ***C. Huygens*** *(1629–1695),* ***Jacob Bernoulli*** *(1654–1705),* ***A. de Moivre*** *(1667–1754),* ***Daniel Bernoulli*** *(1700–1782),* ***T. Bayes*** *(1701–1761),* ***P.S. Laplace*** *(1749–1827),* ***S.D. Poisson*** *(1781–1840).*

- *The works of* astronomers and geodesists *on the solutions of overdetermined set of equations:* ***L. Euler*** *(1707–1783),* ***Tobias Mayer*** *(1723–1762),* ***Ruggiero Boscovich*** *(1711–1787).*

- *The ideas of mathematicians concerned with the* errors of measurement *and* combination of observations: ***C.F. Gauss*** *(1777–1855),* ***Legendre*** *(1752–1833).*

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375 For further reading, see:

- Stigler, S.M., *The History of Statistics*, Harvard University Press, 1986, 410 pp.

- Larsen, R.J. and M.L. Marx, *An Introduction to Mathematical Statistics and its Applications*, Prentice-Hall: Englewood Cliffs, NJ, 1981, 596 pp.

- *The labors of social scientists to extend a calculus of probability to the social sciences:* ***John Graunt*** *(1620–1674),* ***A. Quetelet*** *(1796–1874),* ***A.A. Cournot*** *(1801–1877),* ***Francis Galton*** *(1822–1911),* ***Wilhelm Lexis*** *(1837–1814),* ***F.Y. Edgeworth*** *(1845–1926),* ***Karl Pearson*** *(1857–1936),* ***G.U. Yule*** *(1871–1951),* ***W.S. Gosset*** *(1876–1937),* ***Ronald Fisher*** *(1890–1962).*

Early work in mathematical probability was motivated by problems in the social sciences, annuities, insurance, meteorology, and medicine, but the paradigm for the mathematical development of the field was the analysis of games of chance. Concepts applied there were applied in astronomy. Thus, the consideration of games of chance led to the first mathematical treatment of the quantification of uncertainty.

By the end of the 17^{th} *century the mathematics of many simple games of chance was well understood and widely known.* ***Fermat***, ***Pascal***, ***Huygens***, ***Leibniz*** and ***Jacob Bernoulli*** *— all had examined the ways in which the mathematics of permutations and combinations could be employed in the enumeration of favorable cases in a variety of games of known properties. These works had been concerned with* a priori *computations: given an urn known to contain* Q *white balls and* P *black balls, the chance of a white ball being drawn is* $\frac{Q}{P+Q}$. ***Jacob Bernoulli*** *(1713) was the first to consider the* inverse problem *or the* a posteriori *question: to determine* P *and* Q *from observations of the outcomes of a game.*

In his search for a solution to this problem, Bernoulli developed (1713) the theory of the binomial distribution *(alias 'Bernoulli's Formula'). The physical picture was that of a sequence of experiments satisfying the following conditions (known as 'Bernoullian trials'):*

- *For each experiment, the possible results are classified as either success or failure.*
- *The probability of success is the same for every experiment.*
- *Each trial is independent of all the others.*

Under these conditions, Bernoulli showed (on the basis of the Newtonian binomial theorem) that the probability *of exactly* k *successes in* n *Bernoullian trials is* $C_k^n p^k q^{n-k}$ *where* $q = 1 - p$ *and* $C_k^n = \binom{n}{k}$. *It then followed that the probability of* at least k *successes in* n *trials is* $\sum_{s=k}^{n} \binom{n}{s} p^s q^{n-s}$, *while the probability of* at most k *successes in* n *trials is* $\sum_{s=0}^{k} \binom{n}{s} p^s q^{n-s}$.

The next major step was made by ***de Moivre****. In a work that started in 1721 and culminated in 1733, he succeeded in approximating the terms of a binomial expansion and derive what we now call the* normal approximation *to the binomial distribution. In achieving this goal he used his own approximation to* $n!$ *(factorial* n*) ahead of* ***Stirling*** *(1730). He also recognized that the root mean square deviation is proportional to* $\sqrt{n}$*, and calculated values of the normal integral*[376] $\int_{-a}^{a} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$ *as approximations to the binomial probabilities (e.g. in the case* $p = q = \frac{1}{2}$*)* $P(\frac{1}{2}n - \frac{1}{2}a\sqrt{n} \le X \le \frac{1}{2}n + \frac{1}{2}a\sqrt{n})$.

Looking at de Moivre's work from a perspective of 270 years it is easy to appreciate the profound influence it had upon later mathematical developments and the solution of a wide variety of scientific problems. Yet his contemporaries greeted these advances with indifference and missed the potential in this masterful work. No application or extension of these ideas occurred before 1774, the year that ***Laplace*** *revisited the inverse probability problem of Jacob Bernoulli.*

Here, the analytic superiority of Laplace enabled him to apply probability to statistical inference *for the first time and succeed where Bernoulli and Thomas Bayes had failed.*

In another vein, astronomers and geodesists were struggling with the solution of large sets of overdetermined equations. The outcome of this endeavor was a novel idea that had a profound effect on the theory of statistics.

The method of least squares *was the dominant theme of 19*th *century mathematical statistics. It was to statistics what the calculus had been to mathematics a century earlier. Indeed, disputes on the priority of its discovery signaled the intellectual community's recognition of the method's value. This "calculus of observations", like the calculus of mathematics, did not spring into existence without antecedents, and the exploration of its subtleties and potential took over a century. Throughout much of this time statistical methods were commonly referred to as "the combination of observations".*

The genesis of the method of least squares is anchored in the three major physico-astronomical problems of the 18th *century:*

- *To determine and represent mathematically the motions of the moon.*
- *To account for an apparently nonperiodic (secular) perturbations that had been observed in the motions of the planets Jupiter and Saturn.*

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[376] $P(a) = \int_{-a}^{a} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$ is the probability for a measurement to occur in an interval within $a\sigma$ of the *mean*, where σ is the *variance*. In particular $P(0.6745) = 1/2$ is the median deviation from the mean (probable error).

- *To determine the shape (figure) of the earth.*

These problems involved astronomical observations and the theory of gravitational attraction, and they all presented intellectual challenges that engaged the attention of many of the ablest mathematical scientists of the period.

The chain of development that led to the method of least squares began with **Johann Tobias Mayer** *(1723–1762, Germany), who in 1747 undertook the study of the complex irregular minor perturbations of the moon's motion.*[377] *The specific work of Mayer that most influenced statistical practice was his study (1750) of the librations of the moon. He made numerous observations of the position of several prominent lunar features and in his memoir he showed how these data could be used to determine various characteristics*

[377] In the 18^{th} century the problem of accurately accounting for these minor perturbations in the moon's movement, either by a mathematical formula or by an empirically determined table describing *future lunar positions*, was of great scientific, commercial and even military significance. Its *scientific* importance lay in the general desire to show that Newtonian gravitational theory can account for the movement of our nearest celestial neighbor if allowance is made for the attraction of other bodies (such as the sun), for periodic changes in the earth's and the moon's orbits, and for the departure from sphericity of the shapes of the earth and the moon. But it was the potential *commercial* and *military* usefulness of a successful accounting of the moon (as an aid to navigation) that prompted the widespread attention the problem received. Over the previous 19 centuries, from **Hipparchos** and **Ptolemy** to **Newton** and **Flamsteed**, the linked development of theoretical and practical astronomy had played a key role in freeing ship's navigation from a dependence upon land sightings as a way of determining the ship's position. The developments of better nautical instruments (including the sextant, 1731) and a more accurate understanding of astronomical theory, increasingly enabled navigators to map their ships' courses across previously trackless seas.

By 1700, it had become possible to determine ship's *latitude* at sea with relative precision by the *fixed stars*, simply by measuring the angular elevation of the celestial pole above the horizon. The determination of *longitude*, however, was not so simple. Indeed, in 1714 England established the "commissioners for the discovery of longitude at sea", a group that by 1815 had disbursed £101,000 in prizes and grants to achieve its goal. The two most promising methods of ascertaining longitude at sea were the development of an *accurate clock* (so that Greenwich time could be maintained on shipboard and longitude determined by the comparison of the fixed stars' positions and Greenwich time) and the creation of *lunar tables* that permitted the determination of Greenwich time (and thus longitude) by *comparison of the moons position and the fixed stars.*

of the moon's orbit: all in all he ended up with an overdetermined *system of 27 equations for the calculations of 3 unknown parameters.*

Mayer divided his equations into three groups of nine equations each, added each of the three groups separately, and solved the resulting three linear equations for their unknowns. He then made a numerical estimate of the accuracy of his empirical determination. This way of combining observations and making an error assessment was remarkable for this time.

Mayer's story of statistical success showed that there was a potential gain to be achieved through the combination of observations. Yet the discovery of the method of least squares was not possible in the intellectual climate of 1750 and certain conceptual barriers had to be crossed before this climate became sufficiently supportive for the later advances.

In a widely read treatise, Astronomie *(1771),* ***Joseph Jerome Lalande*** *presented an extensive discussion of Mayer's work for the specific purpose of explaining how large number of observational equations could be combined to determine unknown quantities. It was this exposition that called the method to the attention of contemporary astronomers.: Boscovich, Laplace and Legendre. After it was decided that the earth was oblate (1735) it remained only to establish the size of the oblateness (ellipticity) because different pairs of arcs gave different results.*

In 1760, ***Ruggiero Giuseppe Boscovich*** *introduced a statistical procedure for resolving measurements of the length of a meridian arc. Boscovich was aware, as others before him had been, that to obtain an accurate determination of the figure of the earth it would be necessary to compose measurements widely separated in latitude, as even small errors made in proximate arc measurements would be greatly exaggerated in any pairwise combination of them.*

He thus focused his attention on only five determinations that were made at well-separated locations and were likely to be accurate. He then calculated the inverse ellipticity *and* polar excess *(the amount by which a degree at the pole exceeds a degree at the equator) for all possible ten pairs. He next focused upon the discrepancy between his* average value *of polar excess and the ten components that made up the average[378], coming forth with two conditions that would lead to best choice of the results:*

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[378] As his average yielded the value of $\frac{1}{155}$ for the ellipticity, Boscovich tried to improve this value by rejecting pairs which looked "different from the others". It finally occurred to him that he must subject this arbitrary rejection of data to certain principles.

(i) Since positive and negative error are equally likely, the sum of positive corrections should be equal to the sum of negative corrections.

(ii) The sum of the corrections, taken without regard of sign, was to be a minimum.

In 1770, Boscovich appended the French translation of his 1760 paper with a geometric description of an algorithm which was based on the above two principles. Through this he calculated the ellipticity as $\frac{1}{230}$, in close proximity to Newton's own value. Boscovich gave no further development of the method, no analytic formulation, and no application to problems other than the figure of the earth. The method might thus have faded into obscurity had not a brief reference to its existence, in a 1772 review of the 1770 translation, caught the eye of ***Pierre Simon Laplace****.*

In the course of a memoir on the perturbation of the motions of Saturn and Jupiter (1787), Laplace proposed and extension of Mayer's method of reconciling inconsistent linear equations. In his epochal work, Laplace finally laid to rest what was then a century-old problem by showing that the perturbations were in fact periodic with a very long period.

In 1810, Laplace produced a major result in probability theory, known today as the Central Limit Theorem (CLT)*. Roughly speaking, it states that whenever a random variable X may be expressed as a sum of a very large number of independently varying random variables, then the probability density of X is approximately normal. Combined with his unrivaled ability to derive asymptotic approximations to integrals, the CLT enabled Laplace to show that quite general sums or averages had distributions well approximated by the* normal curve*.*

Thus, two major avenues of attack on statistical problems were at the disposal of mathematicians in 1810:

- *Legendre's method of least squares (1805) and Laplace's way of combining observations in complex situations (1787).*
- *The probability apparatus developed by de Moivre and Laplace for the analysis of binomial distributions and its limiting case of the normal distribution.*

What was missing was any connection between these two lines of work. In 1809 ***Gauss*** *provided the key, and within two years a remarkable synthesis was achieved.*

Laplace must have encountered Gauss' work soon after April 1810, and it struck him like a bolt. Before seeing Gauss' work Laplace had not seen

any connection between his limit theorem and the method of least squares, but almost immediately afterward he could see how it all fit together: If the errors of Gauss' formulation were themselves random variables, then the limit theorem implied they should be approximately distributed as what would later be called normal, or Gaussian curve. And once Gauss' choice of curve was given a rational basis, the entire development of least squares fell into place, just as Gauss had showed.

One of the Gauss' most efficient tools in his research was the method of least squares. When he first developed it (1795), he did not consider it very important. Although formal priority belongs to **Legendre** *(1805), it seems that the motivation, deduction, and systematic application of the method of least squares is more interesting than the problem of deciding who happened to discover, use, and publish it first.*

Suppose we try to measure some quantity x and make M measurements x_i. We do not see x but only measurements x_i with errors of measurement ε_i; that is, we observe $x_i = x + \varepsilon_i, \quad i = 1, 2, ..., M$. We will regard the residuals ε_i as "noise" and call x the true value, whatever that may mean in a situation in which it cannot be measured directly.

The principle of least squares states that the best estimate $\hat{x}$ of x is that number which minimizes the sum of the squares of the deviations of the data from their estimate,

$$f(\hat{x}) = \sum_{i=1}^{M} \hat{\varepsilon}_i^2 = \sum_{i=1}^{M} (x_i - \hat{x})^2. \tag{1}$$

In the final analysis, the usefulness of this principle rests on how useful the results turn out to be in practice and how easy it is to use.

This principle is equivalent to the assumption that the average (sample mean)

$$\bar{x} = \frac{1}{M} \sum_{i=1}^{M} x_i \tag{2}$$

is the best estimate. To prove this equivalence, we first show that the least-squares principle leads to the average. We regard

$$f(\hat{x}) = \sum_{i=1}^{M} (x_i - \hat{x})^2 \tag{3}$$

as a function of $\hat{x}$ to be minimized. Applying the usual calculus rule $\frac{df}{d\hat{x}} = 0$, (3) yields at once $\hat{x} = \bar{x}$ as given in (2). Thus $\bar{x} = \hat{x}$ minimizes the sum of

squares of the residuals. We note also that $\frac{d^2 f}{dx^2} = 2M > 0$ *and hence we have an absolute minimum.*

We have now proved that the principle of the least squares and the choice of the average as the best value are equivalent.

The maximum-likelihood estimator *of an unknown parameter is motivated by the Bayes-type notion that we should select the value that maximizes the likelihood of observing it. If the sample is from some probability density* $\varphi(x;\theta)$, *then the likelihood of the value* θ *for* x *is the product of the individual (independent) observations*

$$L(\theta) = \varphi(x_1;\theta)\varphi(x_2;\theta)\cdots\varphi(x_m;\theta). \tag{4}$$

In the case that the errors come from a normal distribution

$$\frac{k}{\sqrt{\pi}} e^{-k^2(x-\theta)^2},$$

then this leads to

$$L(\theta) = \frac{k^M}{\pi^{M/2}} e^{-k^2 \sum (x_i-\theta)^2}.$$

When this is maximized θ *clearly is the solution to a least-squares problem, and is the maximum likelihood estimator,*

$$\theta = \frac{1}{M}\sum_{i=1}^{M} x_i = \overline{x}. \tag{5}$$

Thus lest squares can be derived from the normal distribution via the (assumed) maximum-likelihood estimator.

In his decisive papers (1821, 1823)[379] *Gauss defines the function* $\varphi(x)$ *as the relative frequency of errors in the observations* X. *Then* $\varphi(x)dx$ *expresses the probability of the error lying between* x *and* $x + dx$. *The function* φ *is required to fulfill the two conditions:*

$$\int_{-\infty}^{\infty} \varphi(x)dx = 1\ ; \quad \int x^2\varphi(x)dx \quad \text{attains a minimum.}$$

These conditions expresses the idea that the squares of the error *is its most suitable weight. This is where Gauss' approach differs from that of Laplace,*

[379] In his several publications Gauss derived the method in substantially different ways. His most mature approach was developed in the two papers "*Theoria combinationis observationum erroribus minimis obnoxiae*", *I and II*.

who earlier tried to use the absolute value of an error *for its weight. This is why Gauss' method is called the method of least squares; computationally, it is clearly superior to Laplace's original method.*

After developing the theoretical basis of his method, a suitable distribution function $\varphi(x)$ had to be found. In general, the distribution of errors will not be known in advance. After some heuristic preparations, Gauss introduced the experimental density[380] $\varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$ *as a particularly natural law according to which errors of observation occur*[381]*. This law was found to represent the errors of observations in astronomy and other physical sciences remarkably well. Hence its name "Law of Errors". This law occupies a central position in statistical theory.*

380 This probability density is often called the *Laplacian* density by the French and the *Gaussian* density by the Germans. *Normal* is British usage. Probability theorists and statisticians use 'normal', while physicists and engineers often use 'Gaussian'. The least-squares criterion is widely used, and often believed to be the "right one" to use.

There is a saying that mathematicians believed that it is a physical principle while physicists believe that it is a mathematical principle.

Either we can assume the principle or we can assume some other principles and deduce that of least squares – something must be assumed in any case. There appears to be a widespread belief that the principle of least squares implies the normal law of errors. This belief is false. Another belief sometimes encountered is that the normal law is "a law of nature". Certainly, the normal law has been found in practice to be a useful model in many applications. Deviations from it usually occur from having more values in the "tail" of the distribution (when x is large) than the model indicates there should be. The reason for this is that often there is a small effect which has a wide variability. In such cases, a mixture of two normal distributions with different parameters sometimes is useful. The theory of *quality control* is , in part, based on the observed excesses in the tails.

381 It can be easily shown that if (4) and (5) are *assumed*, then the Gaussian Law follows. To see this, one differentiates (4) logarithmically, uses (5) and solves the ensuing differential equation $\frac{\varphi'(x)}{\varphi(x)} = \lambda x$, where λ is a constant.

Gauss' 'Clock-arithmetic' [382]

Consider a clock, numbered (in unorthodox fashion) with the hours $0, 1, 2, \ldots, 11$*. Such a clock has its own peculiar arithmetic. For example, since three hours after 5 o'clock is 8 o'clock, we could say that* $3 + 5 = 8$*, as usual. But 3 hours after 10 o'clock is 1 o'clock, and 3 hours after 11 o'clock is 2 o'clock; so by the same token,* $3 + 10 = 1$ *and* $3 + 11 = 2$*. Not so standard!*

Nevertheless, this 'clock arithmetic' has a great deal going for it, including almost all of the usual laws of algebra. Following Gauss, we describe it as arithmetic to the modulus *12, and replace '=' by the symbol '≡' as a reminder that some monkey-business is going on. The relation '≡' is called a* congruence*. In arithmetic* modulo *(that is, to the modulus)* 12*, all multiples of* 12 *are ignored. So* $10 + 3 = 13 \equiv 1$ *since* $13 = 12 + 1$ *and we may set* $12 \equiv 0$.

If a scientist is performing an experiment in which it is necessary for him to keep track of the total number of hours that have elapsed since the start of the experiment, he may label the hours sequentially $1, 2, 3$*, etc. When 41 hours have elapsed, it is 41 o'clock "experimental time." How does he reduce experiment time (e.t.) to ordinary time? If zero hours e.t. corresponded to midnight, his task is easy: he simply divides by* 12 *and the* remainder *is the time of day.* 41 *e.t. is thus 5 o'clock, because* 12 *goes into* 41 *with a remainder of* 5*;* 41 *is congruent to* 5 *modulo* 12*:*

$$41 \equiv 5 (\text{mod } 12).$$

For the purpose of telling the time of day it is not necessary to know how many *times* 12 *is contained in* 41*, but only the remainder,* 5*. Of course if one wants to distinguish between a.m. and p.m., then it would be better to divide by* 24*. We then find that* 41 *is congruent to* 17 *modulo* 24*. This means that* 41 *e.t. is 17 hours (military time), namely 5 p.m.*

Any other number n *may be used as the modulus: now multiples of* n *are neglected. The resulting arithmetical system consists only of the numbers*

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[382] For further reading, see:

- Deskins, W.E., *Abstract Algebra*, Dover: New York, 1955, 624 pp.
- Childs, L.N., *A Concrete Introduction to Higher Algebra*, Springer-Verlag, 1995, 522 pp.
- Littlewood, D.E., *A University Algebra*, Dover Publications: New York, 1970, 324 pp.

$0, 1, 2, \ldots, n-1$, and has its own multiplication and subtraction rules, as well as addition. Division is less pleasant: but if n is a prime *then it is possible to divide by any non-zero number. For example, modulo 7 we find that $3/5 = 2$, since $2 \cdot 5 = 10 \equiv 3$. This bizarre arithmetic was introduced by Gauss because it is ideal for handling questions of* divisibility. *Number theory has used it for this purpose ever since.*

Every integer may be expressed as the multiple of a lower integer plus a remainder. The number n can accordingly be expressed as

$$qm + r$$

where r is the remainder when n is divided by m; qm begin a multiple of m. Where n is expressed as a multiple of m, and m is a factor of n, the remainder will obviously be equal to zero.

The number n will not, however, be the only number which gives the remainder r when divided by m, and this is the basic fact upon which the theory of congruences was founded by Gauss.

The numbers 40 and 64, for instance, give the same remainder of 4 when divides by 12. This is another way of saying that both numbers are equal to multiples of 12 added to 4. They both, therefore have something in common in their relationship to the number 12. In mathematical parlance the latter number is called the 'modulus' with respect to which the two other numbers have similar properties, and these two numbers are said to be 'congruent' to each other for that modulus.

Thus, if

$$a = pm + r$$

and

$$b = qm + r$$

then a is said to be congruent to b for the modulus m, and the relationship is expressed in either of the equivalent forms:

$$\left.\begin{array}{c} a \equiv b \pmod{m}\ ; \quad a - b = km\ ; \quad a - b \equiv 0 \pmod{m} \\ a \equiv b \end{array}\right\} \qquad (1)$$

where k may be positive, zero, or negative.

For example $10 \equiv 3 \pmod 7$, and $13 \equiv 28 \pmod 5$. Also $a \equiv b \pmod p$ if $a = b + kp$.

Sometimes the notation is used in a wider sense: the general solution, in radian units, of $\tan\theta = \tan\alpha$ may be written $\theta \equiv \alpha \pmod \pi$.

It will be seen that the algebra of congruences is so like that of ordinary integer arithmetic that no inconvenience arises from using the same notation

for both. The explicit statement of the modulus at each stage of the work is not necessary when the same modulus is used throughout.

The congruence $a \equiv r \pmod{m}, \quad 0 \le r < m$ *means that*

$$a = r + km\ , \quad r = 0, 1, 2, \dots, m-1.$$

The set of all possible m *remainders (including zero) constitute a complete residue system (mod* m*). On the other hand, since* k *is an arbitrary integer (positive or negative), all integers will group into* m *classes, known as residue classes (mod* m*), where each class is characterized by one of the remainders of the residue system* $r = 0, 1, \dots, m-1$*. For a given* r *and* m*, these classes will be*

$$a = km,\ 1+km,\ 2+km, \dots, (m-1)+km.$$

No number from one class is congruent to any number from another class.

For example for $m = 2, \quad r = 0, 1$ *there are two classes:*

1st class:	$a = 2k$	$\dots,\ -4,\ -2,\ 0,\ 2,\ 4, \dots$	*(even)*
2nd class:	$a = 1 + 2k$	$\dots,\ -3,\ -1,\ 1,\ 3,\ 5, \dots$	*(odd)*

For $m = 3,\ r = 0, 1, 2$

1st class:	$a = 3k$	$\dots,\ -6,\ -3,\ 0,\ 3,\ 6, \dots$
2nd class:	$a = 1 + 3k$	$\dots,\ -5,\ -2,\ 1,\ 4,\ 7, \dots$
3rd class:	$a = 2 + 3k$	$\dots,\ -4,\ -1,\ 2,\ 5,\ 8, \dots$

Clearly, all numbers in a residue class have the same greatest common divisor with the modulus m*.*

Consider next the case $m = 5, \quad r = 0, 1, 2, 3, 4$ *with the corresponding residue classes*

class	*residue class*
$a = 5k$	$\dots, -10,\ -5,\ \boxed{0},\ 5,\ 10, \dots$
$a = 5k + 1$	$\dots, -9,\ -4,\ \boxed{1},\ 6,\ 11, \dots$
$a = 5k + 2$	$\dots, -8,\ -3,\ \boxed{2},\ 7,\ 12, \dots$
$a = 5k + 3$	$\dots, -7,\ \boxed{-2},\ 3,\ 8,\ 13, \dots$
$a = 5k + 4$	$\dots, -6,\ \boxed{-1},\ 4,\ 9,\ 14, \dots$

Where the squares indicate a particular choice of a set of 5 numbers, no two of which are congruent to each other. This can be done in an infinite number of ways by picking just one number from each class. Such a set will be referred to as a complete residue system. Thus, instead of the canonical system $r = 0, 1, 2, 3, 4$*, we may select* $r = 5, 6, 7, 8, 9$*, or (as above)* $r = 0, 1, 2, -2, -1$*. In general, we shall say that the* m *numbers* $\{a_1, a_2, \dots, a_m\}$ *form a complete*

system of residues if every number is congruent to a_i (only one); each residue a_i then represents its class in mod-m arithmetic.

The word 'congruence' can be used synonymously with 'residue class'.

Returning to the basic logic of clock arithmetic, we next consider the operations of addition and multiplication of congruences. One can easily verify the following addition and multiplication tables for modulo 5 *arithmetic.*

Addition mod 5

	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Multiplication mod 5

	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

The following properties of congruences are obvious from their definition:

- *Symmetry:* when $a \equiv b \pmod{m}$ then $b \equiv a \pmod{m}$.
- *Transitivity:* $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$ imply $a \equiv c \pmod{m}$.
- *Residue classes of the same modulus may be added, subtracted, multiplied by an arbitrary integer and multiplied together unambiguously. Namely, if* $a \equiv b \pmod{m}$ and $a' \equiv b' \pmod{m}$ *and if* (q, r) *are integers, then*

$$qa + ra' \equiv qb + rb' \pmod{m}$$
$$aa' \equiv bb' \pmod{m}$$
$$a^n \equiv b^n \pmod{m} \qquad [(a^n - b^n) \text{ is always divisible by } a - b]$$

- *If* $a \equiv b \pmod{m}$ $f(a) \equiv f(b) \pmod{m}$ *for polynomials of integer coefficients* $f(x)$.
- *If any polynomial* $f(x)$ *and* $g(x)$ *have congruent coefficients of corresponding powers modulo* m, *and* $a \equiv b \pmod{m}$, *then* $f(a) \equiv g(b) \pmod{m}$.

The notion of congruence began with **Euler** *(1783). In his treatise Disquisitiones Arithmeticae,* **Gauss** *(1801) developed the systematic algebra of congruences, treating congruence polynomials of the n-th degree with a prime modulus. He showed that congruences w.r.t. the same modulus can be treated like ordinary Diophantine equations: they can be added, subtracted and multiplied, and one can ask for a solution of congruences involving unknowns, e.g.*

$$Ax^n + Bx^{n-1} + \cdots Mx + N = 0 \pmod{p},$$

where p is a prime not dividing A. Gauss proved that this equation cannot have more than n noncongruent roots. A famous example is the Fermat's Little Theorem

$$a^{p-1} - 1 \equiv 0 \pmod{p},$$

where p is prime and a is not a multiple of p. Gauss' book was indeed a new way of looking at old things, introducing the concept of residue classes. *In his own words:*

> *"If a number A divides the difference of two numbers B and C,* B and C are called *congruent* with respect to A, and if not, *incongruent. A is called the* modulus; *each of the numbers B and C* are *residues* of each other in the first case, and *non-residues* in the second."

Does it seem strange that Gauss should write a whole book about the implication of $A \mid (B - C)$? It surely is not clear a priori *why this group of symbols should be worthy of such protracted attention. In fact, these opening sentences are completely unmotivated and hardly understandable, except in the light of historical perspective. But in that light, the time was ripe for such an investigation. Gauss may not have been aware of the underlying structure of the works of Fermat and Euler that evolved from their preoccupation with perfect and Mersenne numbers. But his interest in* periodic decimals *called for a new notation and new* notions *to handle an* algebra of ambiguity *and an* arithmetic *of* remainders.

The power of the congruence algebra is manifested in the following simple examples:

I. *Since $(2n+1)^2 = 4n(n+1) + 1$, every even power of any odd number is congruent to $1 \pmod{8}$. This modulus partitions the integers into 8 residue classes, all the odd numbers being congruent to one of $1, 3, 5, 7$. If we square these four possible congruences, we find that every odd square is congruent to one of the numbers $1, 9, 25,$ or $49 \pmod{8}$. But these all happen to be $\equiv 1 \pmod{8}$. Raising the original residue to an arbitrary even power will not change this congruence.*

II. *A number is divisible by* 3 *or* 9 *if and only if the sum of its digits is divisible by* 3 *or* 9*. For let*

$$N = a_0 + a_1 \cdot 10 + a_2 \cdot 10^2 + \cdots + a_n \cdot 10^n.$$

Multiplying the congruence $10 \equiv 1 \pmod{3 \text{ or } 9}$ *repeatedly, one obtains* $10^k \equiv 1 \pmod{3 \text{ or } 9}$ *and therefore* $a_k \cdot 10^k \equiv a_k$ *for* $k = 1, 2, \ldots, n$*. Then* $N \equiv a_0 + a_1 + a_2 + \cdots + a_n \pmod{3 \text{ or } 9}$*. This means that* N *is divisible by* 3 *or* 9 *iff the sum of its digits is so divisible.*

In a similar way the congruences $10 \equiv -1 \pmod{11}$, $10^k \equiv (-1)^k \pmod{11}$ *yield* $N \equiv a_0 - a_1 + a_2 + \cdots \pmod{11}$*. Hence an integer is divisible by* 11 *iff the sum of its digits with alternating signs is divisible by* 11*.*

III. *Prove that* $999,999$ *is divisible by* 7*:*

$$\begin{gathered} 999,999 = 10^6 - 1 \\ 10 \equiv 3 \pmod{7} \\ 10^6 \equiv 3^6 \pmod{7} \equiv (3^2)^3 \pmod{7} \equiv 9^3 \pmod{7} \\ 9 \equiv 2 \pmod{7} \quad \therefore 9^3 \equiv 2^3 \pmod{7} \equiv 1 \pmod{7} \end{gathered}$$

$$\therefore 10^6 \equiv 1 \pmod{7} \quad \therefore 10^6 - 1 \equiv 0 \pmod{7}.$$

IV. *Prove that* $2^{11} - 1$ *has* 23 *as one of its factors; The steps are:*

$$\begin{gathered} 2^5 = 32 \equiv 9 \pmod{23} \\ 2^{10} \equiv 81 \pmod{23} \equiv 12 \pmod{23} \\ 2 \equiv 2 \pmod{23} \\ 2^{11} = 2^{10} \times 2 \equiv 12 \times 2 \pmod{23} \equiv 1 \pmod{23} \end{gathered}$$

$$\textit{Therefore} \quad 2^{11} - 1 \equiv 0 \pmod{23}.$$

V. *Prove that* $3^{4n+2} + 5^{2n+1}$ *is divisible by* 14*:*

$$\begin{aligned} 3^{4n+2} &= 9 \cdot 81^n \equiv 9 \cdot [11 \pmod{14}]^n \equiv 9 \cdot 11^n \pmod{14} \\ 5^{2n+1} &= 5 \cdot 25^n \equiv 5 \cdot [11 \pmod{14}]^n \equiv 5 \cdot 11^n \pmod{14} \end{aligned}$$

$$\therefore 3^{4n+2} + 5^{2n+1} \equiv 14 \cdot 11^n \equiv 0 \pmod{14}.$$

VI. *Find the remainder when* 2^{1000} *is divided by* 13*.*

$$2^3 = 8; \quad 2^6 = 64 \equiv -1 \pmod{13}; \quad \textit{But since } 1000 = 6 \cdot 166 + 4$$
$$\textit{and} \quad 2^{996} = (2^6)^{166} \equiv (-1)^{166} \pmod{13} \equiv +1 \pmod{13}, \ \textit{we have:}$$

$$2^{1000} \equiv 2^4 \pmod{13} \equiv 16 \pmod{13} = 3 \pmod{13}.$$

The remainder is therefore 3.

VII. *What are the last two digits of* 3^{1234}*? In* mod 100 *arithmetic*

$$\begin{aligned}
&3^2 \equiv 9\ ;\ 3^4 \equiv 81\\
&3^8 \equiv 81^2 \equiv 61\\
&3^{10} \equiv 9 \cdot 61 \equiv 49\\
&3^{20} \equiv 49^2 \equiv 1\\
&1234 = 20 \times 61 + 4 + 10\\
&3^{1234} \equiv (3^{20})^{61} 3^4 3^{10} \equiv 81 \cdot 49 \equiv 69 \pmod{100}
\end{aligned}$$

The last two digits are seen to be 69.

VIII. *Show that* $A = 2903^n - 803^n - 464^n + 261^n$ *is divisible by* 1897 *for any natural number* n.

Write

$$A = (2903^n - 464^n) - (803^n - 261^n).$$

The first group is divisible by $2903 - 464 = 9 \cdot 271$ *and the second by* $803 - 261 = 2 \cdot 271$, *so* A *is divisible by* 271 *But we can also write*

$$A = (2903^n - 803^n) - (464^n - 261^n),$$

where the first group is divisible by $2903 - 803 = 7 \cdot 300$ *and the second by* $464 - 261 = 7 \cdot 29$, *so that* A *is also divisible by* 7. *Since* 271 *is not divisible by the prime* 7, A *is divisible by the product* $271 \cdot 7 = 1897$.

IX. *Prove that the* 5^{th} *Fermat number* $F_5 = 2^{2^5} + 1 = 2^{32} + 1$ *is divisible by* 641 *(Euler's claim, 1732):*

$$\begin{gathered}
640 = 5 \cdot 128 = 5 \cdot 2^7 \equiv -1 \pmod{641}\\
5^4 \cdot 2^{28} = (5 \cdot 2^7)^4 \equiv (-1)^4 \equiv 1 \pmod{641}\\
5^4 = 625 \equiv -16 \pmod{641} \equiv (-2^4) \pmod{641}\\
\therefore \quad -(2^4) \cdot 2^{28} \equiv 1 \pmod{641}\\
-(2^{32}) \equiv 1 \pmod{641}\\
2^{32} \equiv -1 \pmod{641}
\end{gathered}$$

$$2^{32} + 1 \equiv 0 \pmod{641}$$

Alternatively,

$$F_5 = 2^{2^5} + 1 = 2^{32} + 1 \equiv (5^4 \cdot 2^{28} + 2^{32}) - (5^4 \cdot 2^{28} - 1)$$

But

$$\begin{aligned}
5^4 \cdot 2^{28} + 2^{32} &= 2^{28}(5^4 + 2^4) = 641 \cdot 2^{28} \\
5^4 \cdot 2^{28} - 1 &= (5^2 \cdot 2^{14} + 1)(5^2 \cdot 2^{14} - 1) \\
&= (5^2 \cdot 2^{14} + 1)(5 \cdot 2^7 - 1)(5 \cdot 2^7 + 1) \\
&= 641 \cdot (5^2 \cdot 2^{14} + 1)(5 \cdot 2^7 - 1)
\end{aligned}$$

Application - Calendar problems

How does one find the relation between dates and days of the week in the Gregorian calendar?

According to this calendar, the common year consists of 365 days and each leap year of 366 days. Leap years are the years for which the number is divisible by 4*, except the centurial years, which are leap years only if divisible by* 400*. Thus, the first centurial leap year after the reformation of the calendar, which occurred in the catholic countries in 1582, was 1600, but 1700, 1800, 1900 were common years; the next centurial leap year was 2000, and so on.*

It is easy to determine the number of leap years between 1600 exclusive and a given year N inclusive. The number of years divisible by 4 *in the assumed interval is the same as the number of integers x such that*

$$400 < x \leq \frac{N}{4};$$

that is,

$$\left[\frac{N}{4}\right] - 400.$$

But from this we must exclude the number of centurial years not divisible by 400*. The number of all centurial years between 1600 exclusive and N inclusive is*

$$\left[\frac{N}{100}\right] - 16,$$

and among them there are

$$\left[\frac{N}{400}\right] - 4$$

divisible by 400*. Consequently the number of centurial years which are not leap years is*

$$\left[\frac{N}{100}\right] - \left[\frac{N}{400}\right] - 12$$

and the requested number of all leap years between 1600 exclusive and N inclusive is thus

$$T = \left[\frac{N}{4}\right] - \left[\frac{N}{100}\right] + \left[\frac{N}{400}\right] - 388.$$

This expression can be put into more convenient form by setting

$$N = 100C + D$$

where C is the century number and $D < 100$. Then

$$\left[\frac{N}{4}\right] = 25C + \left[\frac{D}{4}\right]; \qquad \left[\frac{N}{100}\right] = C; \qquad \left[\frac{N}{400}\right] = \left[\frac{C}{4}\right]$$

and

$$T = \left[\frac{D}{4}\right] + \left[\frac{C}{4}\right] + 24C - 388.$$

Since in a leap year an additional day is added at the end of February, it is convenient to proceed as if the years begin in March. Then March, April, May, ... will be counted as the first, second, third, ... months of the year N, while January and February of the same year will be considered as the eleventh and twelfth months of the year $N-1$. It will also be convenient to denote days of the week beginning with Sunday by 0, 1, 2, ..., 6.

Now suppose that the first of March of the year 1600 had the weakday number a. Since the next year 1601 was a common year, 365 days elapsed between March 1, 1600, and March 1, 1601. But 365 days consist of 52 full weeks and 1 day; hence March 1, 1601, had the number $a+1$ or this number diminished by 7.

Again, since the years 1602 and 1603 were common years, March 1, 1602, and March 1, 1603, had the numbers $a+2$ and $a+3$ or these numbers diminished by a proper multiple of 7. Between March 1, 1603, and March 1, 1604, since 1604 was a leap year, 366 days or 52 weeks and 2 days elapsed; hence the number of March 1, 1604, was $a+5$ or the least positive residue of it modulo 7.

It is now clear that every common year elapsing augments the number of March 1 modulo 7 by one unit and every leap year by two units. Hence, to find the number of March 1 in the year N, we have to add to a the number of all years between 1600 exclusive and N inclusive and also the number of leap years in the same interval, and to reduce the sum to its least positive residue mod 7. *Thus March 1 of the year N will have the number a' determined by the congruence*

$$a' \equiv a + 100C + D - 1600 + \left[\frac{D}{4}\right] + \left[\frac{C}{4}\right] + 24C - 388 \pmod 7$$

or

$$a' \equiv a + D + \left[\frac{D}{4}\right] + \left[\frac{C}{4}\right] - 2C \pmod 7.$$

For the year 1938, March 1 was on Tuesday, so $a' = 2$*; again for the same year*

$$D = 38, \quad C = 19; \quad D + \left[\frac{D}{4}\right] + \left[\frac{C}{4}\right] - 2C \equiv 6 \pmod 7,$$

whence

$$2 \equiv a + 6 \pmod 7, \quad a = 3.$$

That is, March 1, 1600, was a Wednesday, and the preceding expression for a' *becomes*

$$a' \equiv 3 + D + \left[\frac{D}{4}\right] + \left[\frac{C}{4}\right] - 2C \pmod 7.$$

This congruence determines the day of the week on which March 1 falls in every year after the Gregorian reform.

Origins of the Vector Concept (1589–1831)

The foundations of vector analysis were laid in the 1840*'s. The deep roots of this concept were latent in various forms, in the works of* 10 *men:*

*(1) Parallelograms of velocities and forces (**Galileo**, 1589).*

*(2) 'Geometry of situations' (**Leibniz**, 1693).*

*(3) Geometrical representation of complex numbers (1797–1831). **Caspar Wessel** (1745–1818,* Norwegian Surveyor*, 1797), **Carl Friedrich Gauss** (1777–1855, Germany, ca* 1797*, first published in 1831), **Abbé Buée** (1805,* France*), **Jean Robert Argand** (1768–1822,* Switzerland*, 1806), **John Warren** (1828,* England*), **C.V. Mourey** (1828,* France*).*

(4) 'Sensed magnitudes' first to be employed systematically in projective geometry by ***Lazare Carnot*** *(1803) and later by* ***August Ferdinand Möbius*** *(1827).*

Wessel *was the first to render a clear exposition on the subject in his paper 'On the Analytical Representation of Directions', which he read before the Royal Academy of Science and Letters of Denmark. It contained a complete development of laws governing operations with directed line segments as representation of numbers in the form* $a + b\sqrt{-1}$ *and their applications, as well as a partial theory of rotation.*

Wessel was born in Jonsrud, Norway, to a family blessed with 13 *children. In* 1763 *he went to Copenhagen, and in the following year he was engaged by the Danish Academy of Sciences as an assistant in the preparation of a map of Denmark. He remained in the employment of the Academy until* 1805.

It speaks well for the Academy that they received Wessel's paper sympathetically, since he was neither a member nor was he considered a mathematician. Written in Danish (in Volume 5 *of the Memoirs of the Academy, 1799), it failed to achieve wide accessibility to the mathematicians of other countries — with the result that this excellent and significant work did not become generally known until a French translation of it was published in 1897!*

1795–1805 CE **Mungo Park** (1771–1806, Scotland). African explorer. Explored the course of the Niger River. Born on a farm in Selkirkshire, Scotland, the seventh son in a family of thirteen. In 1791 he obtained a surgical diploma at the University of Edinburgh. Through his connections with **Joseph Banks**, president of the Royal Society, he was sent in 1795 by the African Association with a small expedition to ascertain the course of the River Niger. He ascended the Gambia River, crossed Senegal, followed the course of Niger (1795 to 1796), was captured by an Arab chief, and escaped after four months of imprisonment. In 1797 he reached England, where he had been given up for dead, by way of America. An account of his adventures by his own pen appeared in 1799 (*Travels in the Interior of Africa*). He then married (1799) and settled in Peebles, where he worked as a country doctor. The hardness and monotony of life at Peebles impelled him to accept the government's offer to lead another expedition to the Niger (1805). Park and his European colleagues were either killed or drowned in an encounter with forces of the King of Haoussa, 800 km south of the Niger delta (1806).

1796 CE **Edward Jenner** (1749–1823, England). Physician and discoverer of *vaccination*[383]. Laid the foundation of modern immunology. It was introduced by him as a preventive measure against smallpox. The success of the smallpox vaccine led to the search for vaccines to prevent other serious diseases. Before his time, no mother counted her children safe until all had passed through smallpox.

Smallpox (*Variola major*) replaced the plague as the foremost epidemic disease. The first reasonably effective method of control in the early 18th century was *variolation* — inserting pus from a smallpox pustule into a scratch on someone unaffected. In 1768, the '*inoculator*' **Thomas Dimsdale** (1712–1800) treated the Russian Empress Catherine the Great, her son and her court, and was rewarded with a considerable fee, pension and the rank of a baron. Unfortunately, variolation could sometimes lead to a fatal attack or fail to give protection, and all who underwent it became infectious and had the potential to spread the disease.

It was common knowledge in Jenner's time that a person could catch smallpox only once. Many people tried to inoculate themselves with matter

383 From the Latin *vacca* = cow, since it referred to the injection of cowpox virus to prevent smallpox. In general, *vaccination* is the introduction of dead or weakened viruses or bacteria, or their *toxins* (poisons) into the body to develop resistance to disease. The material introduced is called a *vaccine*. The vaccine causes the body to manufacture substances called *antibodies* which fight the effects of bacteria, toxins and viruses. Vaccines must be strong enough to excite resistance, but too weak to cause serious illness.

from smallpox sores. They hoped to catch a light case of the disease and then be immune for it for the rest of their lives.

Mary Wortley Montagu (1689–1762), an English author, had introduced the practice of inoculation in 1717 from Turkey. She had her own children inoculated, but encountered a vast amount of prejudice against this procedure. The method was, indeed, unsafe.

Jenner was born at Berkeley, Gloucestershire. In 1770 he went to London to study medicine and in 1792 he obtained the degree of doctor of medicine from St. Andrews College. In 1796, Jenner took matter from the hand of Sarah Nelms, a Berkeley dairymaid[384] who caught the cowpox disease while milking the cows. Jenner made two cuts on the arm of James Phipps, a healthy 8-years-old boy, and infected the matter from one of Sarah's sores. The boy then caught cowpox. Six weeks later, Jenner risked his medical reputation by introducing variolous matter into the boy's arm. Ordinarily fatal, the smallpox matter had no effect.

Subsequently the method proved routinely successful[385], and honors began to shower on him from abroad[386]: He was elected a member of almost all the chief scientific societies on the continent of Europe. In his own country his merits were less recognized; In 1813 the University of Oxford conferred on him the degree of M.D. but the college of physicians would not admit him until he had undergone an examination in classics! To which Jenner replied: "*To brush up my classics — I would not do it for a diadem*". He continued to vaccinate gratuitously all the poor who applied to him, so that he sometimes had as many as 300 persons waiting at his door. Only in 1858 was a statue of him erected by public subscription in London.

With Jenner's vaccination, smallpox could be controlled, and by 1975, it had been eradicated.

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[384] Cowpox is a minor disease in humans, that causes a few sores on the hands, but carries little danger of disfiguration and death. People believed that dairymaids who had caught cowpox could not catch smallpox.

[385] An early advocate of vaccination was the physician **Jacob Ezekiel Aronsson** (1759–1845) of Alsace-Lorraine.

[386] In 1796, **Catherine the Great**, the Empress of Russia, caused the first child operated upon to receive the name **Vaccinov**, and to be educated at the public expense. On one occasion, when Jenner was endeavoring to obtain a release of some of the unfortunate Englishmen who had been detained in France on the sudden termination of the Peace of Amiens (1803), **Napoleon** was about to reject the petition, when Josephine uttered the name of Jenner. The Emperor paused and exclaimed: "*Ah, we can refuse nothing to that name*".

1796–1815 CE *The Napoleonic Wars*:

- 1796–1797 *The Italian campaign*: Napoleon defeated the Austrians and the Piedmontese in a series of 8 battles (Millesimo, Mondovi, Ladi, Castiglione, Roverdo, Bassano, Arcola and Rivoli).
- 1798–1799 *The Egyptian expedition*: Napoleon wins the land-battles of the Pyramides (against the Mamluks) and Abukir (against the British and the Turks), but looses the sea-battle of the Nile.
- 1798–1799 *War of the second coalition* (Britain, Russia, Austria, Naples, Portugal and Ottoman Empire): The French are driven out of Italy in a series of five battles (Magnano, Cassano, Zürich, Trebbia and Novi).
- 1800 *Battles of Marengo and Hohenlinden*: France defeats Austria.
- 1805 *Battle of Ulm*: France defeats Austria.
  Battle of Trafalgar: British navy under Nelson defeats the combined French and Spanish fleets.
  Battle of Austerlitz: France defeats the combined armies of Austria and Russia.
- 1806 *Battles of Jena and Auerstädt*: France defeats Prussia.
- 1807 *Battle of Friedland*: France defeats Russia.
- 1808–1814 *The Peninsular War* of the British against the French in Portugal and Spain. A series of 4 battles (Vimerio, Corunna, Talavera and Ocana).
- 1809 *Battle of Aspern and Essling* and *Battle of Wagram* in which Napoleon crushed the Austrians.
- 1812 *Battle of Borodino*: Russian retreated and abandoned Moscow. The retreating French army fought at *Jaroslavetz* and *Viazma* against Kutuzov.
- 1813 *Battle of Dresden*: Napoleon defeated the allied army of Prussia, Russia and Austria.
  Battle of Leipzig: Napoleon is driven out of Germany.
- 1813–1814 Napoleon is driven out of Spain. The allies enter Paris and Napoleon is exiled to Elba.
- 1815 *Battle of Waterloo*.

Impact of Social Revolutions (1775–1814)

From 1775 *to* 1783 *Britain's North American colonies, with a population of well over 2 million, broke away from rule by the mother country. The thirteen ex-colonies formed the United States of America* [Connecticut, Delaware, Georgia, Massachusetts, Maryland, North Carolina, New Hampshire, New Jersey, New York, Pennsylvania, Rhode Island, South Carolina, Virginia]. *In 1778* France *entered the American War of* Independence *in support of the colonists.*

The success of the American Revolution had profound effects on Europe, and eventually on other parts of the world. It passed on to the western world the ideas of constitutional government and popular sovereignty.

Encouraged by the American example, the French Revolution (1789–1799) not only transformed the government of France, but also shook the establishment throughout Europe and led to new ideas that remained influential throughout the next century.

France was the hive of advanced ideas. A widespread rebellion had broken out against the tyranny of the clergy and the monarchy which finally culminated in the French Revolution. It did more to spread political ideas than philosophical ones. Following the U.S. and French example, other parts of America fought to free themselves from Europe and Europe from the Pope. These ideas liberated the world from the yokes of religion and with this liberation came the flowering of new sciences.

Indeed, for the next two centuries, nobody was able to think of the era of the sciences without referring to the French Revolution, when the scientists quite plainly took power. An astronomer was Mayor of Paris, the inventor of topology was at the head of the Committee for Public Health, the scholars occupied the institutions before the people did and in their place, and a geometrician, although a minor one, gained the title of Emperor. The nobility and the clergy collapsed, society no longer lived according to the same divisions or the same offices and scientists at last formed a class, replacing the clerics and forming a new "Church".

Under the influence of the social ferment, the movement of ideas and beliefs which had been dominant until this time – the Enlightenment, with its emphasis on reason and natural law – gave way to the Romantic movement in the arts, which favored emotion before reason and espoused free individual expression.

The French revolution marked the coming of a modern world — a world of class conflict, middle-class ascendancy, acute national conflicts and popular democracy. Together with industrialization, *the revolution reshaped the institutions, the societies and even the mentalities of the European peoples.*

In France itself, the revolution stimulated the rapid growth of science at the turn of the 18^{th} *century. The scientists of France found their activities directed towards practical ends, which appears to have given them a greater taste for experimentation than they previously had. Simultaneously, scientific institutions were established which trained the French talent that was to dominate the cutting edge of science during the early years of the* 19^{th} *century.*

The first practical problem which the revolutionaries posed to the scientists of France was the standardization of weights and measures throughout the country. During the 18^{th} *century weights and measures varied in France from region to region. The meter, for example, measured* 100 *centimeters in Paris, was* 98 *cm at Marseilles,* 102 *cm at Lille and* 96 *cm at Bordeaux. By* 1799 *'the astounding and scandalous diversity in measures' was brought to an end.*

In 1794 *the National Convention founded the École Polytechnique and the École Normale Supériere, which were important institutions devoted to scientific education and research in France throughout the* 19^{th} *century. The Supériere was closed down after* 4 *months, and did not become important until* 1808, *when it was reopened by Napoleon Bonaparte (1769–1821). The Polytechnique, however, flourished from the start. It opened in* 1794 *with* 400 *pupils and a staff composed of the leading scientists of the time: mathematical physics were taught by* ***Laplace*** *and* ***Lagrange****, geometry by* ***Monge*** *and chemistry by* ***Berthollet****. Amongst their students were* ***Poncelet****,* ***Poisson****,* ***Cauchy****,* ***Carnot****,* ***Gay-Lussac****,* ***Dulong*** *(1785–1838) and* ***Petit*** *(1791–1820).*

Napoleon himself encouraged the practical side of science by offering prizes for useful discoveries. He also discouraged the speculative thinkers who continued the tradition of the earlier materialist philosophers. Thus French science became more practical and experimental during the Napoleonic period.

A marked anti-scientific movement arose in the official and fashionable circles in France with the restoration of the Bourbons in 1815. *The movement was particularly opposed to the mathematical tradition of French science. But the scientific institutions set up by the National Convention in* 1794 *had the effect of concentrating the scientific activity of France in the capital, at the Paris schools. During the* 19^{th} *century the Polytechnique and the Supériere became the Mecca of young French scientists from the provinces as well as from the metropolis.*

1796 CE At the age of 19, **Carl Friedrich Gauss** conceived the first proof that a 17-sided regular polygon is constructible by means of a compass and a ruler: The first mathematician to thus go beyond the ancient Greeks.

Cyclotomic Equations and the Roots of Unity – de Moivre to Gauss (1730–1801)

During the 36 centuries that elapsed from he Old Babylonian period to the end of the 19th century, algebra was the science of solution of equations.

The solution of algebraic equations occupied the minds of the finest mathematicians of Europe in the 17th, 18th and 19th centuries and demanded the combined efforts of men like **de Moivre** *(1730),* **Euler** *(1749),* **Lagrange** *(1770),* **Vandermonde** *(1771),* **Gauss** *(1801),* **Ruffini** *(1810),* ***Abel*** *(1824) and* **Galois** *(1831).*

1. Historical overview

Archaeological research in the 20th century has revealed that the people of Mesopotamia around 1700 BCE had an advanced mathematical culture, including a knowledge of the Pythagorean theorem (a millennium before Pythagoras), a *sexagesimal system of arithmetic* and a method of solution of *quadratic equations.*

There was only modest progress in algebra in the 3000 years that followed; During the Middle Ages, Europe had learned about algebra from the Arabs and had begun to improve it by devising new symbols and notations. Then, in the 16th century, the algebraic solution of cubic equations was discovered (1515), and closely thereafter the solution of quartic equations (1544).

It was not until almost 300 years later that it was shown – first by ***Abel*** *(1824), then by* **Galois** *(1831) – that it is impossible to solve the quintic equation in the same manner that the quadratic, cubic and quartic were solved;*

specifically, by using a finite number of additions, subtractions, multiplications, divisions, and the extractions of roots.

However, certain classes of quintic (and higher order) equations can be solved in this manner. Thus, any algebraic equation can be associated with a Galois group, which may be the symmetric (permutation) group S_n, metacyclic group M_n, dihedral group D_n, alternating group A_n, or the cyclic group C_n. Solvability of a quintic is then predicated on its corresponding group being a solvable group. An example of a quintic equation with a solvable cyclic group is

$$x^5 - x^4 - 4x^3 + 3x^2 + 3x - 1 = 0$$

which arises in the computation of $\sin\left(\frac{2\pi}{11}\right)$.

2. Roots of unity

In the 18th century, the problem of solving the n^{th} degree equation centered on the special case $x^n = 1$, called the binomial equation. **Roger Cotes** *(1714) and* **de Moivre** *(1707, 1730) showed, through the use of complex numbers, that the solution of this problem amounts to the division of the circle into n equal parts; hence the alternative name cyclotomic equation. To obtain the roots of this equation by radicals (trigonometric solutions are not necessarily thus expressible) it is sufficient to solve the case of n an odd prime p. Indeed, assume this: Then if $n = pm$, let $y = x^m$. But $y^p - 1$ is solvable. By assumption, and for each such $y = y_j$, $x^m = y_j$ can be solved if m is either prime or, if not, m can be decomposed in the same manner that n is.*

To solve $x^n = 1$ we write

$$x^n = 1 = \cos 2k\pi + i \sin 2k\pi, \qquad k = 0, 1, \dots, n-1 \tag{1}$$

Then, using de Moivre's theorem, we have

$$x_k = (\cos 2k\pi + i \sin 2k\pi)^{\frac{1}{n}} = \cos\left(\frac{2k\pi}{n}\right) + i \sin\left(\frac{2k\pi}{n}\right), \tag{2}$$

which renders all the n^{th} roots of unity when k sweeps the range $k = 0, 1, \dots, n-1$. Denoting

$$R = \cos\frac{2\pi}{n} + i \sin\frac{2\pi}{n}, \tag{3}$$

we use again de Moivres' theorem to obtain

$$R^k = \cos\frac{2k\pi}{n} + i \sin\frac{2k\pi}{n} \qquad (k \text{ an integer}). \tag{4}$$

Comparing with (2) we see that the n^{th} roots of unity are powers of R. The n distinct roots of unity are

$$R, R^2, R^3, \ldots, R^{n-1}, R^n = 1. \tag{5}$$

Since the absolute value (modulus) of R^k is 1, the points representing the nth roots of unity are equally spaced on the circumference of the unit circle. Joining these points by straight line segments, a regular polygon of n sides is formed. The possibility of construction of such regular polygons with the use of straightedge and compass alone is discussed next.

An nth root of unity which is not also a pth root (for some prime $p < n$) is called a primitive root. The number R defined by (3) is a primitive nth root of unity.

Of the numbers (5) the primitive nth roots are those whose exponents are prime to n. To see this we consider the root R^s in (5) $(s < n)$, namely

$$R^s = \cos\frac{2s\pi}{n} + i\sin\frac{2s\pi}{n}.$$

Suppose s and n are not relatively prime. Let k be the g.c.d. of s and n; Then $n = ka$, $s = kb$; and $1 < k < n$ (all lower case Latin letters but i represent natural numbers here, unless stated otherwise). We have

$$(R^s)^a = (R^n)^b = 1^b = 1.$$

Since $a < n$, then R^s is not a primitive n^{th} root of unity by virtue of the definition of a primitive n^{th} root.

But if s and n have no common factor other than unity, then $(R^s)^r \neq 1$ for r a positive integer less than n, since for

$$(R^s)^r = \cos\frac{2rs\pi}{n} + i\sin\frac{2rs\pi}{n}$$

to be unity, $\frac{rs}{n}$ must be an integer. However s is by hypothesis prime to n. Therefore $\frac{r}{n}$ must be an integer; yet this is impossible, since $r < n$.

The primitive n^{th} roots of unity are

$n = 3$	ω, ω^2	$\omega = -\frac{1}{2} + \frac{1}{2}i\sqrt{3}$
$n = 4$	$i, -i$	
$n = 5$	R, R^2, R^3, R^4	$R = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$
$n = 6$	R, R^5	$R = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$
$n = 8$	R, R^3, R^5, R^7	$R = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$

If n is a prime number, all the complex roots are primitive and given by (5). In general, when one primitive nth root is known, all the other are obtainable.

The properties of the roots of the binomial equation are summarized in the following six theorems:

- *If α is a complex root of $x^n - 1 = 0$, then α^m, m integer, is also a root.*
  Proof: $\alpha^n = 1 \therefore (\alpha^n)^m = 1$ *or* $(\alpha^m)^n = 1$; *Therefore α^m is a root of* $x^n - 1 = 0$.

- *If m and n are prime to each other, the equations $x^m - 1 = 0$, $x^n - 1 = 0$ have no common root except unity.*
  Proof: Let α be a common root $\alpha^m = 1$, $\alpha^n = 1 \therefore \alpha^{(mb-na)} = 1$ for integers a and b. But since $(m, n) = 1$ there exist integers a, b such that $mb - na = 1$, so $\alpha = 1$ and 1 is the only common root of the given equations.

- *If k is the greatest common divisor of two integers m and n, the roots common to the equations $x^m - 1 = 0$ and $x^n - 1 = 0$ are roots of the equation $x^k - 1 = 0$.*
  Proof: $m = km'$, $n = kn'$, $(m', n') = 1$. So integers a, b can be found such that $m'b - n'a = 1 \therefore mb - na = k$. Thus if α is a common root then $\alpha^{mb-na} = 1$ and also $\alpha^k = 1$. This proves that α is a root of $x^k = 1$.

- *When n is a prime number, and α any complex root of $x^n - 1 = 0$, all the roots are included in the series,*

$$1, \alpha, \alpha^2, \ldots, \alpha^{n-1} \, .$$

  Proof: By our first theorem, these entities are all roots of $x^n - 1$, and by the second they are all different.

- *When n is a composite number formed of the factors $p, q, r, \ldots$, the roots of the equations $x^p - 1 = 0$, $x^q - 1 = 0$, $x^r - 1 = 0$ etc., all satisfy the equation $x^n - 1 = 0$.*
  Proof: Let α be a root of $x^p - 1 = 0$; then $\alpha^p = 1$. Then

$$(\alpha^p)^{qr\cdots} = 1 \quad \text{or} \quad \alpha^n - 1 = 0.$$

3. Solution by radicals

(a) $n = 3, 4, 5, 6, 8, 10, 12$

The most important result of the previous section is that the primitive n^{th} roots of unity are

$$e^{2k\pi i/n} = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} \tag{6}$$

where k runs over the positive integers which are less than n and relatively prime to n. In particular, when n is prime, then every n^{th} root of unity except 1 is primitive.

Now, Eq. (6) renders the roots of unity in a trigonometric form. Yet, one of the basic problems in the theory of algebraic solution of equations is to give algebraic solutions of the cyclometric equation $x^n = 1$.

In the case $n = 3$, **Lagrange** *(1771) showed how this can be elegantly achieved: starting from*

$$x^3 + qx + p = 0 \tag{7}$$

he substituted $x = y - \frac{q}{3y}$, obtaining the 6^{th} degree equation

$$y^6 + py^3 - \frac{q^3}{27} = 0.$$

Putting $r = y^3$, one derives the quadratic equation $r^2 + pr - \frac{q^3}{27} = 0$ with the explicit solutions $r_{1,2} = \frac{1}{2}\left[-p \pm \sqrt{p^2 + \frac{4}{27}q^3}\right]$. *It then remains to solve*

$$y^3 - r = 0;$$

Clearly

$$y = \sqrt[3]{r_j}; \qquad \omega\sqrt[3]{r_j}; \qquad \omega^2\sqrt[3]{r_j}; \qquad j = 1, 2; \qquad r_1 r_2 = -\frac{q^3}{27},$$

where

$$\omega^3 = 1, \qquad 1 + \omega + \omega^2 = 0; \qquad \omega = \frac{1}{2}(-1 + \sqrt{-3}),$$

and therefore

$$\begin{aligned} x_0 &= \sqrt[3]{r_1} + \sqrt[3]{r_2} \\ x_1 &= \omega\sqrt[3]{r_1} + \omega^2\sqrt[3]{r_2} \\ x_2 &= \omega^2\sqrt[3]{r_1} + \omega\sqrt[3]{r_2} \end{aligned} \tag{8}$$

This solution exhibits the link between the solutions of the general cubic and the cyclotomic equation of degree 3, $\omega^3 = 1$. *The solution of the latter is an immediate result of the factorization*

$$\omega^3 - 1 = (\omega - 1)(\omega^2 + \omega + 1) = 0 \tag{9}$$

and so

$$\begin{aligned} x_1 &= \cos\tfrac{2\pi}{3} + i\sin\tfrac{2\pi}{3} = -\tfrac{1}{2} + \tfrac{i}{2}\sqrt{3} = \omega \\ x_2 &= \cos\tfrac{2\pi}{3} + i\sin\tfrac{2\pi}{3} = -\tfrac{1}{2} - \tfrac{i}{2}\sqrt{3} = \omega^2 \end{aligned} \tag{10}$$

In the case $n = 4$, *the algebraic solution is*

$$x^4 - 1 = (x^2 - 1)(x^2 + 1) = 0,$$

resulting in

$$x_1 = 1, \quad x_2 = -1; \quad x_2 = i, \quad x_4 = -i.$$

The points $(1, 0)$; $(-1, 0)$; $(0, i)$; $(0, -i)$ *then divide the unit circle into four equal parts.*

In the case $n = 5$

$$x^5 - 1 = (x - 1)(x^4 + x^3 + x^2 + x + 1) = 0 \tag{11}$$

To solve Eq. (11), one uses the trick discovered by ***de Moivre*** *(1707) to solve general reciprocal equations,* namely the substitution $x + \frac{1}{x} = u$. *Then, since*

$$\begin{aligned} x^2 + \frac{1}{x^2} &= u^2 - 2, \\ x^3 + \frac{1}{x^3} &= u^3 - 3u, \\ x^4 + \frac{1}{x^4} &= u^4 - 4u^2 + 2, \\ x^5 + \frac{1}{x^5} &= u^5 - 5u^3 + 5u, \quad \textit{etc.}, \end{aligned} \tag{12}$$

the quartic reciprocal equation yields,

$$\begin{aligned} x^4 + x^3 + x^2 + x + 1 \quad &= x^2\left[(x^2 + \frac{1}{x^2}) + (x + \frac{1}{x}) + 1\right] \\ &= x^2(u^2 + u - 1) = 0. \end{aligned}$$

It then remains to solve the pair of quadratic equations:

$$u^2 + u - 1 = 0; \qquad x^2 - ux + 1 = 0, \tag{13}$$

yielding

$$\begin{aligned}
x_1 &= \tfrac{1}{4}[(\sqrt{5}-1)+i\sqrt{10+2\sqrt{5}}] = \cos\tfrac{2\pi}{5}+i\sin\tfrac{2\pi}{5} = e^{\frac{2\pi i}{5}}\\
x_2 &= \tfrac{1}{4}[-(\sqrt{5}+1)+i\sqrt{10-2\sqrt{5}}] = \cos\tfrac{4\pi}{5}+i\sin\tfrac{4\pi}{5} = e^{\frac{4\pi i}{5}}\\
x_3 &= \tfrac{1}{4}[-(\sqrt{5}+1)-i\sqrt{10-2\sqrt{5}}] = \cos\tfrac{6\pi}{5}+i\sin\tfrac{6\pi}{5} = e^{\frac{6\pi i}{5}}\\
x_4 &= \tfrac{1}{4}[(\sqrt{5}-1)-i\sqrt{10+2\sqrt{5}}] = \cos\tfrac{8\pi}{5}+i\sin\tfrac{8\pi}{5} = e^{\frac{8\pi i}{5}}
\end{aligned} \tag{14}$$

The very same result could have been obtained through the factorization

$$x^4+x^3+x^2+x+1=(x^2+Ax+1)(x^2-\frac{1}{A}x+1)=0 \tag{15}$$

where

$$A=\frac{\sqrt{5}+1}{2}, \qquad \frac{1}{A}=\frac{\sqrt{5}-1}{2}.$$

Note that since $x_1=\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}$, *it follows that*

$$u_1=x_1+\frac{1}{x_1}=e^{\frac{2\pi i}{5}}+e^{\frac{-2\pi i}{5}}=2\cos\frac{2\pi}{5}=\frac{1}{2}(\sqrt{5}-1).$$

The geometrical interpretation of this result renders the key to the construction of a regular pentagon *inscribed in a unit circle: one draws a unit circle centered at* O *and two perpendicular diameters* AA' *and* BB'. *Let the midpoint of* OA' *be* C. *Draw an arc with* C *as center and* CB *as a radius, cutting* OA *at* D. *Then, if* S_n *represents a side of a regular polygon of* n *sides,* $S_{10}=OD$ *and* $S_5=BD$.

Apart from $x^5=1$, *there is a whole class of quintic equations solvable by radicals. Furthermore, certain quintics have solutions expressible in terms of the fifth roots of unity. Thus the equation*

$$x^5-5ax^3+5a^2x-2b=0, \tag{16}$$

sometimes known as de Moivre's quintic, *has the explicit simple solutions*

$$x_k=\epsilon^k u_1+\epsilon^{4k}u_2 \qquad k=0,1,2,3,4, \qquad \epsilon=e^{\frac{2\pi i}{5}}. \tag{17}$$

To solve for u_j, *set* $k=0$ *and substitute* $x=u_1+u_2$ *in (16) and obtain*

$$u_1^5+u_2^5+5(u_1+u_2)(u_1^2+u_1u_2+u_2^2-a)(u_1u_2-a)-2b=0.$$

Letting $u_1^5+u_2^5=2b$ *and* $u_1^5u_2^5=a^5$, *the solution is*

$$u_1=\sqrt[5]{b+\sqrt{b^2-a^5}}\,; \qquad u_2=\sqrt[5]{b-\sqrt{b^2-a^5}}\,. \tag{18}$$

It can be shown[387] *that if* a *and* b *are rational numbers such that the quintic trinomial* $x^5 + ax + b$ *is irreducible over the rationals, then the equation* $x^5 + ax + b = 0$ *is solvable by radicals iff there exist a sign* $\theta(=\pm 1)$*, and reals* $c(\geq 0)$ *and* $e(\neq 0)$ *such that* $a = \frac{5e^4(3-4\theta c)}{c^2+1}$, $b = \frac{-4e^5(11\theta+2c)}{c^2+1}$, *in which case the roots of* $x^5 + ax + b = 0$ *are*

$$\begin{gathered}
x_k = e\left[\epsilon^k u_1 + \epsilon^{2k} u_2 + \epsilon^{3k} u_3 + \epsilon^{4k} u_4\right], \qquad k = 0,1,2,3,4 \\
u_1 = \left(\frac{v_1^2 v_3}{D^2}\right)^{1/5}, \quad u_2 = \left(\frac{v_3^2 v_4}{D^2}\right)^{1/5}, \\
u_3 = \left(\frac{v_2^2 v_1}{D^2}\right)^{1/5}, \quad u_4 = \left(\frac{v_4^2 v_2}{D^2}\right)^{1/5} \\
v_1 = \sqrt{D} + \sqrt{D - \theta\sqrt{D}} \qquad v_2 = -\sqrt{D} - \sqrt{D + \theta\sqrt{D}} \\
v_3 = -\sqrt{D} + \sqrt{D + \theta\sqrt{D}} \qquad v_4 = \sqrt{D} - \sqrt{D - \theta\sqrt{D}} \\
D = 1 + c^2
\end{gathered} \tag{19}$$

For $a = 0,\ b = -1$, *we fall back on (14).*

In the case $n = 6$

$$x^6 - 1 = (x^3-1)(x^3+1) = (x-1)(x+1)(x^2+x+1)(x^2-x+1) = 0. \tag{20}$$

So, the only roots of $x^6 - 1 = 0$ *which are not roots of lower order equations are those of* $x^2 - x + 1 = 0$*, namely* $\alpha = \frac{1+i\sqrt{3}}{2}$, $\beta = \frac{1-i\sqrt{3}}{2}$ *with the additional provisions* $\alpha\beta = 1 = \alpha^6$ *or* $\beta = \alpha^5$*. Therefore, the primitive roots of* $x^6 - 1 = 0$ *are*

$$\alpha, \alpha^5 \qquad \text{or} \qquad \beta^5, \beta \quad \text{or} \quad \alpha, \frac{1}{\alpha}\,.$$

In the case $n = 8$

$$x^8 - 1 = (x^4-1)(x^4+1) = (x^2-1)(x^2+1)(x^2 - x\sqrt{2}+1)(x^2 + x\sqrt{2}+1) = 0. \tag{21}$$

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[387] B.K. Spearman and K.S. Williams Am. Math. Monthly **101**, 986–992, 1994.

The eight roots are

$$x_k = \cos\frac{k\pi}{4} + \sin\frac{k\pi}{4} \qquad k = 0, 1, \dots, 7,$$

which correspond explicitly to the series

$$1, \;\; -1, \;\; i, \;\; -i, \;\; \frac{1+i}{\sqrt{2}}, \;\; \frac{1-i}{\sqrt{2}}, \;\; \frac{-1+i}{\sqrt{2}}, \;\; \frac{-1-i}{\sqrt{2}}.$$

The case $n = 10$, yields

$$\begin{aligned} x^{10} - 1 \quad &= (x^5 - 1)(x^5 + 1) = (x^5 - 1)(x + 1)(x^4 - x^3 + x^2 - x + 1) \\ &= (x^5 - 1)(x + 1)(x^2 + ax + 1)(x^2 + bx + 1) \end{aligned} \tag{22}$$

with

$$a = \frac{\sqrt{5} - 1}{2}, \qquad b = -\frac{\sqrt{5} + 1}{2}.$$

There are four primitive roots, corresponding to the roots of the quadratic equations $x^2 + ax + 1 = 0$ *and* $x^2 + bx + 1 = 0$. *The division of the circle into* 10 *equal parts is then feasible with a compass and a ruler.*

The case $n = 12$, likewise, lends itself to the factorization

$$\begin{aligned} x^{12} - 1 = \quad &(x^6 - 1)(x^6 + 1) \\ &x^6 - 1 = (x - 1)(x + 1)(x^2 + x + 1)(x^2 - x + 1) \\ &x^6 + 1 = (x^4 - x^2 + 1)(x^2 + 1) \end{aligned} \tag{23}$$

Clearly, there are only 4 *primitive roots corresponding to the roots of*

$$x^4 - x^2 + 1 = 0 \qquad \therefore x^2 = \frac{1}{2}(1 \pm \sqrt{-3}),$$

namely

$$x_{1,2,3,4} = \pm\sqrt{\frac{1}{2}\left(1 \pm \sqrt{-3}\right)}.$$

But since

$$\frac{1 \pm \sqrt{-3}}{2} = \left(\frac{\sqrt{3} \pm i}{2}\right)^2, \qquad x_{1,2,3,4} = \pm\frac{\sqrt{3} \pm i}{2},$$

(the two signs are independent), the 4 primitive roots are:

$$\alpha = \frac{\sqrt{3}+i}{2}, \quad \frac{1}{\alpha} = \frac{\sqrt{3}-i}{2}; \quad \alpha_1 = \frac{-\sqrt{3}+i}{2}, \quad \frac{1}{\alpha_1} = \frac{-\sqrt{3}-i}{2}.$$

Division of the circle into 12 equal parts is enabled simply because $12 = 2 \cdot 2 \cdot 3$. For the same reason, the division of the circle into $16 = 2^4$ equal parts is possible, and also into $15 = 3 \cdot 5$ parts, because $\frac{2\pi}{15} = \frac{\pi}{3} - \frac{\pi}{5}$.

(b) $\boldsymbol{n = 7, 9, 11}$

The Greeks, as well as succeeding mathematicians, tried in vain to construct a regular heptagon. It was especially frustrating since regular polygons of $3, 4, 5, 6, 8, 10$ sides could be constructed with a compass and a ruler.

The source of the difficulty becomes obvious once we try to solve

$$x^7 - 1 = (x-1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1) = 0. \tag{24}$$

Applying the substitution $y = x + \frac{1}{x}$ and using (12), we end up solving a pair of equations:

$$y^3 + y^2 - 2y - 1 = 0, \qquad x^2 - xy + 1 = 0. \tag{25}$$

The three solutions of the cubic are obtained via the irreducible Cardano solution

$$\begin{aligned} y_1 &= 2\cos\tfrac{2\pi}{7} = u + v - \tfrac{1}{3} \\ y_2 &= 2\cos\tfrac{4\pi}{7} = \omega u + \omega^2 v - \tfrac{1}{3} \\ y_3 &= 2\cos\tfrac{6\pi}{7} = \omega^2 u + \omega v - \tfrac{1}{3} \end{aligned} \tag{26}$$

$$u = \tfrac{1}{3}\sqrt[3]{\tfrac{7}{2}}\,\left[1 + 3i\sqrt{3}\right]^{1/3}, \qquad v = \tfrac{1}{3}\sqrt[3]{\tfrac{7}{2}}\,\left[1 - 3i\sqrt{3}\right]^{1/3}$$
$$\omega = -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i, \qquad \omega^2 = -\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i$$

It follows from (26) that

$$2\cos\frac{2\pi}{7} = \frac{1}{3}\sqrt[3]{\frac{7}{2}}\left[(1+3i\sqrt{3})^{1/3} + (1-3i\sqrt{3})^{1/3}\right] - \frac{1}{3}\,. \tag{27}$$

Since the separation of the real and imaginary parts of $(1 \pm 3i\sqrt{3})^{1/3}$ *leads back to the cubic in (25), it is impossible to express* $\cos\frac{2\pi}{7}$ *by an algebraic expression involving real radicals of any kind. Indeed, it can be rigorously proved that an irreducible cubic equation with rational coefficients is not solvable by real roots. This explains why it is not possible to construct a regular polygon with seven sides with a compass and a ruler.*[388]

The same fate befalls the case $n = 9$, *leading to the cyclotomic equation*

$$x^9 - 1 = (x^3-1)(x^6+x^3+1) = 0. \tag{28}$$

The trigonometric solution is

$$x_k = \cos\frac{2k\pi}{9} + i\sin\frac{2k\pi}{9}\,, \qquad k = 0, 1, \ldots, 8\,,$$

and yields $x_1 = \cos 40^\circ + i\sin 40^\circ$. *The corresponding algebraic solution is* $x_1 = \frac{1}{\sqrt[3]{2}}(-1+i\sqrt{3})^{1/3}$. *This implies that*

$$\cos 40^\circ = \frac{1}{2\sqrt[3]{2}}\left\{(-1+i\sqrt{3})^{1/3} + (-1-i\sqrt{3})^{1/3}\right\}. \tag{29}$$

However, the complex–algebra process indicated on the r.h.s. of this equation leads again to a cubic equation, the solution of which is $2\cos 40^\circ$. *It is thus impossible to express* $2\cos 40^\circ$ *by an expression involving real radicals of any kind.*

Note that here again $x^6 + x^3 + 1 = x^3(y^3 - 3y + 1)$ *with* $y = x + \frac{1}{x}$, *where* $y^3 - 4y + 1 = 0$ *is an irreducible cubic. Ergo – a regular polygon of* 9 *sides cannot be constructed with a compass and a ruler.*

Until 1771, no one knew how to solve by radicals the equation $x^n - 1 = 0$ *for* $n > 10$. *A paper by* ***Alexandre-Théophile Vandermonde*** *(1735–1796), in which he solved (1775) the case* $n = 11$ *was regarded as an important advance. In his paper, Vandermonde had pinpointed (without himself being aware of that!*[389]*) the very basic idea of Galois theory (1831), namely, that in*

[388] **Archimedes**, however, devised a method for trisecting an angle using *a pair* of compasses and a ruler with two marks on it which enables a most ingenious method for constructing a regular heptagon using the same instruments.

[389] On this missed opportunity **Lebesgue** said:

order to determine the 'structure' of an equation, deciding eventually whether is solvable by radicals, one has to look at the permutations of the roots; but one needs only consider those permutations which preserve the relations between the roots.

Vandermonde started from the factorization

$$x^{11} - 1 = (x-1)(x^{10} + x^9 + \cdots + x + 1) = 0. \tag{30}$$

Substituting $u = -(x + \frac{1}{x})$ and using (12), the solution of (30) lead him to the irreducible quintic (solvable!)

$$u^5 - u^4 - 4u^3 + 3u^2 + 3u - 1 = 0, \tag{31}$$

which he set to solve by radicals over the field of complex numbers. His novel idea was to generalize the solutions of the cubic and the quartic equations [see (8)] by the introduction of the auxiliary entity, known today as the Lagrange resolvent:

$$t_2 = a + \alpha b + \alpha^2 c + \alpha^3 d + \alpha^4 e. \tag{32}$$

Here $(1, \alpha, \alpha^2, \alpha^3, \alpha^4)$ are the primitive roots of $x^5 = 1$, [given explicitly in Eq. (14)] and (a, b, c, d, e) are the solutions of (31), yet undetermined.

Now, in (32) we effect the permutation

$$a \to b \to d \to c \to e \to a \tag{33}$$

and consequently define

$$\left.\begin{aligned} t &= a + b + c + d + e \\ t_1 &= a + \alpha b + \alpha^2 c + \alpha^3 d + \alpha^4 e \\ t_2 &= b + \alpha d + \alpha^2 e + \alpha^3 c + \alpha^4 a \\ t_3 &= d + \alpha c + \alpha^2 a + \alpha^3 e + \alpha^4 b \\ t_4 &= c + \alpha e + \alpha^2 b + \alpha^3 a + \alpha^4 d \end{aligned}\right\} \tag{34}$$

"Surely, any man who discovers something truly important is left behind by his own discovery; he himself hardly understands it, and only by pondering over it for a long time. But Vandermonde never came back to his algebraic investigations because he did not realize their importance in the first place, and if he did not understand them afterwards, it is precisely because he did not reflect deeply on them; he was interested in everything, he was busy with everything; he was not able to go slowly to the bottom of anything. To assess exactly what Vandermonde saw, understood and what he did not catch, one would have to reconstruct not only the mind of a man from the eighteenth century, but Vandermonde's mind, and at the moment when he had a glimpse of genius and went ahead of his age."

Since

$$x_k = \cos\frac{2\pi k}{11} + i\sin\frac{2\pi k}{11} = e^{\frac{2\pi k i}{11}}, \qquad k = 1, 2, \dots, 10$$

we have

$$\begin{aligned} u_1 &\equiv a = -2\cos\tfrac{2\pi}{11} \\ u_2 &\equiv b = -2\cos\tfrac{4\pi}{11} \\ u_3 &\equiv c = -2\cos\tfrac{6\pi}{11} \\ u_4 &\equiv d = -2\cos\tfrac{8\pi}{11} \\ u_5 &\equiv e = -2\cos\tfrac{10\pi}{11} \end{aligned} \tag{35}$$

Vandermonde then used the trigonometric formula

$$2\cos\theta_1\cos\theta_2 = \cos(\theta_1+\theta_2) + \cos(\theta_1-\theta_2) \tag{36}$$

to obtain relations between a, b, c, d *and* e*. For instance, substituting* $\frac{2\pi}{11}$ *for* θ_1 *and* θ_2*, one obtains* $2\cos^2\frac{2\pi}{11} = \cos\frac{4\pi}{11} + \cos 0$*, leading to the relation* $a^2 = -b + 2$*. Likewise, substituting* $\frac{2\pi}{11}$ *for* θ_1 *and* $\frac{4\pi}{11}$ *for* θ_2 *one finds that* $ab = -c - a$*. Altogether one can easily verify the relations*

$$\begin{array}{llll} a^2 = -b+2 & ab = -a-c & bc = -a-e & cd = -a-d \\ b^2 = -d+2 & ac = -b-d & bd = -b-e & ce = -b-c \\ c^2 = -e+2 & ad = -c-e & be = -c-d & \\ d^2 = -c+2 & ae = -d-e & & de = -a-b \\ e^2 = -a+2 & & & \end{array} \tag{37}$$

Note that the relations in (37) are preserved under the permutation (33).

Next, Vandermonde created the algebraic expressions for the fifth powers $t^5, t_1^5, t_2^5, t_3^5, t_4^5$*. Using (37) repeatedly and the known relations between the roots of (31) and its coefficients, namely*

$$\begin{aligned} & a+b+c+d+e = 1 \\ & abcde = 1 \\ & abcd + bcde + cdea + deab + eabc = 3 \\ & abc + abd + cda + cdb + eab + ecd + eac + ead + ebc + ebd = -3 \\ & ab + bc + cd + de + ea + ac + ce + eb + ad + bd = -4, \end{aligned} \tag{38}$$

it is possible to greatly simplify the algebraic expressions for the fifth powers [in fact $t = 1$*, by (31)]. It is thus finally shown that fifth root of* t_j^5*,* $j = 1, 2, 3, 4$*, are linear combinations in the roots* a, b, c, d, e*. It then follows, after some algebra, that*

$$a = \frac{1}{5}\left[1 + \sqrt[5]{t_1^5} + \sqrt[5]{t_2^5} + \sqrt[5]{t_3^5} + \sqrt[5]{t_4^5}\right] \tag{39}$$

Using the above simplifications for t_j^5, we can express them in terms of the coefficients of the quintic in (31). We then find

$$\begin{aligned}
(t_1)^5 &= \tfrac{11}{4}\left[(89 + 25\sqrt{5}) + i(45\sqrt{5 - 2\sqrt{5}} - 5\sqrt{5 + 2\sqrt{5}})\right] \\
(t_2)^5 &= \tfrac{11}{4}\left[(89 + 25\sqrt{5}) - i(45\sqrt{5 - 2\sqrt{5}} - 5\sqrt{5 + 2\sqrt{5}})\right] \\
(t_3)^5 &= \tfrac{11}{4}\left[(89 - 25\sqrt{5}) - i(45\sqrt{5 + 2\sqrt{5}} + 5\sqrt{5 - 2\sqrt{5}})\right] \\
(t_4)^5 &= \tfrac{11}{4}\left[(89 - 25\sqrt{5}) + i(45\sqrt{5 + 2\sqrt{5}} + 5\sqrt{5 - 2\sqrt{5}})\right]
\end{aligned} \tag{40}$$

Eq. (39) for the first root is augmented with similar ones for the roots b, c, d, e. Note that since (t_1^5, t_2^5) and (t_3^5, t_4^5) are complex conjugate in pairs, the expression for a in (39) is real, as it should be.

Note also that to obtain the solutions x_k for $x^{11} - 1 = 0$, one must yet solve the quadratic equation

$$x_k^2 + x_k u + 1 = 0$$

for $u = a, b, c, d, e$. Thus for $k = 1$:

$$x_1 = -\frac{a}{2} + i\sqrt{1 - \left(\frac{a}{2}\right)^2} = \cos\frac{2\pi}{11} + i\sin\frac{2\pi}{11} \tag{41}$$

etc.

(c) Gauss and $n = 17$ (heptadecagon)

Apart from the three classical outstanding ancient Greece non-construction problems (squaring the circle, trisection of an angle and the doubling of the cube), Greek geometers also focused their interest on several other problems, among them the construction of regular polygons *and platonic bodies (regular polyhedra).*

The number of regular polygons which can be constructed in 2-dimensional space is unlimited. The number of regular convex polyhedra in a space of 3 dimensions is five. The Pythagoreans, who were interested in such matters, regarded the dodecahedron *as begin worthy of special respect. By extending the sides of one of its pentagonal faces to form a star, they arrived of the* pentagram, *or* triple triangle, *which they used as a symbol and badge of the Society*

of Pythagoras. By this sign they recognized a fellow member. The construction of the pentagon[390] *and the pentagram discovered by Pythagoreans (and given by Euclid), is directly based on the Golden Section ratio* $\frac{\sqrt{5}+1}{2}$ *and on the formulas:*

$$p_r = \textit{side of the regular pentagon} = \frac{R}{2}\sqrt{10-2\sqrt{5}}$$

$$p_s = \textit{side of the star-pentagon} = \frac{R}{2}\sqrt{10+2\sqrt{5}}$$

$$\frac{p_s}{p_r} = \frac{\sqrt{5}+1}{2} = \frac{\textit{diagonal of regular pentagon}}{\textit{side of a regular pentagon}}$$

where R is the radius of the circumscribed circle.

The construction of the regular n-gons with $n = 3, 4, 5, 6, 8, 10, 12, 15, 16$ *were known to the mathematicians of ancient Greece.*[391]

It was not until 1796 that any further constructions of regular polygons were discovered. In that year, Gauss, a student of mathematics at Göttingen who had just turned 19, proved that it is possible to construct the regular 17-gon with ruler and compass.

In the 7^{th} and last section of Disquisitiones arithmeticae (1801), Gauss turns to the general problem of constructing regular polygons by compass and ruler[392]*. The geometric entities that are constructible from known data by means of compass and ruler correspond algebraically to those expressions that*

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390 No pentagon or decagon is to be found in *Egyptian* monuments, although it is easy enough to divide a circle into 5 equal parts *without* geometrical consciousness of any kind. Pentagonal ornaments occur in *Mycenaean* art, prisms of heptagonal shape were found in Babylon, and regular dodecahedron of *Etruscan* and *Celtic* origins were discovered. Thus, elaborate geometrical ornaments can be drawn without explicit geometry.

391 The case $n = 15$ is interesting: knowing how to construct geometrically the angle for the equilateral triangle (120°) and the regular pentagon (72°), the angle for the regular 15-gon (24°) is half the difference angle $\left[\frac{120^\circ - 72^\circ}{2} = 24^\circ\right]$. It can also be constructed as the sum $\frac{60^\circ}{4} + \frac{72^\circ}{8} = 15^\circ + 9^\circ = 24^\circ$.

392 It is assumed that each of these two instruments be used only for a single, specific operation: with the compass, circles with *given* center and circumference-point can be drawn; with the ruler, a straight line can be drawn through two *given* points. Thus, marking on the ruler cannot be utilized. Any construction that can be performed by compass and ruler can be made by compass alone, and, also, if a fixed circle has been drawn, the construction may be achieved by ruler alone!

Since the algebraic calculation of the intersection points of a circle and a straight

may be deduced from given numbers by repeated use of the four rational operation and square root extraction. Thus, in principle, construction problems are transliterated into questions in the theory of equations.

Therefore, to decide on the possibility of solving a construction problem, one must examine first whether the quantity to be found satisfies an algebraic equation that is rational in the given quantities, and second whether this equation has a constructible solution, i.e. whether it is solvable by square roots.

There may be several such equations, but among them there is one of minimal degree, which cannot be factored further with rational coefficient (known as irreducible), and it divides all other equations of the same kind. For this minimal equation to be solvable by square roots it must have very special properties. One of these is that its degree must be a power of 2*. Indeed, the unsolvable problem of the duplication of the cube leads to the cubic equation* $x^3 - 2 = 0$*. Similarly, the trisection of any angle* α *leads to the cubic equation* $4x^3 - 3x - \cos\alpha = 0$ *and, in general, one cannot decompose this equation further into factors whose coefficients depend rationally on* $\cos\alpha$*.*

A regular polygon with n *sides has its vertices equidistant on a circle (say of unit radius, without loss of generality). Since each side of the polygon corresponds to a central angle of* $\frac{360^\circ}{n}$*, the problem is to divide a full angle of* 360° *into* n *equal parts. Now, any angle can be bisected, so when a regular polygon with* n *sides has been obtained, one can successively construct a polygon with* $2^m n$ *sides. On the other hand, from a polygon with* $2n$ *sides, one can draw one with* n *sides by joining every second vertex by a side. Consequently, one can limit the considerations only to regular polygons with an odd number of sides. From the fact that regular polygons with* $3, 4, 5$ *sides can be easily constructed, it follows that all polygons with* 2^m, $3 \cdot 2^m$, $5 \cdot 2^m$ *sides are constructible. Furthermore, given the constructability of polygons with sides* a *and* b*, where* a *and* b *are relatively prime, a polygon with* ab *sides is obtainable. Clearly, the side of a regular* n*-polygon inscribed in unit circle with* n *sides is* $s_n = \sqrt{2 - 2\cos\frac{2\pi}{n}}$*.*

All this was known before Gauss. But instead of dealing with these quantities directly, Gauss took a step that at the time was innovative: he used

line, or of a circle with another circle leads to a second degree equation, the coordinates are obtained as the sum of a rational expression and the *square root* of such an expression. The distance between two points is also expressible as a square root. Since all other allowed constructions can be composed of a series of these simple operations, those magnitudes that can be constructed from given ones may be computed algebraically by repeated operations of the four arithmetic operations and by extracting square roots (and vice versa).

the unit circle in the complex plane. In the circle he inscribed a regular polygon with n *sides such that one vertex lies on the positive real axis at the point* $x = 1$*. The next vertex will correspond to the complex number* $R = R_1 = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ *and the subsequent ones to* $R_k = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}$, $k = 1, 2, 3, \dots, n-1$ *with* $R_k = R^k$ *by the theorem of de Moivre. This means that* R *as well as its powers are roots of the algebraic equation* $x^n - 1 = 0$.

Gauss then proved that a necessary and sufficient condition that a regular polygon with n *sides could be inscribed in a circle (constructed) by compass and ruler is that*

$$n = 2^m p_1 p_2 \dots p_n,$$

where the prime factors are also Fermat numbers $p_k = 2^{2^k} + 1$.

Since $17 = 2^{2^2} + 1$*, a regular polygon of* 17 *sides can be inscribed in a circle by ruler and compass. The possibility of this construction is proved if we show that* $\cos\frac{2\pi}{17}$ *can be constructed.*

Starting from the equation $x^{17} - 1 = 0$*, we arrive at*

$$x^{16} + x^{15} + x^{14} + \dots + x^2 + x + 1 = 0,$$

with

$$R = \cos\frac{2\pi}{17} + i\sin\frac{2\pi}{17},$$

Gauss arranged the roots in the order

$$R, R^3, R^9, R^{10}, R^{13}, R^5, R^{15}, R^{11}, R^{16}, R^{14}, R^8, R^7, R^4, R^{12}, R^2, R^6,$$

each of which is the cube of the preceding, the first being the cube of the last.

Set

$$y_1 = R + R^9 + R^{13} + R^{15} + R^{16} + R^8 + R^4 + R^2$$
$$y_2 = R^3 + R^{10} + R^5 + R^{11} + R^{14} + R^7 + R^{12} + R^6.$$

Then $y_1 + y_2 = -1$, $y_1 y_2 = -4$, *and each* y *satisfies the equation* $y^2 + y - 4 = 0$ *whose roots are*

$$y = \frac{\pm\sqrt{17} - 1}{2}.$$

But

$$y_1 = (R + R^{16}) + (R^2 + R^{15}) + (R^4 + R^{13}) + (R^8 + R^9)$$
$$= 2\cos\frac{2\pi}{17} + 2\cos\frac{4\pi}{17} + 2\cos\frac{8\pi}{17} + 2\cos\frac{16\pi}{17} > 0$$

$$\therefore \quad y_1 = \frac{\sqrt{17}-1}{2}, \qquad y_2 = \frac{-\sqrt{17}-1}{2}\,. \tag{42}$$

Now set

$$\begin{aligned} z_1 &= R + R^{13} + R^{16} + R^4 = 2\cos\frac{2\pi}{17} + 2\cos\frac{8\pi}{17} > 0 \\ z_2 &= R^9 + R^{15} + R^8 + R^2 = 2\cos\frac{4\pi}{17} + 2\cos\frac{16\pi}{17} < 0. \end{aligned}$$

Then $z_1 + z_2 = y_1$, $z_1 z_2 = -1$, *and each* z *satisfies the equation* $z^2 - y_1 z - 1 = 0$, *whose roots are*

$$\begin{aligned} z_1 &= \frac{\sqrt{17}-1}{4} + \frac{\sqrt{34-2\sqrt{17}}}{4}; \\ z_2 &= \frac{\sqrt{17}-1}{4} - \frac{\sqrt{34-2\sqrt{17}}}{4}. \end{aligned} \tag{43}$$

Now set

$$\begin{aligned} w_1 &= R^3 + R^5 + R^{14} + R^{12} = 2\cos\frac{6\pi}{17} + 2\cos\frac{10\pi}{17} > 0 \\ w_2 &= R^{10} + R^{11} + R^7 + R^6 = 2\cos\frac{12\pi}{17} + 2\cos\frac{14\pi}{17} < 0. \end{aligned}$$

Then $w_1 + w_2 = y_2$, $w_1 w_2 = -1$, *and each* w *satisfies the equation*

$$w^2 - y_2 w - 1 = 0,$$

whose roots are

$$\begin{aligned} w_1 &= \frac{-\sqrt{17}-1}{4} + \frac{\sqrt{34+2\sqrt{17}}}{4}; \\ w_2 &= \frac{-\sqrt{17}-1}{4} - \frac{\sqrt{34+2\sqrt{17}}}{4}. \end{aligned} \tag{44}$$

Now set

$$u_1 = R + R^{16} = 2\cos\frac{2\pi}{17}, \qquad u_2 = R^4 + R^{13} = 2\cos\frac{8\pi}{17}.$$

Then

$$u_1 > u_2 > 0, \qquad u_1 + u_2 = z_1, \qquad u_1 u_2 = w_1.$$

Whence each u *satisfies the equation* $u^2 - z_1 u + w_1 = 0$ *whose roots are*

$$u_1 = \frac{z_1 + \sqrt{z_1^2 - 4w_1}}{2} = 2\cos\frac{2\pi}{17}, \quad u_2 = \frac{z_1 - \sqrt{z_1^2 - 4w_1}}{2}, \tag{45}$$

and we see that $u_1/2 = \cos\frac{2\pi}{17}$ *can be obtained by a finite number of rational operations and extraction of square roots of real numbers. Hence the regular polygon of* 17 *sides is constructible.*

Gauss' final algebraic expression, as obtained by the substitution of (43) and (44) into (45) is

$$\begin{aligned}\cos\tfrac{2\pi}{17} = -\tfrac{1}{16} + \tfrac{1}{16}\sqrt{17} + \tfrac{1}{16}\sqrt{34 - 2\sqrt{17}} \\ + \tfrac{1}{8}\sqrt{17 + 3\sqrt{17} - \sqrt{34 - 2\sqrt{17}} - 2\sqrt{34 + 2\sqrt{17}}};\end{aligned} \tag{46}$$

Note that the problem is not entirely *algebraic because the* sign *of the radicals in Gauss' formula (46) must be determined by* nonalgebraic *means.*

The construction of regular polygons had interested Gauss since 1796 when he conceived the first proof that the 17-sided polygon is constructible. There is a story about this discovery. One day Gauss approached his professor A.G. Kästner at the University of Göttingen with the proof that this polygon is constructible. Kästner was incredulous and sought to dismiss Gauss, much as university teachers today dismiss angle-trisectors. Rather than take the time to examine Gauss' proof and find the supposed error in it, Kästner told Gauss the construction was unimportant because practical constructions were known. Of course Kästner knew that the existence of practical or approximate constructions was irrelevant for the theoretical problem. To interest Kästner in his proof Gauss pointed out that he had solved a seventeenth degree algebraic equation. Kästner replied that the solution was impossible. But Gauss rejoined that he had reduced the problem to solving an equation of lower degree. "Oh well," scoffed Kästner, "I have already done this."

Gauss also proved the following important theorems:

- *For every prime* p*, the cyclotomic polynomial*

$$f_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1 \tag{47}$$

  is irreducible over the field of rational numbers.

- *For* every *integer* n*, the* n^{th} *roots of unity have expressions by radicals.*

4. Circulant matrices and the roots of unity

There is an interesting and useful connection between matrix algebra and

roots of polynomials through a special kind of matrices known as circulants[393]. *An $n \times n$ circulant matrix is defined by*

$$C_n = \begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_n \\ c_n & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_n & c_1 & \cdots & c_{n-2} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ c_2 & c_3 & c_4 & \cdots & c_1 \end{bmatrix}, \tag{48}$$

where the c's are real or complex. Each row in (48) consists of the elements of the preceding row shifted one position to the right, with the 'overflow' element begin moved to the first position. The matrix is entirely determined by its first row. Three properties can be seen immediately by inspection of (48):

- *The elements along each diagonal line parallel to the principal diagonal (including the principle diagonal itself) are equal.*
- *Transpose of a circulant is also a circulant.*
- *C_n is symmetric w.r.t. its secondary diagonal (the line from top right corner to bottom left corner.*

Using the notation $C_n = \mathrm{circ}(c_1, c_2, \ldots, c_n)$, an important special circulant matrix is

$$W_n = \mathrm{circ}(0, 1, 0, \ldots, 0)_n = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}_n, \tag{49}$$

known as the permutation matrix of order n or the shift matrix, because the postmultiplying of any matrix by W_n shifts its columns one place to the right (a similar shift is applied to rows on premultiplying by W). Clearly

$$C_n = c_1 I + c_2 W + c_3 W^2 + \cdots + c_n W^{n-1} = \sum_{k=1}^{n} c_k W^{k-1} \tag{50}$$

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393 *Circulants* were introduced by **Catalan** (1846) and further investigated by **Bertrand** (1850), **Sylvester** (1855), **Cremona** (1856), **Bellavitis** (1857) and **Souillart** (1858).

and

$$C_n W_n = W_n C_n \tag{51}$$

For example, with $n = 3$:

$$C_3 = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix} = a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \quad aI_3 \quad + \quad bW_3 \quad + \quad cW_3^2$$

Using (50), it can be shown that the inverse of C_n is also a circulant matrix. Note that if we denote $C = [c_{ij}]$, then $c_{ij} = c_{j-i+1}$ with $c_{-k} = c_{n-k}$ for $k \geq 0$. With this notation, one easily shows that the matrix product of any two circulant matrices is also a circulant. In other words, circulants form a group under multiplication. It also follows directly from (50) that any two circulants of the same order commute

$$C_1 C_2 = C_2 C_1 \tag{52}$$

Eigenvalues of a circulant

It is known from the theory of matrices that if λ is an eigenvalue of an $n \times n$ matrix A i.e. $A\overrightarrow{x} = \lambda \overrightarrow{x}$ or $|A - \lambda I| = 0$, then the Cayley-Hamilton theorem guarantees that $P(A)\overrightarrow{x} = P(\lambda)\overrightarrow{x}$, *where $\overrightarrow{x}$ is an eigenvector and P is any polynomial. Let us first calculate the eigenvalues of W_n, through the equation*

$$|W_n - \lambda I_n| = \begin{vmatrix} -\lambda & 1 & 0 & \cdots & 0 \\ 0 & -\lambda & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ & & & & 1 \\ 1 & & \cdots & & -\lambda \end{vmatrix}_n \tag{53}$$

Clearly, the characteristic polynomial *of W_n is $\lambda^n = 1$ and therefore its eigenvalues are the n^{th} roots of unity. Because of (50) and on the strength of the Cayley-Hamilton theorem, the k^{th} eigenvalues of a circulant C_n is $(\lambda_1 = 1)$*

$$r_k = c_1 + c_2\lambda_k + c_3\lambda_k^2 + \cdots + c_n\lambda_k^{n-1}, \qquad k = 1, 2, \ldots, n \tag{54}$$

This implies that the determinant *of C_n can be expressed in the compact form*

$$|C_n| = \prod_{k=1}^{n} (c_1 + c_2\lambda_k + c_3\lambda_k^2 + \cdots + c_n\lambda_k^{n-1}). \tag{55}$$

For example

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ x_3 & x_1 & x_2 \\ x_2 & x_3 & x_1 \end{vmatrix} = (x_1 + x_2 + x_3)(x_1 + \omega x_2 + \omega^2 x_3)(x_1 + \omega^2 x_2 + \omega x_3),$$

where ω and ω^2 *are the complex cube roots of unity.*

We note that (54) can be recast in the matrix relation

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \lambda & \lambda^2 & \cdots & \lambda^{n-1} \\ 1 & \lambda^2 & \lambda^4 & \cdots & \lambda^{2(n-1)} \\ \vdots & & & \cdots & \\ 1 & \lambda^{n-1} & \lambda^{2(n-1)} & \cdots & \lambda^{(n-1)^2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_n \end{bmatrix}. \tag{56}$$

The $n \times n$ *matrix on the l.h.s. is a Vandermonde type matrix* $V(1, \lambda, \lambda^2, \ldots, \lambda^{n-1})$. *When multiplied by* $\frac{1}{\sqrt{n}}$ *it is known as the Fourier-Transform matrix* $F_n = \frac{1}{\sqrt{n}} V$. *It is a unitary matrix, its inverse being equal to its conjugate transpose*

$$F^{-1} = F^*.$$

Let

$$D_n = \begin{bmatrix} 1 & & & & 0 \\ & \lambda & & & \\ & & \lambda^2 & & \\ & & & \ddots & \\ 0 & & & & \lambda^{n-1} \end{bmatrix}, \qquad \lambda = e^{-\frac{2\pi i}{n}}. \tag{57}$$

Using (50), (56) (or 54), one readily proves the important relations

$$\begin{aligned} W_n &= F_n^{-1} D_n F_n & D_n &= F_n W_n F_n^{-1} \\ C_n &= F_n^{-1} \Delta_n F_n & \Delta_n &= F_n C_n F_n^{-1} \end{aligned} \tag{58}$$

where

$$\Delta_n = c_1 I + c_2 D + c_3 D^2 + \cdots c_n D^n. \tag{59}$$

This means that all elements of a circulant C_n are simultaneously diagonalized by the same unitary matrix.

A special type of circulant matrix is defined as

$$C_n = \begin{bmatrix} 1 & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{n-1} \\ \binom{n}{n-1} & 1 & \binom{n}{1} & \cdots & \binom{n}{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{bmatrix} , \tag{60}$$

where $\binom{n}{k}$ is a binomial coefficient. The determinant of C_n is given by the formula

$$C_n = \prod_{j=0}^{n-1} [(1+\lambda_j)^n - 1] \,. \tag{61}$$

Thus, the computation of a circulant's eigenvalues is actually quite trivial: simply generate a polynomial from the first row

$$q(t) = c_1 + c_2 t + \cdots + c_n t^{n-1}$$

and then evaluate the polynomial at $t = \lambda_k$. For example, if

$$C_4 = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 3 & 1 & 2 & 1 \\ 1 & 3 & 1 & 2 \\ 2 & 1 & 3 & 1 \end{bmatrix}$$

Since $q(t) = 1 + 2t + t^2 + 3t^3$ and $\lambda_1 = 1$; $\lambda_2 = -1$; $\lambda_3 = i$; $\lambda_4 = -i$, we have for the eigenvalues of C:

$$q(1) = 7; \qquad q(-1) = -3; \qquad q(i) = -i; \qquad q(-i) = i \,.$$

Suppose that $\mu_1, \mu_2, \ldots, \mu_{r-1}$ are the roots of the polynomial

$$q(z) = c_1 + c_2 z + \cdots + c_n z^{n-1} \tag{62}$$

known as the representer of the circulant (to be distinguished from the eigenvalues of C_n). Then

$$C_n = q(W) = a_r (W - \mu_1 I)(W - \mu_2 I) \cdots (W - \mu_{r-1} I) \,. \tag{63}$$

This gives us a factorization of any circulant into a product of circulants $(W - \mu_k I)$ that are of a particular elementary type.

Let C_n be nonsingular. i.e.: non of the eigenvalues of C_n is zero, namely, $\lambda_j = q(W^{j-1}) \neq 0, \quad j = 1,2,\ldots,n$. This will be true iff $\mu_k^n \neq 1, \quad lk = 1,2,\ldots,r-1$. Then, from (63) one has

$$C_n^{-1} = a_r^{-1}(W-\mu_1 I)^{-1}(W-\mu_2 I)^{-1}\cdots(W-\mu_{r-1}I)^{-1}\ . \tag{64}$$

It can be shown that for $\mu^n \neq 1$

$$(W-\mu I)^{-1} = \frac{1}{1-\mu^n}\left[\mu^{n-1}I + \mu^{n-2}W + \mu^{n-3}W^2 + \cdots + W^{n-1}\right]\ . \tag{65}$$

The characteristic polynomial of C_n

Given a $n \times n$ circulant matrix C_n, the polynomial

$$P_n(x) = \det(xI - C_n) = x^n + p_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \tag{66}$$

is the characteristic polynomial of C_n. Its roots r_k, as determined in (54), are the eigenvalues of C_n. Clearly

$$p_{n-1} = -(r_1 + r_2 + r_3 + \cdots + r_n) = \text{trace of } C_n = -nc_1\ . \tag{67}$$

If we effect the transformation $y = x - \frac{1}{n}p_{n-1}$ in (66), the term of degree $n-1$ is eliminated and this operation corresponds to making the trace of C_n vanish. Thus, the circulant matrix for the modified polynomial has vanishing diagonal and trace; such a matrix is called a traceless circulant.

Suppose we wish to obtain expressions for the roots of $p_3(x) = x^3 + \beta x + \gamma$ as the eigenvalues of a circulant matrix

$$C_3 = \begin{bmatrix} 0 & b & c \\ c & 0 & b \\ b & c & 0 \end{bmatrix} \tag{68}$$

The characteristic polynomial of C_3 is $x^3 - 3bcx - (b^3 + c^3)$. This equals $p_3(x)$ if $b^3 + c^3 = -\gamma; \quad 3bc = -\beta$. To complete the solution of the original equation, we must solve this system for b and c, and then apply $q(x) = bx + cx^2$ to the cube roots of unity. That is, for any a and b satisfying $b^3 + c^3 = -\gamma$ and $3bc = -\beta$, we obtain the roots of p as

$$\begin{aligned} q(1) &= b + c \\ q(\omega) &= b\omega + c\omega^2 \\ q(\bar\omega) &= b\bar\omega + c\bar\omega^2 \end{aligned} \tag{69}$$

Solving, then, the above equations for the unknowns b^3 and c^3 we obtain

$$b = \left[\frac{-\gamma + \sqrt{\gamma^2 + 4\beta^3/27}}{2} \right]^{1/3} , \qquad c = -\frac{\beta}{3b} , \tag{70}$$

which is essentially the Cardano solution. What distinguishes this approach is the role of the roots of unity and the immediate extendability of the circulant approach to the quadric equation and solvable polynomial equations of higher degree.

Circulant matrices have important applications to diverse disciplines including physics, image processing, probability and statistics, numerical analysis, number theory and geometry. The built-in periodicity also means that circulants are closely related to Fourier analysis and group theory.

1796–1825 CE **Georges (Léopold Chrétien Frédéric Dagobert) Cuvier** (1769–1832, France). Geologist and paleontologist. Author of the geological-historical concept of world revolutions in nature. Founder of comparative anatomy and paleontology (1805). Attributed fossil succession to extinction caused by a series of *natural catastrophes*[394] rather than to evolution (1812–1825). Developed a method of classifying mammals (1796) and gave an account of the whole animal kingdom, dividing it into four groups (1817).

[394] *Crisis*: an event that occurs in the history of a system, when stress is sufficient to cause the imminent alteration of the system's principal structures, but, through the absorption of this stress into its subsystems, the system survives.

Catastrophe: an event that occurs in the history of a system, when stress is sufficient to cause the imminent alteration of the system's principal structures; and the subsystems fail to absorb all of the stress but survive, although the system fails. In such cases, a new and modified system is then formed to take the place of the failed system.

Cataclysm: an event that occurs in the history of a system, when stress is sufficient to cause the imminent alteration of the system's principal structures, and both the system and its subsystems fail.

In each of the three events just described, the source of the stress is left undefined; but, for the present, it can be inferred to be external. Crises occur often, catastrophes happen less often, and cataclysms rarely occur on a grand scale.

Cuvier was born at Montbéliard, in Wurthenburg (now a part of Burgundy) of a poor Lutheran military family of Huguenot stock. After spending four years at the Caroline University near Stuttgart, he accepted the position of tutor in the family of the Comte d'Hericy. Like **Laplace**, he was appointed (1795) assistant at the Muséum d'Histoire Naturelle. He later became professor of natural history in the Collège de France (1799), and titular professor at the Jardin des Plantes (1802).

During the early years of the 19^{th} century Cuvier was a man of considerable influence, earning for himself in the sciences the title of 'the dictator of biology.' In 1808 he was placed by Napoleon upon the council of the Imperial University, assisting the latter in the reorganization of higher education. In 1831 he was raised by Louis-Philippe to the rank of peer of France.

Cuvier lived through turbulent times: the fall of the nobility, the French Revolution, the reign of Napoleon, the return of the nobility, the fall of the Church, and the resurgence of its influence. Like **Laplace**, he was a "survivor", who died rich, famous and powerful. His vanity was boundless, as was his hunger for honors and praise. He was said to have had an exceptional memory and to have known the contents of all 19,000 books in his library.

His life story and character may explain why he chose the catastrophic point of view.

1796–1826 CE **Aloys Senefelder** (1771–1834, Germany). Inventor of *lithography*. In 1826 he invented a process of lithographing in color. Born in Prague. Director of the royal printing office in Münich (1809).

1796–1833 CE **Samuel Hahnemann** (1755–1843, Germany). Physician. Founded *homeopathic medicine*. This medical system is based on simple remedies (exercise, a nourishing diet, and pure air), and on two fundamental principles:

- diseases are cured by drugs which produce in healthy persons the symptoms found in those who are ill;
- the smaller the dose, the more efficacious the medicine.

Hahnemann was born in Meissen, Saxony. He studied medicine at Leipzig and Vienna and received his M.D. at the University of Erlangen (1779). Through his practice he quickly discovered that the medicine of his day (purgatives, emetics, blistering, cupping, sweating, bloodletting and huge doses of calomel and other mineral drugs) did as much harm as good.

He then gave up his practice and made his living as a writer and translator. While translating William Cullen's *A Treatise on the Materia Medica*,

Hahnemann encountered the claim that Cinchona, the bark of a Peruvian tree, was effective in treating malaria because of its astringency. Hahnemann realized that other astringent substances are not effective against malaria and began to research cinchona's effect on the human organism very directly: by self-application. He discovered that the drug evoked malaria-like symptoms in himself, and concluded that it would do so in any healthy individual. This led him to postulate a healing principle: "*that which can produce a set of symptoms in a healthy individual, can treat a sick individual who is manifesting a similar set of symptoms.*" This principle, '*like cures like*', became the first of a new medicinal approach to which he gave the name *homeopathy*.

Hahnemann began systematically testing substances for the effect they produced on a healthy individual and trying to deduce from this the ills they would heal. He quickly discovered that ingesting substances to produce noticeable changes in the organism resulted in toxic effects. His next task was to solve this problem, which he did through exploring dilutions of the compounds he was testing. He discovered that these dilutions, when done according to his technique of succussion (systematic mixing through vigorous shaking) and potentization, were still effective in producing symptoms.

Hahnemann began practicing medicine again using his new technique, which soon attracted other doctors. He first published an article about the homeopathic approach to medicine in a German medical journal in 1796; in 1810, he wrote his *Organon of the Medical Art*, the first systematic treatise on the subject.

Hahnemann continued practicing medicine, researching new medicines, writing and lecturing to the end of a long life. He died in 1843 in Paris, 88 years of age, and is entombed in a mausoleum at Paris' Pére Lachaise cemetery.

1797 CE **Lorenzo Mascheroni** (1750–1800, Italy). Mathematician and poet. Published a variety of mathematical works. He shares with Euler the name of the number $\gamma = \lim_{n\to\infty}\left[\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log_e n\right]$, known as the *Mascheroni-Euler number*.

$\gamma = 0.577\ 215\ 664\ 901\ 532\ 860\ 606\ 512\ 090\ 082\ 402\ 431\ 042\ 159\ 335\ 939\ 92\ldots$

is one of the most mysterious of all arithmetic constants. It appears unexpectedly in several places in number theory.

In *Geometria del compasson* (1797) he showed that every compass-and-straightedge (unmarked ruler) construction can be done with a compass

alone[395] (it is assumed that two points, obtained by arc intersections, define a straight edge). Among the problems solved by Mascheroni, using the compass alone, were:

- Locating the center of a given circle.
- Finding a point midway between two given points A and B.
- Dividing a circle, its center given, into four equal arcs (known as "Napoleon problem"[396]).

Mascheroni was ordained as a priest at the age of 17. At first he taught rhetoric, then, from 1778, he taught physics and mathematics at the seminary at Bergamo. In 1786 he became professor of algebra and geometry at the university of Paris. He later became rector of that University.

1797–1808 CE **Joseph Louis Proust** (1754–1826, France). Chemist. Discovered the quantitative nature of chemical combination through the *Law of Definite Proportions*. He also was first to distinguish a chemical *compound* from a simple *mixture* of elements. Identified the sugars: *glucose*, *fructose* and *sucrose* in plant juices (1808).

1798 CE **Benjamin Thompson** (1753–1814); **Count Rumford**. British-American scientist, adventurer and political figure. In "*An Inquiry*

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395 A Danish geometer **George Mohr** with no other claim to fame, published this surprising result already in 1672 in a 24-page booklet. It was issued in a Danish edition under the name *Euclides Danicus* and a Dutch edition (1673) under *Compendium Euclidis Curiosi*. The Dutch edition, published anonymously, was translated (1674) into English, but the Danish book was discovered only in 1928 in Copenhagen.

Jean Victor Poncelet suggested a proof (1822) that *all* compass-and-straightedge constructions are possible with a straightedge and a *fixed compass*. But again, a little-known geometer, **Servais**, published this result earlier (1805). Incidentally, the first systematic effort to go beyond the Greek by imposing more severe restrictions on instruments used in construction problems, is ascribed to the Persian mathematician Abu al-Wafa (ca 970 AD). In his work he described constructions possible with a straightedge and a fixed compass.

396 Young Mascheroni was an ardent admirer of Napoleon and the French Revolution. His book *Problems for Surveyors* (1793) was dedicated in verse to Napoleon. The two men met and became friends in 1796, when Napoleon invaded Northern Italy. A year later, when Mascheroni published his book on constructions with the compass alone, he again honored Napoleon with a dedication in a lengthy ode.

Concerning the Source of the Heat Which is Excited by Friction", he reported his experiments which discredited the caloric theory of heat and established heat as a form of energy rather than a substance. Since heat was being introduced by motion, he suggested that heat is a form of motion itself. In 1797, Rumford conjectured the existence of large-scale convection currents in the world oceans. In his essay "*On the propagation of heat in fluids*", he concluded that the existence of cold water at depth in the tropics implies a meridional circulation, transporting deep water from the polar regions toward the equator.

Thompson was born in Woburn, Massachusetts, to a family of wealthy farmers that had settled in New England around 1650. His father died when he was very young and his mother speedily remarried. At the age of 14 he was already versed in algebra, geometry, astronomy and higher mathematics. In 1768 he was apprenticed to a storekeeper at Salem, and occupied himself in chemical and mechanical experiments.

He began his checkered career when at the age of 19 he married a wealthy widow (from the township of Rumford), 14 years his senior. He was allegedly engaged in spying for the British during the American Revolution and had to leave America when the British troops left Boston in 1776. On his arrival in London he entered the civil service and within 4 years rose to a rank of under-secretary of state. His official duties, however, did not interfere with the prosecution of his scientific pursuits, and in 1779 he was elected a fellow of the Royal Society. He then left the civil service and joined the cavalry, which he quit in 1783 at the rank of lieutenant-colonel. He then joined the Austrian army, for the purpose of campaigning against the Turks. At Strasbourg he was introduced to Prince Maximilian, afterwards elector of Bavaria, and was invited by him to enter the civil and military service of that state.

During 1787–1798 he remained in Münich as a minister of war, minister of police and grand chamberlain to the elector. His work to improve the living conditions of the poor in Münich gained him the title of Count of the Holy Roman Empire in 1791. His political and courtly employments, however, did not absorb all his time, and during his stay in Bavaria he contributed a number of papers to the *Philosophical Transactions*. In 1798 he made his greatest contribution to science while supervising the boring of cannons.

The death of the elector Karl Theodor, the rise of Napoleon and the fact that the Bavarians were beginning to find him tiresome, led Rumford to return to England in 1799. In 1800 he helped found the British Royal Institution. In 1804 he established himself in Paris, where he was married briefly and unhappily to the widow of **Lavoisier**. He died at Auteuil, near Paris, at the age of 61.

1798–1801 CE *Napoleon's Egyptian campaign*. A French expedition of 151 scientists, engineers, medical men and scholars created the first modern vision of Egyptian antiquity and its natural history. It resulted in a monumental encyclopedia *Le Description de L'Égypte*, printed between 1809 and 1828 in ten folio volumes of plates (50 cm by 65 cm), three atlases (65 cm by 100 cm) and nine accompanying volumes of text comprising approximately 7000 pages of memoirs, description and commentary.

Serving as permanent secretary of the project was **Jean Baptist Fourier**, who had yet to invent the analysis that bears his name. Among the scientists in the expedition were **Gaspard Monge** (exact sciences), **Claude Louis Berthollet** (physical chemistry), **Étienne Geoffray Saint-Hilaire** (vertebrate zoology), **Jules César Lelorgne de Savigny** (invertebrate zoology, ornithology), **Francois-Michel de Rozière** (mineralogy), **Dominique Jean Larrey** (medicine) and **Jean Francois Champollion** (archaeology).

On the first of July 1798, an armada of 400 ships appeared of the coast of Alexandria. By the end of the day, an army of 36,000 men, under the command of Napoleon Bonaparte landed ashore. On July 21, this army defeated the Mamelukes in the *Battle of the Pyramids*. Ten days later, Admiral Horatio Nelson destroyed the French fleet, marooning the expeditionary force for the next three years[397].

The most famous discovery of the expedition remains the *Rosetta stone*; it was only in 1822 that Champollion succeeded in matching the name Ptolemy in the three scripts — hieroglyphic, demotic and Greek — inscribed on the Rosetta stone, and not until the 1850's were scholars able to construe whole texts.

Another archaeological feat was the excavation of the route of the canal that had linked the Red Sea to the Mediterranean in ancient times.

As for science is the ordinary sense, the Egyptian environment created exceptional opportunities. Some of these achievements are:

(1) **Monge**'s explanation of *mirages* as the effect of light rays from beyond the horizon, reflected from the surface of a layer of air superheated at

[397] In 1798, when all of England's allies in the war against France had been defeated and when the Spaniards had changed sides, the Royal Navy was obliged to withdraw from the Mediterranean. Bonaparte then set out from Toulon with a powerful fleet with the intension of conquering Egypt and attacking the British in India. He had captured Malta, stormed Alexandria and taken Cairo. On Aug. 01, Nelson discovered the French fleet in the Bay of Aboukir and although outmanned and outgunned, he attacked without delay. His victory shattered Napoleon's scheme in Egypt and India and had great political influence in Europe.

ground level by the sun-soaked sand. Although modern optics attributes the effect to dual refraction within the surface layer, Monge had the underlying physics right.

(2) Motivated by the *natural* occurrence of the reaction

$$CaCo_3 + 2NaCl \xrightarrow[\text{humidity}]{\text{heat}} Na_2Co_3 + CaCl_2$$

in limestone formations surrounding saline lakes, **Berthollet** advanced (1803) his theory that the course of reactions is determined not only by the relative concentration of reagents but also by exterior physical factors such as pressure, heat and light. This, in retrospect, is considered as the point of departure for *physical chemistry.*

(3) The zoological studies of the naturalists **Geoffray** and **Savigny** moved beyond *taxonomy* (classification) to *morphology* (form and structure); the former had been the main preoccupation of the natural history of the 18^{th} century; the latter became important subdiscipline of the emerging science of biology in the 19^{th} century.

(4) In a monograph '*On the Physical Constitution of Egypt and its Relation with the Ancient Institutions of the Country*' **Rozière** observed that in no other country has a highly developed society such as that of ancient Egypt, ever exhibited such dependence on a single set of physical factors: everything in the laws of the land and the customs of the people derived from the behavior of the Nile. The rise and fall of the river not only shaped the civilization of Egypt but also accounted for the influence of its culture on the theogonies, the sciences, and the arts and crafts of all antiquity.

(5) **Larrey** gave clinical descriptions of trachoma, bubonic plague, tetanus, yellow fever, leprosy, elephantiasis and gigantism. In his view, the etiology of some of these diseases involved a specific external agent, for which he sometimes used the word *virus* and sometimes *germ*.

With the French conquest of Egypt, began the spread of European science and its appurtenances to African and Asian societies under the aegis of military conquest and political power.

1798–1816 CE *Extinction of German universities*. The political storms which marked the turn of the 19^{th} century dealt a death-blow to some 15 of the old universities in Germany. Among them: **Mainz** (1476–1798), **Cologne** (1388–1798), **Bamberg** (1648–1804), **Altdorf** (1580–1807), **Frankfurt a.o.** (1506–1809), **Wittenberg** (1502–1815) and **Erfurt** (1379–1816).

Rumford and Caloric

Many earlier experimenters believed that heat was a weightless, highly elastic, self-repellent fluid, indestructible and uncreatable. This fluid they called caloric[398]*. The caloric theory offered an explanation of the facts then known: bodies emitted heat because the particles of caloric repelled one another strongly. Differences in specific heats were due to the different attracting powers of different substances for the fluid. Expansion occurred because the self-repellent fluid tended to increase the volume of any body in which it was lodged. Latent heat was supposed to enter into combination with the particles of the material: thus water = ice + latent heat. If a simple theory of this kind served to explain all the observed facts, it was quite reasonable to look no further. But the generation of heat by percussion and by friction presented difficulties.*

The theory stated that the rise of temperature of a block of lead, when hammered, was due to the extrusion of caloric under pressure, much as water issues from a sponge when it is squeezed. The rise in temperature of two bodies when rubbed together was due to the diminution of the bodies themselves: as the small particles rubbed off, the bodies' overall power of attracting caloric, decreased, and some of it was thereby freed — that is, the specific heat of a finely powdered substance was less than that of the same substance in one solid mass. No attempt seems to have been made to detect this diminution of specific heat, and this argument could not possibly explain the generation of heat by the churning of a liquid.

Rumford performed a series of experiments (1798) at the Münich military arsenal, Germany, in which heat was generated by rotating a blunt cannon borer in a large mass of gun-metal. He observed that a large quantity of heat (sufficient to raise nearly 12 kg of water from the freezing point to the boiling point in one experiment) was released from the abrasion of a very small quantity of metallic dust, and he found that the specific heat of this dust was not appreciably different from that of the solid material, which showed that the caloric theory was false, on this point at least. Further, the supply of heat appeared to be inexhaustible, and it was clear that no closed system could supply unlimited amounts of any material substance.

In a paper published in 1799, ***Humphry Davy*** *(1778–1829, England) reported that rubbing two blocks of ice together in a vacuum, using a clockwork*

[398] The term "phlogiston" was used before "caloric", in a similar but not identical manner.

mechanism, caused the ice to melt at the surface in contact. As it was well known that considerable latent heat is required to turn ice into water, and as no other possible source of heat was available, this experiment was claimed to demonstrate that heat was evolved here by mechanical action only. Also, the specific heat of the product (water) is approximately double that of the solid used. Although this work has been accepted for many years, it seems very doubtful if Davy, then 19 years of age, could have carried out such an experiment, which would tax the ingenuity of any trained physicist.

In spite of the works of Rumford and Davy, the caloric theory remained in favor for some 50 *more years, until finally wiped out by the theories and experiments of* ***Mayer*** *(1840),* ***Joule*** *(1847),* ***Helmholtz*** *(1847),* ***Kelvin*** *(1852) and* ***Rankine*** *(1853).*

1798–1820 CE **Thomas Robert Malthus** (1766–1834, England) Economist. Aroused controversy by his *Essay of the Principles of Population Theory*, based on the premise that population, when unchecked, tends to increase in a geometrical progression (doubling every 25 years), whereas the means of subsistence tend to increase only in an arithmetic progression. From this Malthus concluded that:

- population always increases as the means of subsistence increase.
- population is limited by the means of subsistence.
- population is kept from overgrowing the means of subsistence by two kinds of checks: *positive checks* (disease, war, famine etc.) and *preventive checks* (voluntary abstinence from sex indulgence).

The Malthusian population theory went hand in hand with *Wage Theory*: If laborers receive wages affording them more than mere subsistence, they will raise more children. The number of people will thus increase until there are more than can be fed and the population will reduce to numbers that can just be supported by the available means of subsistence. It is thus useless to attempt to relieve the laboring classes of their misery: in the struggle for survival the fittest come out on top, the unfit perish. This is better than to keep the unfit alive through charity and to let the fit die instead. **Darwin** built his theory of evolution on this Malthusian idea.

Although it contains much that is true, the theory has been criticized in several counts: (1) The ratios of increase of population and the means of subsistence are hypothetical, not actual; (2) Population does not always increase to the full extent of its biological capacity[399]; (3) Means of subsistence may increase faster than population due to *technological improvements*; the experience of the last 150 years bears this out.

The doctrine of Malthus was a corrective reaction against the superficial optimism diffused by the school of Rousseau and its blindness to the real conditions that circumscribe human life.

Malthus was born near Guilford, Surrey. He was educated by private tutors and went to Cambridge (1784–1797). He then became a curate at a small parish in Albury, Surrey (1797); During 1805–1834 he was professor of history and political economy at the East India Company's college at Haileybury.

1798–1805 CE **Johann Wilhelm Ritter** (1776–1810, Germany). Physicist. His suggestion that the galvanic current was due to a *chemical* interaction between the metals (1798), was the first electrochemical explanation of this phenomenon. Discovered the process of *electroplating* (1800); discovered existence of ultraviolet radiation through its effect of darkening a silver chloride film (1801); observed thermoelectric currents (1801); invented the *dry voltaic cell* (1802) and the electrical storage battery (1803).

Ritter was born in Samitz, Silesia (now Poland) and began his career as an apothecary. He then studied at the University of Jena (1796). The basic concept of electrolysis and electroplating was discovered by Ritter at the same time or in some cases earlier than the experiments of **Carlisle**, **Nicholson** and **Davy**.

In 1801 he observed thermoelectric currents, anticipating the discovery of *thermoelectricity* by **Seebeck** (1821). In 1805, Ritter moved to Munich to take a position at the Bavarian Academy of Science. He died at the young age of 33, as a direct result of exposing his body to very high voltages in his experiments on the electrical excitation of muscle and sensory organs.

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[399] On the eve of the *Agricultural Revolution* (10,000 BCE), the human species numbered about 4 million people. On the eve of the *Industrial Revolution* (ca 1750 CE), the total world population was estimated at 800 million people. In 1950 world population reached 2485 ($\pm$ 5%) millions, and in 1990 it was about 5300 million. Projections to the years 2000 and 2050 CE yield the respective estimates of 6200 and 10,000 million.

1797–1815 CE **Johann Friedrich Pfaff** (1765–1825, Germany). Mathematician. Presented the theory of *Pfaffian forms* and *equations*[400] (1797). His work constituted the starting point of a basic theory of integration of PDE which, through later work of **Jacobi**, **Lie** and others, has developed into the modern **Cartan**'s exterior calculus of differential forms.

Pfaff was born in Stuttgart to a distinguished family of Württemberg civil servants. At age of 9 he went to the Hohe Karlsschule in Stuttgart, a school with harsh military discipline, serving chiefly to train servile government officials. Pfaff completed his legal studies there in 1785 and then spent a few years in travel and study at the Universities of Göttingen, Berlin, Vienna, Halle, Jena and Prague.

He finally settled down as a professor of mathematics at the University of Helmstadt. Gauss, after completing his studies at Göttingen (1795–1798), lived in Pfaff's house. Pfaff recommended Gauss' doctoral dissertation and, when necessary, greatly assisted him. Gauss always retained a friendly memory of Pfaff both as a teacher and as a man.

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[400] The expression $\sum_{i=1}^{n} F_i(x_1, x_2, \ldots, x_n)dx_i$ in which the F_i $(i = 1, 2, \ldots, n)$ are functions of the n independent variables $x_1, x_2, \ldots, x_n$, is called a *Pfaffian differential form* in n variables. Similarly, the relation $\sum_{i=1}^{n} F_i dx_i = 0$ is called a *Pfaffian differential equation*.

In the case of 2 variables, the form $P(x, y)dx + Q(x, y)dy = 0$ is equivalent to $\frac{dy}{dx} = f(x, y) = -\frac{P}{Q}$. If $\{P, Q\}$ are single-valued functions, then $\frac{dy}{dx}$ is single-valued, and the solution to the above ODE which satisfies the boundary condition $y_0 = y(x_0)$ consists of a curve which passes through this point and whose tangent at each point is defined by the DE. Thus the original Pfaffian equation defines a one-parameter family of curves in the xy-plane. It can be shown that a Pfaffian DE in 2 variables always possesses an *integrating factor*, i.e. there exist $\mu(x, y)$, $\phi(x, y)$ such that $0 = \mu(Pdx + Qdy) = d\phi$ or $\frac{1}{P}\frac{\partial\phi}{\partial x} = \frac{1}{Q}\frac{\partial\phi}{\partial y} = \mu$.

When there are 3 variables, the Pfaffian DE is of the form $Pdx+Qdy+Rdz = 0$. If we introduce the vectors $\mathbf{x} = (P, Q, R)$ and $d\mathbf{r} = (dx, dy, dz)$, we may write this equation in vector notation as $\mathbf{x} \cdot d\mathbf{r} = 0$. A necessary and sufficient condition that the Pfaffian DE $\mathbf{x} \cdot d\mathbf{r} = 0$ should be integrable is that $\mathbf{x} \cdot \operatorname{curl} \mathbf{x} = 0$ [this theorem figures prominently in theoretical thermodynamics. Moreover, the exterior calculus of **Cartan** can be used to extend it to criteria for solvability of systems of ODE's].

The representation of a given vector-field $\overrightarrow{f}$ in the form $\overrightarrow{f} = \nabla w + u\nabla v$, where (u, v, w) are scalar functions of the coordinates, is known as the *Pfaff Problem*.

In 1810, the University of Helmstadt was closed and Pfaff went to Halle, where he stayed to the end of his life.

1798–1817 CE **Louis Nicolas Vauquelin** (1763–1829, France). Chemist. Discovered *chromium* (1798), *beryllium*[401] (1798) and [with Pierre Jean Robiquet (1780–1840)] first isolated an amino acid, *asparagine*[402], from asparagus (1806).

1799 CE **Marc Antoine Parseval des Chênes** (1755–1836, France). Mathematician. His reputation rests on a single formula for summing special cases of series of products. Since its appearance in print in 1800, dozens of equations have been called Parseval equations, theorems or identities both in the theory of Fourier series and the theory of the Fourier integral; most of them only remotely resemble the original. In his memoirs, which were presented at the Academy of Sciences, Parseval applied his theorem[403] to

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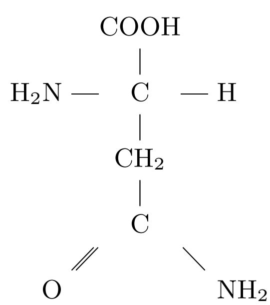
401 He discovered it in the gems *beryl* and *emerald*, but did not isolate the element. Beryllium was finally isolated by **Friedrich Wöhler** (1828).

402 *Asparagine* was synthesized by **Wilhelm Körner** (1839–1925) in 1887. It has the structural formula



Asparagine is important in the metabolism of nitrogen and the anabolism of nitrogen-containing compounds.

403 In modern notation: If in the series

$$M = A_0 + A_1 s + A_2 s^2 + \cdots$$

and

$$m = a_0 + a_1 s + a_2 s^2 + \cdots ,$$

s is replaced by e^{iu} and the real and imaginary parts are separated so that $M = P + iQ$ and $m = p + iq$, then

$$\frac{2}{\pi}\int_0^{\pi} P p du = 2A_0 a_0 + A_1 a_1 + A_2 a_2 + \cdots .$$

Today, *Parseval's theorem* (or *identity*) for Fourier series states: Let $f(x)$

the solution of certain differential equations suggested by **Lagrange** and **d'Alembert**.

Little is known of Parseval's life or work; he was a member of a distinguished French family. An ardent Royalist, he was imprisoned in 1792 and later fled the country when Napoleon ordered his arrest for publishing poetry against the regime.

1799–1813 CE **Paolo Ruffini** (1765–1822, Italy). Physician and mathematician. Taught mathematics as well as clinical medicine at the University of Modena. In his book *Teoria generale dell equazioni* (1799), and later in 1813, he continued the thread of thought of **Lagrange** (1770), giving an incomplete proof that virtually established the unsolvability of the quintic equation by means of algebraic functions of the coefficients involving radicals.

Previously, **Euler**'s attempts (1750) to reduce the solution of the quintic equation to that of a quartic equation met with total failure. Ruffini's proof was later improved by Abel (1824).

1799–1831 CE **Aimé Jacques Alexandre Bonpland** (1773–1858, France and South America). Naturalist, botanist, horticulturist, agricultural experimenter and physician. As a botanist of the Humboldt expedition to the Spanish territories of South America (1799–1804), he garnered and described some 60,000 plant specimens, which he personally managed to collect in the equatorial swamps and rain forests. Later, during his stay in Argentine, Brazil, Uruguay and Paraguay (1816–1858) he continued to enrich European science with new floral specimens. He collected thousands of specimens of new plants and diagnosed them. Being a physician he had a special interest in plants which might have medicinal virtues and he sent many of them to the Paris Muséum for chemical analysis. He was first to investigate the culture of certain herbs and try to improve them in a scientific manner. He was one

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and $g(x)$ be bounded and integrable in $(-\pi, \pi)$ such that

$$f(x) = A_0 + \sum_{n=1}^{\infty} [A_n \cos nx + B_n \sin nx],$$

$$g(x) = a_0 + \sum_{m=1}^{\infty} [a_m \cos mx + b_m \sin mx].$$

Then

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)dx = 2A_0B_0 + \sum_{k=1}^{\infty} (A_k a_k + B_k b_k).$$

of the first botanists to observe one of the marvels of the floral world — the giant water lily (1819).

Bonpland was born in the parish of St. Bartholomew of La Rochelle. He received a medical education in Paris, and became a surgeon in the French navy. Under the influence of **Lamarck**, he had developed a deep interest in natural history, chiefly botany, which needed only a little encouragement to flare out. Having completed his naval service (1795), he returned to Paris to continue his medical studies. He then acquainted **Alexander von Humboldt**, and the two friends embarked together on their famous expedition to South America.

After their triumphal return to Paris (1804), Humboldt secured for him an appointment as botanist to the empress Joséphine[404] (1763–1814) who had a deep interest in flowers and was an ambitious horticulturist. After the empress' death he emigrated to Buenos Aires (1816), where he established a plantation on the Paraná River. There he undertook agricultural experiments on a large scale. However, in 1821 his plantation was sacked by troops of the Paraguayan dictator Francia, and he himself was imprisoned in Paraguay for more than seven years. Finally, thanks to the intervention of Humboldt, he was freed and settled (1831) in San Borja, on the eastern shore of the Uruguay River. He established there a large plantation, where he continued to conduct horticultural experiments, especially with regard to citrus fruits.

1799–1839 CE **Augustin-Pyrame de Candolle** (1778–1841, Switzerland). Botanist. Laid the foundation for modern studies on plant evolution and classification in his *Théorie Elementaire de la Botanique* (1813); *Regni Vegetabilis Systema Naturale* (1817–1821), and *Prodromus Systematis Naturalis Regni Vegetabilis* (1824–1839).

De Candolle was born in Geneva. Studied in Paris (1796); professor, Montpellier (1808–1817), Geneva (1817–1841). Coined the term "taxonomy" as a method of classifying plants by structure (1818). His son **Alphonse-Louis-Pierre-Pyrame de Candolle** (1806–1893) succeeded him as professor in Geneva (1842–1893) and continued the *Prodromus* to 17 volumes; author of *Geographic botanique raisonnée* (1855).

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[404] Born at Martinique, she accompanied her father to France (1779) and married there the viscount Alexandre de Beauharnais; he was beheaded in July 1794. In 1796, she married general Bonaparte and was crowned with him on Dec. 2, 1804. Napoleon divorced her in 1809. However, she was richly endowed and was able to keep a royal establishment at Malmaison near Paris, which she had bought during Napoleon's absence in Egypt. She died there in 1814.

1799–1858 CE **Alexander von Humboldt** (1769–1859, Germany). Naturalist, geographer and explorer. The first modern geographer to become a great traveler, and thus to acquire an extensive stock of first-hand information on which an improved system of geography might be founded. Laid the foundation to physical geography and meteorology, and pioneered in plant geography and climatology. The theory of geography was advanced by Humboldt mainly by his insistence on the great principle of the *unity of nature*. He brought all the "observable things" which the eager collectors of the previous century had been heaping together regardless of order or system, into relation with the vertical relief and the horizontal forms of the earth's surface. Thus he demonstrated that the forms of the land exercise a directive and determining influence on climate, plant life, animal life and on man himself[405].

He traveled extensively in the Spanish territories of America (1799–1804), and the results of this voyage were published by him in 23 volumes during 1804–1823. Among his novel contributions to science throughout this expedition one may mention his delineation of *isothermal lines*, through which he devised the means of comparing the climatic conditions of various countries.

He first investigated the rate of decrease in mean temperature with increase of elevation above the sea-level, and afforded, by his inquiries into the origin of *tropical storms*, the earliest clue to the detection of the more complicated law governing atmospheric disturbances at higher latitudes. Studied meteorite showers, volcanoes, the earth's magnetic field and communication between the water-systems of the Orinoco and Amazon rivers, and introduced the fertilizing properties of the guano into Europe.

Humboldt was born in Berlin. During 1788–1792 he studied geology, biology, and political science at the University of *Göttingen*, mining and metallurgy at the School of Mines in *Freiburg*, commerce and foreign languages at *Hamburg*, and anatomy and astronomy at *Jena*. His studies were directed, with extraordinary insight and perseverance, to the purpose of preparing himself for his distinctive calling as a scientific explorer. Through the years 1799–1804 he was engaged in the scientific exploration of Central and South America.

In 1808 he settled in Paris, then a center of geographical learning, and lived there for the next 20 years. In 1811 he speculated about the mechanism that

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405 This in itself was no new idea; it had been familiar for centuries in a less definite form, deduced from a prior consideration, and so far as regards the influence of surrounding circumstances upon man, **Kant** had already given it full expression (1765). Humboldt's concrete illustrations and the remarkable power of his personality enabled him to enforce these principles in a way that produced an immediate and lasting effect.

drives the current flowing along the coast of Peru, later named the *Humboldt current* in his honor. In 1827 he settled permanently in Berlin. In 1829 he was the first scientist ever to organize an all-European research program in earth-magnetism and meteorology in which Germany, France, Britain and Russia participated. In 1845 he began the publication of his five-volume treatise *Kosmos*, in which he tried to unify all of the physical science of his day.

In 1829 he made a voyage for the Russian czar, who sent him to the Ural Mount and Central Asia to report on mineral resources. Between May and November of that year he traversed, with his associates, the wide expanse of the Russian Empire from Neva to the Yenesei, accomplishing in 25 weeks a distance of some 16,000 km. One of the most important fruits of this journey were the correction of the prevalent exaggerated estimate of the height of the Central Asian plateau.

The last decade of his life was devoted to the continuation of his *Kosmos*. The scope of this remarkable work may be briefly described as the representation of the unity amid the complexity of nature. In it the large and vague ideals of the 18^{th} century are sought to be combined with the exact scientific requirements of the 19^{th} century. And, in spite of inevitable shortcomings, the attempt was quite successful. The science historian **Agnes Mary Clerke** summed up his personality and lifework in the statement:

"*After every deduction has been made, he yet stands before us as a colossal figure, not unworthy to take his place beside Goethe as the representative of the scientific side of the culture of his country*".

1800 CE **Louis-Francois-Antoine Arbogast**[406] (1759–1803, France). Mathematician. Introduced discontinuous functions and conceived the calculus as algebra of operational symbols. In his book *Calcul des Derivations* (1800) introduced integer powers of $D = \frac{d}{dx}$ as operational symbols for differentiation and integration. In this respect he was far ahead of his time.

1800–1802 CE **Karl Friedrich Burdach** (1776–1847, Germany). Physiologist. Introduced the term *biology* (1800), using it in a restricted sense to denote the combined morphological, physiological, and psychological study of human beings (1800). A broader definition was given in 1802 by **Gottfried Treviranus** (1776–1837, Germany) and **J.B. Lamarck** (1744–1829, France) to signify the study of life in general.[407]

406 Arbogast was a name of a Frankish general in the Roman army (c. 334–394 CE), one of the greatest soldiers of the late empire, and one of the most interesting personalities of the 4^{th} century.

407 The word biology is formed by combining the Greek $\beta\iota o\varsigma$ (bios), meaning "life", and the suffix '-logy', meaning "science of", "knowledge of", "study of", based

1800–1804 CE **Richard Trevithick** (1771–1833, England). Engineer and inventor. A great rival of **James Watt** in improvement on the steam engine. His earliest invention of importance was his improved plunger pole pump (1797) for deep mining, and in 1798 he applied the principle of the plunger pole pump to the construction of a water-pressure engine, which he subsequently improved in many ways. In 1800 he built a *high-pressure non-condensing steam engine*, which became a successful rival of the low-pressure steam-vacuum engine of Watt. He was a precursor of **George Stephenson** in the construction of *locomotive engines* and introduced *rails* into steam transportation.

In February 1804 he invented and constructed the first *steam locomotive* (railway) in South Wales which was able to haul twenty tons of iron and 70 men. It traveled at 8 km/h on the 16 km track. He was the first to recognize the importance of iron in the construction of large ships, and in various ways his ideas also influenced the construction of steamboats.

Trevithick was born in the parish of Illogan, Cornwall, where his father was manager of important Cornish mines. He had little formal education and was a big man of exceptional physical strength. At the age of 18 he began to assist

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on the Greek verb $\lambda\varepsilon\gamma\varepsilon\iota\nu$, 'legein' = "to select", "to gather" (cf. the noun $\lambda o\gamma o\varsigma$, 'logos' = "word"). The term "biology" in its modern sense appears to have been introduced independently by :

- **Karl Friedrich Burdach** in 1800,
- **Gottfried Reinhold Treviranus** (*Biologie oder Philosophie der lebenden Natur*, 1802) and
- **Jean-Baptiste Lamarck** (*Hydrogéologie*, 1802).

The word itself appears in the title of Volume 3 of **Michael Christoph Hanov**'s *Philosophiae naturalis sive physicae dogmaticae: Geologia, biologia, phytologia generalis et dendrologia*, published in 1766.

Before biology, there were several terms used for study of animals and plants. *Natural history* referred to the descriptive aspects of biology, though it also included mineralogy and other non-biological fields; from the Middle Ages through the Renaissance, the unifying framework of natural history was the *scala naturae* or *Great Chain of Being*. *Natural philosophy* and *natural theology* encompassed the conceptual basis of plant and animal life, dealing with problems of why organisms exist and behave the way they do, though these subjects also included what is now geology, physics, chemistry, and astronomy. Physiology and (botanical) pharmacology were the province of medicine. *Botany*, *zoology*, and (in the case of fossils) *geology* replaced *natural history* and *natural philosophy* in the 18^{th} and 19^{th} century before *biology* was widely adopted.

his father and soon showed considerable aptitude for mechanical invention. He went to work in Peru and Costa Rica (1814–1826), but was financially ruined in the Peruvian revolution of the 1820's. He returned to England in 1827, and in 1828 petitioned parliament for a reward for his inventions, but without success. He died penniless, at Dartford. A *Live of Richard Trevithick, with an account of his Inventions* was published in 1872 by his third son, Francis Trevithick (1812–1877).

1801 CE **Johann Georg von Soldner** (1776–1833, Germany). Mathematician. Used classical Newtonian gravitation theory to calculate the bending of starlight rays in the sun's gravitational field, based on the assumption that light consists of particles moving with velocity c, scattered by the sun. The correct answer[408] was given by Einstein in 1915 in the framework of General Relativity. Einstein was not aware of this work and it was rediscovered only in 1921!

1801–1808 CE **John Dalton** (1766–1844, England). Chemist and physicist. Introduced atomic theory into chemistry. Revived, sharpened and

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[408] Soldner's derivation is as follows: By Newtonian mechanics, a *hyperbolic orbit* of a small mass about a massive star, is subjected to the relation $\sin(\frac{\delta}{2}) = \frac{1}{e}$, where $e > 1$ is the eccentricity of the hyperbola, and δ is the angle between the asymptotes, which in turn represents the *angle of deflection* of the orbiting mass. The eccentricity, however, can be shown to be expressible in the form $e = 1 + 2r_{\min}E/GM$, where G is the universal gravitational constant, M is the star's mass, $r_{\min}$ the distance of closest approach and $E = \frac{1}{2}v^2$ the energy per unit mass of the orbiting body, whose orbital velocity is v. For a light particle, $v = c$ and hence $e = 1 + c^2r_{\min}/GM \approx c^2r_{\min}/GM$, since $c^2r_{\min}/GM \gg 1$ in all practical cases. Thus e is very large and δ is very small, leading to the final result

$$\delta \approx 2GM/c^2r_{\min}.$$

For light grazing the surface of the sun, $r_{\min} = R_\odot = 6.960 \times 10^{10}$ cm, $G = 6.672 \times 10^{-8}$ cgs, $M = M_\odot = 1.989 \times 10^{33}$ g, yielding the "*classical*" value $\delta = 0.87$ seconds of arc. Note that, although the mass of the photon cancels out in the derivation, the *mass must be finite* (i.e., $m \neq 0$); and this was *not* established in prerelativity physics. Moreover, if we had used the *special theory of relativity* instead of Newtonian mechanics, we would have found $\delta = 0$, because light tracks are null geodetics, and in flat spacetime these are straight lines! Shortly after developing the general theory Einstein calculated the deflection and got the same answer. However, he later further developed the theory and found that GTR actually predicts a value twice as large, or 1.75 seconds of arc. The experiment was first performed in 1919 and the result was 1.7 seconds of arc.

quantified the atomic theory of matter, expounded by Leucippos and Democritos 23 centuries before. Prepared the first list of *atomic weights* (1803).

In 1801 Dalton formulated his *law of partial pressures* for gases. [It states that the pressure exerted by a mixture of gases in a closed vessel held at a fixed temperature, is the sum of the pressures which each gas alone would exert if separately confined in the whole volume occupied by the mixture.]

Dalton's atomic theory (1803) supposes that:

(1) all matter is made up of minute particles, called *atoms*;

(2) all atoms of the same element are identical in all respects, particularly in weight or mass. Different elements have atoms differing in weight, and each element is characterized by the weight of its atom[409];

(3) in chemical compounds, a *whole* number of atoms of one element is associated with a *whole* number of atoms of another element, to form a *molecule* of the compound;

(4) each kind of atom has a definite small weight or mass.

Realizing that the absolute weights of atoms are very small, Dalton directed his attention to the determination of the *relative weights*, taking the weight of the lightest atom, that of hydrogen, as unity[410]. With his simple

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[409] Dalton's second assumption has been considerably modified by the discovery of *isotopes*, and can no longer be maintained. One element, e.g. chlorine, may have atoms differing in mass, and the atoms of such an element are not necessarily all the same, since an ordinary element may be a mixture of isotopes. It is the *atomic number* (net positive charge on the nucleus of the atom) rather then the *atomic weight*, which characterizes an element. Distinct isotopes of the same elements differ in atomic weight and have minutely different chemistries (quantitatively as well as qualitatively). Only at temperatures close to absolute zero (0° K $\approx -273.15^\circ$ C) can isotopic differences be (sometimes) very significant (as in e.g. superfluid Helium).

[410] Since the relative average *masses* of atoms are in the ratio of nearly integral numbers in most cases, it is convenient to define an *atomic weight* scale that specifies the weights of all atoms relative to an arbitrary standard [the *relative weights* of two objects are always the same at any given common altitude, and are thus always equivalent to the *relative masses*].

We could choose this standard as the (standard isotope) hydrogen atom H, and we could arbitrarily choose the atomic weight of this species to be a dimensionless number 1 (exactly). Suppose we now express this atomic weight in units of gram, which is the most convenient mass unit for most chemical purposes. Since the mass of the hydrogen atom has been found to be 1.67×10^{-24} g, the number of H atoms in 1.00 g of hydrogen is approximately $\frac{1.00\ \text{g}}{1.67\times 10^{-24}\ \text{g/atom}} = 5.99 \times 10^{23}$ atoms. [Therefore, for a pure sample of *any*

assumptions Dalton could explain the basic laws of stoichiometry, such as the law of constant proportions (**Proust**, 1799), the law of multiple proportions (**Dalton**, 1802), and the law of equivalent proportions (**Cavendish**, 1788).

John Dalton was born at Eaglesfield, a village near Cockermouth in Cumberland. His father was a poor weaver and a Quaker. As a boy he earned a living partly by teaching rustic youth, and partly as a farm laborer. He had received some instruction in mathematics from a distant relative, and in 1781 left his native village to become an assistant to his cousin who kept a school at Kendal. There he passed the next 12 years, becoming in 1785 a joint manager of the school with his brother Jonathan. In 1793 he moved to Manchester, where he spent the rest of his life. Mainly through **John Gough** (1757–1825), a blind philosopher (to whose aid he owed much of his scientific knowledge), he was appointed teacher of mathematics and natural science at Manchester's New College.

Apart from his work on atomic theory, he published papers on meteorology and *color vision*[411]. Altogether Dalton contributed 116 memoirs to the Manchester Literary and Philosophical society. In 1822 he was elected to the fellowship of the Royal Society and in 1830 he became a corresponding foreign member of the French Academy of Sciences. He never married, but there is evidence that he delighted in the society of women of education and refinement.

1801–1814 CE **William Hyde Wollaston** (1766–1828, England). Physician, chemist and physicist. Discovered the elements *palladium* and *rhodium* (1804) and isolated the second amino acid, *cystine*, from a bladderstone (1810). Invented the total reflection refractometer (1802). First to

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element, the mass corresponding to the number of grams given by the atomic weight of the element, would contain about 6×10^{23} atoms (*Avogadro's number*, 1811).] The international atomic weight scale presently in use is based on the exact number 12 for the atomic weight of the dominant carbon-12 isotope, ^{12}C (the atomic weight of *natural carbon* on earth is 12.01115 due to the presence of stable isotope ^{13}C (1.11% in abundance) as well as the unstable (but constantly replenished) radioactive ^{14}C isotope.

411 Dalton suffered from red-green color-blindness. He published (1794) the earliest scientific description of the condition in a paper "*Extraordinary Facts Relating to the Vision of Colors*" [before him, **Joseph Huddart** (1741–1811, England) described the condition in a letter to the chemist **Joseph Priestley** (1777)]. Dalton believed that color blindness is caused by coloration of the eye's vitreous humor. He bequeathed his eyes for science, but upon a post mortem examination it was found to be normal. In 1995, his eye's DNA were examined and the cause of his color-blindness was finally discovered — a genetic defect.

observe (1802) Fraunhofer lines in the spectrum. First to fix a meniscus lens to a Camera Obscura (1812). Secretary of the Royal Society (1804–1816).

Before the invention of *photography*, a *Camera Obscura* was a blackened box-like apparatus which was used by painters for the reproduction of complex images (the image was inverted on the base and it simply had to be sketched by hand). Initially, simple biconvex lenses had been used in these dark chambers, but the problem with them was that the images they produced were clear in the middle and blurred around the edges. Wollaston therefore introduced the *meniscus* which was convex on one side and concave on the other, and to which he adopted a diaphragm on the concave side. With its equal focal distances the meniscus gave much better definition[412].

Wollaston also proved the elemental nature of *niobium* and *titanium*. He developed a method of making platinum malleable. His consideration of geometrical arrangements of atoms led him into crystallography and the invention of the reflecting goniometer to measure angles of crystal faces. The mineral *Wollastonite* was named in his honor.

Though he was formally educated as a physician, his great curiosity led him into study and research in the fields of chemistry, physics, astronomy, botany, physiology, pathology, and crystallography. He was one of the most influential scientists of his time.

Wollaston was born in East Derham, Norfolk and died in London.

1802 CE **William Symington** (1763–1831, Scotland). Engineer. Built the tug *Charlotte Dundas*, the first practical *steamboat* equipped with stern paddle.

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[412] It still had appreciable *lateral chromatic aberration* which produced colored fringes on the outer parts of the field. Then there was the *astigmatic deformation* beyond the half-field of 20^0 and the optical distortions that made straight lines appear curved. Finally, the last fault, which appeared when the Wollaston device was used in photographic instruments, was that the aperture was limited to $f/11$ due to *spherical aberrations*. In 1821 the Frenchman **Charles Chevalier** made a positive lens using crown-glass which has a low refractive index and a weak dispersion. He stuck it on to a negative lens made of lead glass, (flint glass), which has a high refractive index and high dispersion. This was the first *double lens objective* which eliminated the *chromatic aberration*, but did not eliminate astigmatism, which remained in the edges. The field distance also remained limited to $f/11$. Finally, **Joseph Petzval** (1841) added a modified telescope lens to the Chevalier lens and thus noticeably eliminated their spherical distortions. It allowed an aperture of $f/3.6$, a previously unknown photographic speed. Even at apertures as large as $f/1.6$ this objective maintained an excellent definition up to 5^0 off the axis

1802 CE **Charles-Francois Brisseau de Mirbel** (1776–1854, France). Botanist. Founder of plant-cytology and physiology (1802).

Mirbel concluded from his numerous observations of plant structure that "*the plant is wholly formed of a continuous cellular membranous tissue. Plants are made up of cells, all parts of which are in continuity and form one and the same membranous tissue*".

Mirbel worked at the Musée d'Histoire Naturelle (1798–1803) and was director of gardens at Malmaison from 1803.

1802–1816 CE **Lorenz Oken (Ockenfuss)** (1779–1851, Germany). Naturalist and philosopher. Sought to unify the natural sciences. Contended that natural sciences can only offer partial knowledge and demand completion by a metaphysical – idealistic interpretation of nature. [His book: *Philosophy of Nature* (1802).] In his speculations, he foreshadowed theories of the cellular structure of organisms, the protoplasmic basis of life (1805), and that light is a state of stress of the ether (1808).

Okenfuss was born at Bohlsbach, Swabia. He changed his name to Oken upon his appointment to privatdocent at Göttingen (1801). His reputation at Göttingen has reached the ear of Goethe, and in 1807, Oken was appointed an associated professor of medical sciences in the University of Jena. In 1808 he advanced the preposition that "light could be nothing but a polar tension of the ether". Founded the influential periodical *Isis* (1816). He then became a professor at Munich (1829), and Zurich (1833).

1802–1826 CE **Heinrich Wilhelm Matthias Olbers** (1758–1840, Germany). Astronomer and physician. Pointed out that in an infinite, homogeneous, static Newtonian universe the mean radiation density would be as high as on the surface of a star.[413]

Born at Arbergen near Bremen, where his father was a minister. He studied medicine and mathematics at Göttingen during 1777–1780. In 1779 he devised a method of calculating cometary orbits.

Olbers settled as a physician in Bremen (1781) and practiced medicine actively for about 40 years — during day-time only. The greater part of each night (he never slept more than 4 hours) was devoted to astronomy, the upper portion of his house being fitted up as an observatory. He paid special attention to comets, and that of 1815 (period = 74 years) bears his name.

[413] For further reading, see:

- Harrison, E., *Darkness at Night*, Harvard University Press, 1987, 293 pp.

On 28 March 1802 he discovered the asteroid *Pallas* and later the minor planet *Vesta*.

While watching the sky for many years, it occurred to him to ask the naive question (1826): "*Why is the night sky dark, away from the Milky way*?" This question is known as '*Olbers' Paradox*', since according to the classical Newtonian cosmology [eternal-infinite-Euclidean-static-homogeneous universe] there are enough stars to fill the sky, and their light should be sufficiently intense to set fire to the earth[414].

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[414] In the absence of absorption, the apparent luminosity of a star of absolute luminosity L at a distance r in a Newtonian universe model will be $\frac{L}{4\pi r^2}$. If the density of such stars is a constant N, then the number of stars at distances between r and $r + dr$ is $4\pi N r^2 dr$, so that the total radiant energy density due to all stars at a given locus in the universe, should be proportional to $\int_0^\infty \left(\frac{L}{4\pi r^2}\right) 4\pi N r^2 dr = LN \int_0^\infty dr = \infty$. *In words*: doubling the distance of a star reduces the light received from it to one quarter. At the same time, doubling the radius of the shell increases the number of stars fourfold; therefore, we should receive from each concentric equi-thickness celestial shell-volume about us the same amount of starlight. A distant shell of many faint stars gives as much starlight as a nearer shell of fewer and brighter stars.

Olbers attempted to resolve this paradox by suggesting that space was filled with a tenuous absorbing medium. This explanation is invalid, however, as the intervening gas would be heated by the radiation it absorbs until it attained a temperature such that it radiates as much energy as it received, and so no reduction in the average radiation intensity would result.

The stars themselves are of course opaque, and block out the light from sufficiently distant sources. This however does *not* resolve the Olbers' Paradox since every line of sight must terminate at the surface of a star, so the intensity would tend to the average surface brightness of the stars — that is to say, comparable to the surface brightness of the solar disk.

Modern cosmological models avoid the Olbers' Paradox. In such models the mean total energy density of starlight anywhere at epoch t_0 is proportional to $\int_{-\infty}^{t_0} \mathcal{L}(t_1) \left[\frac{R(t_1)}{R(t_0)}\right]^4 dt_1$, $\mathcal{L}(t_1) \equiv \int n(t_1, L) L dL$, where: L is the absolute luminosity of a star as reckoned in a comoving coordinate system, t_0 is the time (epoch) the star is *observed*, t_1 is the time the light is *emitted*, $n(t_1, L)dL$ is the number density of stars of luminosity between L and $L + dL$ at time t_1, and $R(t)$ is the *radial scale factor* of the spacetime metric. In a "big-bang" cosmology there is obviously no paradox, since the integral is cut off at a lower limit $t_1 = 0$, and the integrand vanishes at $t_1 = 0$ roughly like $R(t_1)$. The question of an Olbers' Paradox arises only in models such as the (now defunct) *steady state cosmology*, in which the universe is supposed to have existed for an infinitely long time. In such models, a necessary condition for avoidance of the Olbers' Paradox is that $t_1 R_1^4(t_1)\mathcal{L}(t_1) \to 0$ for $t_1 \to -\infty$. In the case of the

The same question was already asked in 1744 by the Swiss astronomer **Jean Philippe Löys de Chéseaux**. The astronomer **Edmund Halley** stumbled on the paradox even earlier (1721) and **Johannes Kepler** wrote about it in 1610. The first to worry about the problem seems to have been **Thomas Digges** in 1576. Olbers had a copy of Chéseaux's book in his library but apparently never read it, and most scholars credit him with having arrived at the idea on his own.

This paradox, which is based on a superficial observation, has a deep cosmological significance and suggests that the naive Newtonian cosmology is wrong.

Modern cosmological models, based on GTR, avoid the Olbers' Paradox, and the threatened fiery furnace is transmuted into the tepid 2.7 °K microwave background. To see this we note that the divergence of the total energy integral can be avoided by assuming that the stars had not been shining forever but had turned on at some finite time in the past. In that case, the absorbing matter might not have heated up yet or the light from distant stars might have not reached us yet. Even in an infinitely old *expanding*, steady-state universe model with a constant average luminosity per unit volume, Olbers' Paradox is avoided on account of the *red-shift*, which weakens the contribution of distant galaxies over and above the inverse square law, such that the integrated radiant energy remains finite and even negligible. In the favored Big-Bang model, both mechanisms are operative, with the finite age of stars being quantitatively more important in avoiding Olbers' Paradox[415].

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oscillating model, absorption is needed during the highly contracted state and *redshift* during the expansion stage, to save the phenomenon (as the universe expands and galaxies drift apart owing to the expansion of intergalactic space, white light emitted by stars in galaxies far away and long ago arrives feeble and red, and the feeblest starlight arriving from the farthest galaxies is red-shifted into invisibility).

[415] The idea that the universe had a finite lifetime also existed in the mid-19th century, although only on the popular fringes of science. The first suggestion that the universe originated in a creative explosion — the first Big Bang — actually came from the pen of **Edgar Allan Poe** (1809–1849, U.S.A.). Poe was not only a well-known writer and poet, but also a scientific popularizer who kept abreast on the latest in astronomical research. In the book-length essay *Eureka* (1848) Poe rejected the idea of an infinite universe, citing Olbers' objections. He reasoned that a universe governed by gravitation would collapse in a heap if not kept apart by some form of repulsion. He postulated that God had, in an enormous explosion at the creation, thrust all the stars apart. Like a rocket racing into the sky, the stars and galaxies would first expand, and then contract into a final catastrophe, the end of the world.

One may use Olbers' Paradox to provide significant constraints on the luminosity that can be attained by very remote galaxies. In this way astronomers were compelled to conclude that most of the luminosity from these young remote galaxies must have been greatly red-shifted.

In *Eureka* Poe offers a solution to *Olbers' Paradox* in which he proposes that the light from very distant stars has not yet reached us. This solution requires only slight amendments to fit in with the standard modern cosmology. In his own words:

"*Were the succession of stars is endless, then the background of the sky would present us a uniform luminosity, like that displayed by the Galaxy — since there could be absolutely no point, in all that background, at which would not exist a star. The only mode, therefore, in which, under such state of affairs, we could comprehend the voids which our telescopes find in innumerable directions, would be by supposing the distance of the invisible background so immense that no ray from it has yet been able to reach us at all*".

It is remarkable that the same man that solved Olbers' Paradox with such brilliant intellectual aplomb, had the capability to match it with an equally grand poetic soul:

"*Deep into the darkness peering, long I stood*
there, wondering, fearing,
Doubting, dreaming dreams no mortal ever
dared to dream before"

Poe was born in Boston. his father deserted the family and his mother died before Poe was three years old. He was raised as a foster child and lived with his new family in London (1815–1820). In 1826 he entered the University of Virginia, but soon left to pursue a literary career. Served in the US Army (1827–1829) and attained the rank of sergeant major. Published his first poetry volume (1827), and married his cousin Virginia Clemm (1836), who at the time was only 14 years old. His most productive years as a fiction writer (1837–1845) were spent in New York City and Philadelphia. His wife died (1847) of tuberculosis and he was engaged (1849) to marry his childhood sweetheart when he suddenly died in Baltimore, apparently of hydrophobia caused by a cat's bite a year earlier.

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Meteorites[416]

"And it came to pass that the Lord cast down great stones from heaven upon them".

Joshua **10**, *11*

It was once thought that the world was small and flat. ***Columbus*** *sailed across the Atlantic (1492) and enlarged it,* ***Magellan*** *(1522) sailed around the world and made it round. It was thought that the earth was at the center of the universe until* ***Copernicus****(1543) restored the sun to its proper place and set the earth in orbit around it. We thought that the earth and its moon were unique until* ***Galileo*** *(1609) showed us that there were similar planets and moons elsewhere in the heavens. And we thought, until the turn of the* 19^{th} *century, that not even a stone could fall from the sky, that nothing could shatter our isolation from the cosmos.*

Fallen meteorites have been recovered throughout history and description of meteorites appear in ancient Hebrew, Chinese, Greek and Roman literature. There are numerous examples of meteorite veneration, such as the black stone of Kaaba enshrined at Mecca.

Yet, the extraterrestrial origin of meteorites was rejected by most people, scientists included, prior to 1800. *Upon hearing a lecture by two Yale professors, President Thomas Jefferson (1743–1826) is said to have remarked: "I could more easily believe that two Yankee professors could lie than that stones could fall from Heaven". The mere fact that meteorite falls had been widely witnessed and specimens had been collected (as, for example, in the* 1751 *fall*

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416 *Meteor*: Greek $\mu\epsilon\tau\epsilon\omega\rho\alpha$, literally: "things in the air", from $\mu\epsilon\tau\alpha$ = beyond and $\alpha\epsilon\iota\rho\epsilon\iota\nu$ = to lift up. Hence *meteorology* — the study of the atmosphere and the weather. A chunk of matter in space is called a *meteoroid*. A *meteor* (shooting star) is a brief flash of light that is visible at night when a meteoroid strikes the earth's atmosphere. When it reaches the ground it becomes a *meteorite*. This happens with extreme rarity at any one locality, but over the entire earth probably about 500 meteorites fall each year. The total mass of the meteor-producing meteorites that enter the atmosphere each day is estimated to be from 10 to 100 tons. Of these, only about 10 tons reach the ground each year. For further reading, see:

- Heide, F., *Meteorites*, The University of Chicago Press, 1957, 144 pp.

near Zagreb, Yugoslavia) did not tip the scales in favor of the extraterrestrial theory of meteorite origin.

In 1772, the distinguished French chemist ***Antoine Lavoisier*** *wrote a memorandum in which he concluded that stories of stones falling from the sky are mere fabrications — since meteorites could not possibly come from outside the earth. He was later beheaded by the revolutionaries' guillotine, although not for that reason.*

In that year (1794), ***Ernst Florens Friedrich Chladni*** *(1756–1827, Germany) openly suggested that meteorites are not terrestrial but cosmic in origin. Conclusive evidence came on April* 26, 1803, *when many witnesses observed the explosion of a bolide that pelted the French village of l'Aigle (Orne). Afterwards, many meteorite stones were found, reportedly still warm, on the ground. The austere French Academy, whose members were among the last diehards, sent the noted physicist* ***Jean Baptiste Biot*** *to investigate the matter and collect evidence to refute the rumors about stones falling from the sky. In his exhaustive report he stated that he believed the witnesses and finally convinced the scientific community of the extraterrestrial nature of meteorites.*

One of the barriers to accepting the existence of meteors and meteorites was the absence of a theory which predicted they exist. About the time ***Biot*** *made his observations,* ***Laplace*** *put forward his* Nebular Hypothesis *for the formation of the solar system. In the* Nebular Hypothesis*, one would expect debris in the form of stones in interplanetary space remaining to this day. Thus, the science of astronomy was transformed from a bastion of powerful empirical arguments against their existence into a strong supporter.*

Occasionally, scientists are wrong, and the illiterate peasants reporting observations of exotic events are correct. But in most cases it is the other way round. An interesting experiment was conducted in 1962 by the astronomer ***Frank D. Drake****; in 1962 two very bright fireballs (a type of meteor) burst over West Virginia, USA, at about 10 P.M. about a month apart. Astronomers were sent to collect meteorite bits and interview people about what they saw. We know what they* should have seen*, since fireballs are well-studied physical phenomena, so the interviews were a* test *of observation by inexpert witnesses who suddenly were exposed to unfamiliar phenomena.*

It turned out that 14 % of witnesses reported hearing a loud noise at the same time they saw the fireball, despite the fact that the witnesses had no contact with each other. A few simple calculation show that the visual stimulus could not have been accompanied by any sound whatsoever. Drake

suggested this auditory delusion to be due to a crossover in the brain when strong uninterpreted stimuli occur[417].

Diogenes and the French Academy, or – History Lessons

During the first five centuries BCE, Greek philosophers established the foundations of much of modern science. ***Aristarchos*** *of Samos proposed that the earth moves around the sun;* ***Democritos*** *and* ***Leucippos*** *described atomic structure;* ***Herophilos*** *of Thrace described the brain as the organ of thought;* ***Empedocles*** *proposed the idea of primal elements and forces;* ***Pythagoras*** *and* ***Euclid*** *developed geometry;* ***Diogenes*** *proclaimed that meteors move in space and frequently fall to earth; and* ***Archimedes*** *founded the subjects of mechanisms and hydrostatics.*

Many of these enduring ideas were not easily accepted; Pythagoras, for example, was forced to flee Magna Graecia because of his bizarre suggestion that numbers constitute the true nature of things.

By the 6^{th} *century CE, the Greek and the Roman civilizations had declined. Much of Greek science had been recast into theological scripture, which rapidly crystallized into dogma. Deviations from the official view were not tolerated.*

During the "Dark Ages", not much happened. With the exception of a few enlightened individuals such as ***Leonardo da Vinci*** *in the* 15^{th} *century, scientific and scholarly development in the Western world was rather quiet*

[417] Drake's findings also revealed that a witness's memory of exotic events fades very quickly: after one day, about half of the reports are clearly erroneous; after two days, about 3/4 are clearly erroneous; after four days, only 1/10 are good; after five days, people report more imagination than truth. Later they were *reconstructing* in their imagination an event based on some dim memory of what happened. We know today that *collective delusions* are extremely common in UFO 'observations'.

for about a thousand years. Because mirrors were not readily available in the dark Ages, da Vinci survived persecution for his dangerously heretical ideas by writing his notes backward. By contrast, the Eastern world continued to develop a thriving civilization and enjoy many ingenious fireworks displays.

By the beginning of the 16^{th} century, as astronomical measurements improved, astrologers and calendar makers had grown increasingly dissatisfied with the inaccuracy of Ptolemy's geocentric (earth-centered) cosmology. ***Copernicus*** *and others proposed a heliocentric (sun-centered) cosmology, which directly challenged scriptural doctrine.*

Despite attacks by theological authorities, the heretical Copernican model proved to be more accurate than the Ptolemaic and was eventually adopted, but not without causing a profound shock to Western's society's metaphysical assumptions. The new cosmology dethroned the earth, and by implication, human beings, from the center of the universe.

By the end of the 17^{th} century, Renaissance luminaries such as ***Newton****,* ***Galileo****, and* ***Descartes*** *had changed the course of Western civilization by splitting the world into two distinct realms. Theology became the authoritative voice for spirituality, morality, and the mental world, and science became the authority for the material world. Although the effects of their revolution would not directly affect most people for a century or more, the Church's reaction to the growing scientific revolution was brutal and persistent.*

Galileo was persecuted for his audacious proposal that moons orbit the planet Jupiter. ***Kepler*** *was accused of blasphemy in suggesting that the moon controlled the motion of the tides. Many Renaissance scientists were charged with heresy; some survived the wrath of the orthodox, many did not.*

While not fully appreciated for several centuries, Newton's famous paper on the nature of light, published in 1671 in the Philosophical Transactions of the Royal Society*, described an experiment that could not have worked the way he said it did. He was well aware that his experiment, involving the refraction of white light through a glass prism, was an idealization, but he did not acknowledge this until he was challenged by a contemporary who tried and failed to repeat his experiment. Newton was apparently so convinced that his theory about light was correct that he fabricated an experiment to confirm it. Fortunately for Newton, his intuition was correct.*

By the middle of the 18^{th} century, there were as many theories about the nature of electricity as there were experimenters. Most of the theories had something in common – the Newtonian-Cartesian concept of a mechanical-corpuscular world – but although all the experiments involved electricity and the experimenters read each other's works, their theories bore no more that a meager resemblance.

Meanwhile, the Royal Society in the United Kingdom was suppressing evidence that supported the existence of phenomena they interpreted as "witchcraft". Centuries later, those phenomena would be largely understood in terms the psychological concepts of suggestion, hypnosis, and hysteria.

By the end of the 18^{th} century, the Newtonian-Cartesian worldview, with its underlying principles of positivism (what is real is measurable), reductionism (complex systems can be understood by reducing them to their individual parts) and materialism (everything real is made of matter), had created an outstanding success of modern science. But the new emergence of successful scientific theories carried a severe price – systematic exclusion and denial of natural phenomena that did not fit the prevailing theories.

By paying homage to prevailing theories, the French Academy (of Science) soundly denounced public reports of "hot stones" falling from the sky, because both common sense and science agreed that there were obviously no stones in the sky, thus there was nothing to fall. The reported phenomena were declared as delusions, and therefore the witnesses of such phenomena were officially pronounced mentally deranged. A few radical scientists suggested that possibly a few stones might be cast into the sky by distant volcanic eruptions, but the prestige of the French Academy was so great that museums all over Western Europe threw away their specimens of rocks that fell from the sky. As a result, there are very few preserved meteorite specimens in France that date prior to 1790.

In 1879, Thomas Edison developed the first successful electric light bulb. He was already famous for many other successful inventions. But when Edison announced his new invention, scientists worldwide were incredulous. In response to the critics, Edison wired up the streets of Menlo Park, New Jersey, the location of his famous laboratory, and artificially illuminated the night sky for the first time in history. A professor, Henry Morton, who lived nearby and personally knew Edison, did not bother to view the evening exhibition, which went on night after night. Instead, he was so confident that the claimed invention was impossible that he offered the sober opinion that Edison's experiments were a "conspicuous failure, trumpeted as a wonderful success. A fraud upon the public".

Of course, Edison had already been denounced as a fraud for his invention of the phonograph years earlier, so one can imagine his amusement upon reading the opinion of Edwin Weston, a respected specialist in arc lighting, who asserted that Edison's claims were "so manifestly absurd as to indicate a positive want of knowledge of the electric circuit and the principles governing the construction and operation of electrical machines".

Meanwhile, back in Britain, after the lime was proposed as a cure for scurvy (a serious disease caused by malnutrition due to depletion of vitamin

C) the British medical establishment declared the proposal laughable and refused to put limes aboard ships. It took another 50 years to convince an entirely new generation of physicians that the cure actually worked, and over that sad half-century, thousands of sailors needlessly lost their lives.

About the same time, prior to the radical "germ theory" of disease, a Viennese physician named ***Semmelweiss*** *reported that washing one's hands before obstetrical assistance could prevent what was then a widespread disease threatening newborns: childbed fever. He was viciously scorned and rejected by his contemporaries and died a broken man several years later in an insane asylum.*

By the middle of the 19^{th} *century, Scottish physicist* ***James Clerk Maxwell*** *had brilliantly synthesized 150 years of unorganized empirical observations about electricity and magnetism, and biologist* ***Charles Darwin*** *had described his theory of evolution. These and other significant developments aroused great hostility among mainstream scientists of the day. Maxwell's theory was called "scandalous"; Darwin's theory was condemned as absurd by both scientists and theologians;* ***von Helmholtz****'s idea that physical experimentations could teach us how the human body worked was severely denounced.*

By the end of the 19^{th} *century, physics professors were so confident in the highly accurate results of Newtonian physics that they began to discourage their best students from pursuing careers in physics because most of the difficult problems had already been solved. Most of the rest of physics was expected to be little more than a "mopping up" operation – adding a few more decimal places to the known physical constants and resolving a few minor questions about puzzles known as the "ultraviolet catastrophe" and the "photoelectric effect".*

Year before the Wright brothers flew their airplane at Kitty Hawk, Rear-Admiral George Melville, chief engineer of the US Navy, declared that attempting to fly a heavier-than-air aircraft was simply "absurd". A few weeks before the airplane flew, ***Simon Newcomb****, a distinguished professor of mathematics and astronomy at Johns Hopkins University, stated that heavier than air powered human flight was, in scientific terms, "utterly impossible". According to Newcomb, any form of powered flight would require the discovery of entirely new force. With such eminence behind these statements, the mainstream media of the day meekly followed the lead of the authorities, and sneered at the ridiculous notion of the powered flight.*

To add injury to insult, more than two years after *the Wright brothers had first flown their aircraft, and in spite of the fact that dozens of eyewitnesses had actually seen them fly, the popular* Scientific American *magazine continued to ridicule the "alleged" flights.*

Many years later, when the editor of the Wright brothers' hometown newspaper was asked why he had refused to publish anything about their amazing accomplishment, he replied "We just didn't believe it. Of course, you remember that the Wrights at that time were terribly secretive." The interviewer responded incredulously, "You mean they were secretive about the fact that they were flying over an open field?" The editor considered the question and replied sheepishly, "I guess the truth is we were just plain dumb."

*At about the same time as the Wright brothers were flying their impossible machine, **Einstein**, **Bohr**, **Heisenberg** and others had begun to revolutionize physics with the quantum theory. Einstein's theories were vigorously attacked on the basis that their acceptance would throw back science to the Dark Ages.*

*The inventor **Lee De Forest**, was prosecuted for fraud in 1913 for claiming that if was possible to transmit the human voice across the Atlantic by radio.*

The Atoms of Leucippos and Dalton (460 BCE–1803 CE)

The empirical laws of chemical combination, particularly the law of multiple proportions, suggest that the chemical elements react together as though the matter of which they are composed is parceled out into exceedingly minute portions. Each such particle is incapable of further subdivision, so that when two elements combine they do so in masses which are whole multiples of the masses of these individual portions.

Two possible guesses as to the ultimate structure of matter present themselves. The first saw matter as a continuous *structure, completely filling the space occupied by bodies in the same way as jelly fills a mould. The second saw matter filling space discontinuously, with interstitial gaps, much as small pellets fills a barrel. The first view is associated with the Elea school of Greek philosophy, founded by **Xenophanes** (ca 530 BCE); the second is the atomic*

hypothesis, due to **Leucippos** (ca 460 BCE), but particularly developed by **Democritos of Abdera** (ca 420 BCE). This assumes the division of matter into exceedingly small particles, or *atoms*, incapable of further division by physical means. It is uncertain whether it was independently proposed in India, but it certainly appears there in a novel form in later Buddhist and Jainist treatises. The idea of divisible atoms (i.e., molecules) was put forward by **Asclepiades of Bithynia** (100 BCE).

Epicuros (310 BCE) and **Lucretius** (57 BCE) adopted the theory, and we possess a long Latin poem of the latter, *De Rerum Natura*, dealing principally with the atomic theory. **van Helmont**, **Nicolas Lémery** (1645–1715, France, 1675), **Hermann Boerhaave** (1668–1738, Holland, 1724), **Boyle** and **Newton**, made use of the hypothesis. Indeed, the latter gave a mathematical demonstration of Boyle's law on the basis of the hypothesis that gases consist of atoms repelling one another with forces inversely proportional to the distances. **Ruggiero Boscovich** (1711–1787, 1758) also made extensive application of a similar theory, but considered the atoms as mere points, centers of attractive and repulsive forces, and endowed with mass.

Bryan Higgins and **William Higgins**[418] (1762–1825, Ireland), in 1777 and 1789 respectively, made some applications of Newton's atomic theory to chemistry, but the merit of having independently elaborated a *chemical atomic theory* capable of coordinating all the known facts, and of being modified and extended with the progress of the science, belongs unquestionably to **John Dalton** (1776–1844).

Dalton's theory provided no means of determining even the *relative* weights of atoms. Although 7.94 parts of oxygen combine with 1 part of hydrogen, we do not know how many atoms of each element the molecule of water contains. If it contains one atom of each element (as Dalton supposed), the atomic weight of oxygen is 7.94, but if it contains 2 atoms of hydrogen to 1 atom of oxygen, as **Davy** supposed from the combining of volume ratio, the atomic weight of oxygen is $2 \times 7.94 = 15.88$. Thus, Dalton's assumption that the particles of elements in the free state are single atoms was the main source of the difficulties of the earlier theory.

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[418] William Higgins' book '*Comparative View of the Phlogistic and Anti-Phlogistic Theories*' anticipated the atomic theory later developed by **Dalton** (1803); however, the book was not widely distributed, and Dalton probably never saw it. Higgins also anticipated later chemical symbolism developed by **Berzelius** when he used first letter abbreviations for many elements.

1802–1815 CE **Joseph Louis Gay-Lussac** (1778–1850, France). Chemist and physicist. Among the distinguished chemists of the early 19^{th} century. Discovered (independently of **J. Charles** and **J. Dalton**) that a volume of gas at constant pressure is proportional to the absolute temperature (1802; known as *Charles-Gay Lussac law*). Deduced the equation for alcoholic *fermentation*. Made balloon ascents to investigate the effects of terrestrial magnetism, composition, temperature and moisture of air at altitudes as high as 7016 meters (1804).

Enunciated the law of volumes (or *Gay-Lussac law*), stating that two gases combine chemically such that the volumes involved are in ratio of small numbers (1808). Investigated the composition of water (1805; with **A. von Humboldt**). Established the properties of potassium (1808; with **L.J. Thenard**). Isolated *boron* (in the same year that **Humphry Davy** did), and devised improved methods for analyzing organic compounds (1809). Demonstrated that *sulfur* is an element (1810). Conducted studies of fermentation and improved processes for manufacturing of sulfuric acid, oxalic acid, etc. (1811–1815). Showed that *iodine* is an element and was first to predict the existence of *isomers* (1814). Discovered the gas *cyanogen* and was first to recognize the importance of radicals in chemical reactions (1815).

Gay-Lussac was born at St. Léonard-le-Noblat. In 1797 he was admitted to the École Polytechnique, and in 1800 became an assistant to **C.L. Berthollet** at the École de Ponts at Chaussées. He was a professor of physics at the Sorbonne (1808 to 1832) and subsequently a professor of chemistry at the Jardin des Planets.

Gay-Lussac will be remembered as a bold and energetic scientist. His early adventures heralded the fearless aeronauts: on Sept. 16, 1804 with the thermometer marking $9\frac{1}{2}$ degrees below freezing, he ascended in a balloon, unaccompanied to the altitude of 7 km above sea-level. He remained at this dizzying height, for a considerable time. A year later he investigated at close range the volcanic eruption of Vesuvius. He exhibited great fortitude of spirit and will power throughout a health crisis and under the laboratory accidents that befell him. Only at the very end, when the disease from which he was suffering left him no hope, did he complain with some bitterness of the hardship of leaving his world while the many discoveries being made pointed to yet greater discoveries to come.

1803–1806 CE **Lazare Nicolas Marguerite Carnot**[419] (1753–1823, France). French army general and geometer who took a leading part in the revolutionary changes at the end of the 18^{th} century and the revival of projective geometry. He was also one of the harbingers of the vector concept.

Carnot was born at Nolay in Burgundy. Following the custom of many of the sons of well-to-do French families, he prepared himself for the army and was thus led to the military school at Mézières, where he studied under Monge, becoming a Captain of engineers in 1782. In 1783 he published his first work '*Essai sur les machines en genéral*' in which he proved that kinetic energy is lost in the collision of imperfectly elastic bodies.

The Revolution drew him into political life. As a republican member of the Assembly he voted in 1793 for the execution of Louis XVI as a traitor. In the same year he undertook the organization of the French army to oppose the million-man force that Europe launched against France. In this capacity he was technically responsible for the acts of the Reign of Terror. In 1796 he opposed Napoleon's coup d'etat, and had to flee to Geneva, where he wrote a semi-philosophical work on the metaphysics of the calculus.

In 1800 he became Minister of War, but opposed the increasing monarchism of Napoleon who, however, gave him a pension and commissioned him to write a book on fortification for the military school at Metz. It is during these years that he published *Géometrie de position* (1803) and *Essai sur la théorie des transversals* (1806). In these works the influence of **Desargues** (1593–1662) and **Pascal** is evident, but Carnot went beyond them in the extension of well-known theorems of geometry as well as the development of various coordinate system that are independent of any particular choice of axes.

In 1814, when France was once more in danger, Carnot offered his services and was made a general of a division. He joined Napoleon during the Hundred Days and was made minister of the interior. On the second restoration he was exiled and lived in Magdeburg, occupying himself with science.

His son Sadi became a celebrated physicist. Of his other son Hippolyte he had one grandson, Sadi, who became the 4^{th} president of the 3^{rd} French Republic (1837–1894) and a second grandson, Adolpe, who became an eminent chemist.

1803–1808 CE **William Henry** (1774–1836, England). Chemist. Formulated *Henry's law*: the weight of a gas dissolved by a liquid is proportional

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[419] For further reading, see:

- Gillispie, C.C., *Lazare Carnot – Savant*, Princeton University Press: New Jersey, 1971, 359 pp.

to the pressure of the gas over the liquid. It contributed directly to the atomic theory of **John Dalton**, who extended the law to mixtures of gases, in conjunction of his own *Law of Partial Pressures*.

1803–1838 CE **Adelbert von Chamisso** [(Louis Charles Adelaide de Chamisso de Boncourt), 1781–1838, Germany]. Botanist, explorer, novelist and poet of French origin. Some of his poems are known in their musical settings, such as the touching song cycle "Frauenliebe und Leben" ("A Woman's Love and Life"), memorably set by Schumann. Chamisso is also known for the novella "Peter Schlemihl's Remarkable Story," the tale of a man who sells his shadow to the devil for a bottomless purse.

Indeed, this trade brings wealth to Peter Schlemihl, but also exclusion from society and he ends in despair. To end his ordeal, the demon offers him a second deal: his shadow against his soul. But Peter declines, although it means that he looses the woman he loves. With the aid of a pair of magic boots he is wandering the world searching for peace of mind; he finds it as a naturalist, and not in endless wealth. But his magic boots cannot bring him everywhere, he regrets not being able to visit the Pacific islands, notably the coral islands. So even his magic tools, do not enable him to get the ultimate knowledge.

This admirable story of Peter Schlemihl certainly reflects the life and struggles of the naturalist Adelbert Chamisso.

The secret of his popularity lies in his poised blend of the conservative and the progressive, the intellectual and the emotional. Though disguised as a simple tale, it has a profound psychological significance that has kept it all as classic, until the present day.

Chamisso was born at the Chateau of Boncourt in Champagne, France, the ancestral seat of the family. Driven out by the French Revolution, his parents settled in Berlin. During 1798–1808 Chamisso served in the Prussian infantry regiment as lieutenant.

He lived in Berlin (1808–1810) and Switzerland (1810–1812), studying botany, natural science and medicine. He continued his botanical research in Berlin and during 1815–1818 served as a botanist aboard the Russian ship "Rurick" which Otto von Kotzebue commanded on a scientific voyage round the world. His diary of the expedition (1821) is a fascinating account of the expedition to the Pacific Ocean and the Bering Sea[420] During this trip Chamisso

[420] For further reading, see:

- Von Kotzebue, Otto, *A New Voyage Round the World, in the Years 1823, 24, 25, and 26*, Henry Colburn & R. Bently: London, 1830, Two volumes. vol.1:(6), 341 pp. plus three maps (two folding); vol.2(2), 362 pp.

described a number of new species found in what is now the San Francisco Bay Area. Several of these, including the California poppy, Eschscholzia californica, were named after his friend Johann Friedrich von Eschscholtz, the Rurik's entomologist. In return, Eschscholtz named a variety of plants, including the genus Camissonia, after Chamisso. On his return in 1818 he was made custodian of the botanical gardens in Berlin, and was elected a member of the Academy of Sciences, and in 1820 he married.

1803–1843 CE **Marc Isambard Brunel** (1769–1849, England). Inventor and civil engineer. Best known for the construction of the *Thames tunnel* (1825–1843).

Brunel was born in France. Arrived in New York (1793) as a refugee of the French Revolution and practiced there as architect and civil engineer. Sailed to England (1799) in order to submit to the British government his plans for the mechanical production of ships' blocks, instead of the manual processes then employed. His proposals were adopted and the machinery was installed at the Portsmouth dockyard (1803–1806).

He erected many sawmills, experimented with steam navigation, invented a knitting machine (1816), a timber bending machine, etc. He also invented a tunneling shield (1818) and with it bore a tunnel under the Thames river between Wapping and Rotherhithe. He used Portland-cement concrete (1828) for filling in the river bed over this tunnel. (It came into large-scale use a generation later, when 70,000 tons of it went into the making of the London main drainage system.)

- Von Chamisso, Adelbert, *A Sojourn at San Francisco Bay 1816*, Book Club of California: San Francisco, 1836, Folio, v+16 pp+8 plates.

- Von Chamisso, Adelbert, *A voyage around the world with the Romanzov exploring expedition in the years 1815–1818 in the Brig Rurick, Captain Otto von Kotzebue*, translated by H. Kratz University of Hawaii Press: Honolulu, 1986.

- Von Chamisso, Adelbert, *The Alaska Diary 1815–1818*, translated by Robert Fortuine, Cook Inlet Historical Society, 1986.

- Fischer, Robert, *Adelbert of Chamisso: Citizen of the World, Natural Scientist and Poet*, Klopp, 1990.

Brunel got into financial difficulties and was thrown into prison (1821), from which he was freed by his friends, who secured for him a grant of £5000 from the government.

Many difficulties were encountered during the Thames tunnel construction; the river broke through the roof of the tunnel (1827), and after a second irruption (1828), work was discontinued for lack of funds. Seven years later it was resumed with the aid of money advanced by the government, and after three more irruptions the tunnel was completed and opened in 1843. Aided by his son, Brunel displayed immense determination and extraordinary skill and resourcefulness in the various emergencies with which he had to deal, but the anxiety broke down his health.

The second phase of the industrial revolution was affected almost continuously by the economy of the Great French Wars (1792–1815). Even while Napoleon was banished to Elba, Britain was still at war with America. This war regime reached a climax in the years 1806–1811, when Napoleon attempted to exclude British trade from the Continent and Britain counteracted his so-called "continental system" blockade by depriving the Continent of all trade which did not pass through British ports. Although this situation created a strain felt by the munitions industries, including the building, arming and servicing of war ships (on which the safety of the island nation was seen to depend), these wars served to boost technological progress[421] — as the block-making machinery introduced by **Brunel** at Portsmouth dockyard clearly illustrates. In general, the increased use of steam-power and machinery enabled Britain to press home the advantage which she derived from her superiority in iron production. The wars also encouraged their use in the textile industries, amplifying (for instance) the insatiable demand for American cotton which, in turn, bolstered the slavery-based culture of the American South and made inevitable the U.S. Civil War.

1804 CE **Nicolas Théodore de Saussure**[422] (1767–1845, Switzerland). Chemist and naturalist. Laid the foundations of *plant chemistry* (phy-

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421 There has always been a strong underlying relationship between man's general history and the history of his technological progress. The Roman empire, for example, rested upon the achievements of its engineers, including the great road-makers, as truly as it did upon its more abstract concepts of law and duty. The expansion of Europe in the 16^{th} century depended upon the existence of new means of crossing the oceans. In the same way, the bewilderingly rapid and numerous political changes during 1750–1900 influenced, and were in turn influenced by the technological revolutions of the time.

422 Name of a distinguished Swiss family including **Nicolas** (1709–1790),agriculturist). His son **Horace Benedict** (1740–1799), was a professor of philosophy and physics at Geneva who developed the first *electrometer* (1766), invented the

tochemistry); made pioneering work on nutrition and respiration in plants, on fermentation, germination, composition of alcohol, and transformation of starch into sugar. Asserted the importance of CO_2 and soil nitrogen to green plants, and demonstrated that plants absorb water. He showed that plants receive their carbon from atmospheric CO_2, not from the soil as earlier theorists had supposed. Moreover, his experiments represent the first treatment of the subject of *photosynthesis*, using quantitative methods and modern chemical terminology.

1804–1806 CE *Lewis and Clark Expedition*: A group of US frontiermen sent by president **Thomas Jefferson** to reconnoiter the vast new territories west of the Mississippi River acquired by the United States in the Louisiana Purchase. The Journals of the expedition provided valuable scientific information about Indian tribes and the natural wealth of the Western lands.

The expedition included about 5000 frontiermen under the leadership of **Meriwether Lewis** (1774–1809) and **William Clark** (1770–1838). The expedition set out on May 14, 1804 from St.Louis on a round trip of 13,000 km to the Pacific Ocean. They went up the Missouri River across the Rocky Mountains, then down the Columbia River to the Ocean in canoes. On the return trip Clark went down the Yellowstone River, reaching St. Louis on Sept. 23, 1806.

1804–1812 CE **Nicolas-Francois Appert** (1750–1841, France). Chef and inventor. Produced a process for preserving food in hermetically sealed containers: he put precooked foods in sealed glass bottles and heated them in boiling water. Appert published his reports in 1810 – a pioneering enterprise in heat sterilization of food. He later established the first commercial cannery (1812).

1804–1820 CE **Jean-Baptiste Biot** (1774–1862, France). Physicist, mathematician and astronomer. Discovered with **Felix Savart** that the intensity of the magnetic field set up by a current flowing through a wire varies inversely with the distance from the wire (known as *Biot-Savart Law*). This law is fundamental to modern electromagnetic theory. Made a balloon ascension with **Gay-Lussac** (1804) to study the upper atmosphere and terrestrial magnetism. Collaborated with **Arago** to study refractive properties of gases.

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world's first *Solar Collector* (1767), built first *hygrometer* utilizing a human hair (1783) and introduced the term *geology* (1799). Nicholas Théodore was the eldest son of Horace. Horace's grandson, **Henri** (1829–1905) was an entomologist. **Ferdinand** (1857–1913), son of Henri, was a linguist, regarded as the father of modern linguistics.

Investigated polarization of light passing through chemical solutions (1815). Became a professor of Mathematical Physics at the College de France (1800).

1804–1834 CE **Louis Poinsot** (1777–1859, France). Mathematician. Contributed significantly to *analytical* and *geometrical mechanics*. Introduced the concept of a *force-couple*, and proved that every system of forces is equivalent to a system consisting of a sum (resultant) acting at an arbitrary point O and a couple whose moment is equal to the moment of the system about O. In 1834, Poinsot gave a geometrical description of the force-free motion of a rigid body about a fixed point, known as *Poinsot's construction*[423].

Poinsot (1809) wrote an important book on polygons and polyhedra [*Mémoire sur les polygons et les polyèdres*], discovering four new regular polyhedra[424]. Two of these appear in **Kepler**'s work *Harmonice Mundi* (1619), but Poinsot was unaware of this. On the subject of polygons, he determined the number of n-pointed regular (noncompound) polygons that can be drawn around the circumference of a circle[425]. Today, this enumeration has become important in electrical network theory, statistical mechanics and numerical analysis.

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423 Describes the motion of the *inertia-ellipsoid* relative to a plane perpendicular to the angular momentum vector of the body about the fixed point (*invariable plane*). According to Poinsot, the motion of the body is equivalent to the rolling the momental ellipsoid on the fixed tangent plane.

424 A simple polyhedron is a closed shape enclosed by faces, all of which are plane polygons (e.g. pyramid, prism, frustum). A convex polyhedron is said to be regular if its faces are regular and equal (e.g. tetrahedron, cube etc.). Non-simple polyhedra can have holes. **Kepler** described the small and the great *stellated dodecahedra*, which do not fit Euler's relationship $(F = 12,\ V = 12,\ E = 30)$, since their faces intersect themselves. **Cauchy** (1810) proved that any regular polyhedron must have the same face planes as the 5 platonic solids and thus deduced that no further regular polyhedra can exist.

425 n points are drawn at evenly spaced intervals on the circumference of a circle. These points are then joined with line segments to form a polygon. The points can be joined consecutively or skipped over any fixed interval $(d-1)$. There are three possible classes of outcomes:
(a) regular convex polygons $(d = 1)$ for any n;
(b) $d > 1$ but prime to n, resulting in non-convex regular polygon (sides have equal length and consecutive sides form equal angles; e.g. the 5-pointed star or the 6-pointed star of David);
(c) $d > 1$, but n and d have common factors. The degenerate form is known as a *compound polygon*. Cases (a) and (b) together yield non-compound *polygons*. Poinsot asked: how many *non-compound* polygons are generated by n points.

Poinsot was born in Paris. He studied at the École Polytechnique during 1794–1797, and left in order to enter the École des Ponts et Chaussées. He eventually gave up the idea of becoming an engineer. From 1809 until 1826 he was both inspector general of the Université de France and teacher and examiner at the École Polytechnique. From 1839 until his death he worked at the Bureau des Longitudes. He showed no interest in algebra and was one of the principal leaders of the revival of geometry in France during the first half of the 19^{th} century.

1805 CE, Oct. 21 *Battle of Trafalgar* (sandy cape on Spain's southern coast, at the western entrance to the Strait of Gibraltar). British navy under Horatio Nelson defeated a combined French and Spanish fleets in one of the greatest naval battles in history. The victory ended the invasion threat of Napoleon and gave England undisputed domination of the seas throughout the 19^{th} century.

1805 CE The German pharmacist **Friedrich Sertürner** (b. 1783) extracted *morphine* from opium (1805) and used it to relieve pain. [It was not until 1925 that the chemical structure of morphine and other alkaloids were fully known.]

Sertürner named the new crystalline substance *morphium*, but the name soon changed to morphine. In 1817 he determined the alkaloid nature of morphine, thus marking the beginning of alkaloid chemistry. His isolation is the first of an alkali with a vegetable origin.

In 1905, the physician **Carl Gauss** of Freiburg, brought down the wrath of the Lutheran Church when he used morphine to induce *dammerschlaf* (the twilight sleep treatment) in women experiencing difficult births. The Church fathers — non of them mothers — declared that Dr. Gauss has fallen from

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His answer was

$$N = \begin{cases} \frac{1}{2}(n-1) & \text{if } n \text{ is prime} \\ \frac{n}{2}\left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\left(1-\frac{1}{m_3}\right)\cdots\left(1-\frac{1}{m_k}\right) & \text{if the } \textit{different} \text{ prime} \\ \qquad \text{factors of } n \text{ are } m_1, m_2, m_3, \ldots, m_k, & \text{excluding } n \text{ and unity.} \end{cases}$$

Thus for $n = 7, 8, 9, 10$ we have respectively $N = 3, 2, 3, 2$.

The preceding relationship between the *geometry* of star polygons and the *theory of numbers* (1809) followed the earlier discovery by **Gauss** (1801) that polygons with a prime number of sides could be constructed (using only a compass and a straightedge) if and only if the number of sides was a *prime* of a special form. Both examples hint to a close relationship between geometry and the theory of numbers.

grace and was near unto heresy because the Bible says (*Gen* **4**, 16) that women were to bring forth in pain.

1805 CE **Joseph Marie Jacquard** (1752–1834, France). Inventor. A silk weaver who perfected the loom with an attachment that made the loom weave patterns *automatically*. The attachment automated the weaving of fabric through use of a series of cards with punched holes, forerunners of the punched cards used later for input to early computers.

1805–1831 CE **Sophie Germain** (1776–1831, France). A mathematician, contemporary of **Gauss**, **Cauchy** and **Legendre**, with whom she corresponded. Contributed to number theory, acoustics and elasticity. Proved a *restricted* form of Fermat's conjecture: The equation $a^p + b^p = c^p$ has no solution in integers prime to p, if p is an odd prime and $2p + 1$ is also a prime.

It follows from this theorem that the equation $a^p + b^p = c^p$ has no solution in integers not divisible by p. The proof of the theorem is quite simple and shows how far one can go with very elementary arguments. In 1831 Germain introduced the notion of *mean curvature* of a surface, $M = \frac{1}{2}(k_1 + k_2)$.

Germain took correspondence courses from the Ecole Polytechnique in Paris since women were not allowed in the building of the school. During 1811–1816 she presented memoirs on the theory of vibrating plates based on the experimental work of **Chladni**. **Gauss** was so impressed by her work that he recommended her for honorary degree from the University of Göttingen. Unfortunately Germain died before the degree could be awarded.

1805–1814 CE **Francois Joseph Servois** (1767–1847, France). Mathematician. Published ideas on 3-dimensional vectorial systems, and developed the first elements of what became known as the *operational calculus* (1814). **Hamilton** later attributed to him the nearest approach to an anticipation of vectors and quaternions. Developed the notion of a mathematical '*operator*'. Introduced the term '*pole*' in projective geometry, and was one of the chief precursors of the English school of symbolic algebra.

Servois was born at Mont-de-Laval, Doubs, France, a son of a merchant. He was ordained a priest at Besancon at the beginning of the Revolution, but in 1793 gave up his ecclesiastical duties in order to join the army. In 1794, after a brief stay at the artillery school of Chalons-sur-Marne, he was made a lieutenant. With the support of **Legendre**, he was appointed professor of mathematics at the artillery school of Besancon (1801). He later served in this capacity at Metz (1802–1808), and finally was appointed curator of the artillery museum at Paris (1816–1827).

Although Servois did not produce a major body of work, he made a number of original contributions to various branches of mathematics and paved the road for important later developments in vector theory, operational calculus, projective geometry and symbolic algebra. Thus, his memoirs directly inspired the work of **George Boole** (1847).

1805–1814 CE **William Congreve** (1772–1828, England). Inventor and pioneer of military rocketry. Developed a war rocket that could carry explosives and was driven by gun-powder. It was used by British troops in the Napoleonic wars [the shelling of and burning of Boulogne (1806), Copenhagen (1807) and Leipzig (1813)] and in their war against the United States Army (1812–1814)[426]. The Congreve rocket was in use by the British army until 1860, when cannons became more accurate.

Congreve was a versatile man of science: he invented a process of color printing (1821), water-marks on banknotes, and was first to suggest the armoring of battleships.

He was educated at Trinity College, Cambridge.

1806–1820 CE **Charles Julien Brianchon** (1785–1864, France). Mathematician. Contributed mainly to projective geometry. Discovered jointly with **J.V. Poncelet** (1820) the *nine-point circle*[427].

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426 After watching the rocket attack of British troops on Fort McHenry in Maryland (1812), **Francis Scott Key** described *the rocket's red glare* in "*The Star-Spangled Banner*".

427 *Brianchon's Theorem*: if all the sides of a hexagon are tangent to a conic, then the diagonals joining opposite vertices are concurrent.

Brianchon discovered it when he was a 21-year-old student and published it in 1806 in the Journal of l'École Polytechnique.

The nine-point circle: the mid-points of the sides, the feet of the altitudes, and the mid-points of the lines joining the orthocenter to the vertices of the triangle are concyclic.

The theorem concerning this circle is named for neither **Brianchon** nor **Poncelet** (joint paper in Gergonne's *Annales* for 1820–1821), but for yet a third mathematician **Karl Wilhelm Feuerbach** (1800–1834) who in 1822 published this and some related theorems. Feuerbach showed that the center of the circle lies on the *Euler line* and is midway between the orthocenter and the circumcenter. *Feuerbach's Theorem* then states that the *nine-point circle* of any triangle is tangent internally to the *inscribed circle* and tangent externally to the three *escribed circles*, possibly one of the most beautiful theorem in elementary geometry that has been discovered since Euclid.

In the 19^{th} century the *geometry of the triangle* made noteworthy progress by

Brianchon was born at Sevres. During 1804–1808 he studied at the École Polytechnique under **Monge**. He then became a lieutenant of artillery in the armies of Napoleon (1808–1813). In 1818 he was appointed professor at the Artillery School of the Royal Guard.

The 'Elastic Skin' of Liquids

If a thin glass tube (a fraction of a millimeter in diameter) is lowered into water, then in violation of the law of communicating vessels, the water in it will begin to rise rapidly, and its level will become considerably higher than that of the large vessel. The discovery of this phenomenon, known as capillarity[428]*, is attributed to* ***Leonardo da Vinci*** *(ca 1500).*

What forces are supporting the weight of the column of liquid that has risen up? The answer was given some 300 years later by ***John Leslie***[429] *(1802): the rise is accomplished by the forces of adhesion between the water*

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the above authors and **Steiner**. But it was many years before the subject attracted much attention. **Lemoine** (1840–1912) was the first (1873) to take up the subject in a systematic way and to contribute extensively to its development. **Henri Brocard** (1845–1922; France) discovered certain critical points of the triangle that bear his name.

428 From the Latin *capilla* = thin as a hair.
Plants and trees have an entire system of long ducts and pores. The diameters of these ducts are less than a hundredth of a millimeter. Because of this, capillarity forces (aided by negative osmotic pressures) raise soil moisture to a considerable height and distribute water through the plant.
If a sheet of blotting paper is observed through a microscope, it is seen to consists of a sparse network of paper fibers, forming thin and long ducts that play the role of capillary tubes. Capillarity causes kerosene to rise through the wick of a lamp, and in the technology of the dyeing industry, frequent use is made of a fabric's ability to draw in a liquid through the thin pores formed by its threads.

429 **John Leslie** (1766–1832, Scotland). Mathematician and physicist. In 1805 he was elected to succeed **John Playfair** to the chair of mathematics at Edinburgh, and in 1819 was promoted to the chair of natural philosophy.

and the glass (for a diameter of 0.01 *mm, the height of the rise is about* 15 *cm).*

The physicist ***Francis Hauksbee*** *(1666–1713, England) made first accurate observations of the capillary action of tubes and glass plates, and ascribed the action to an* attraction *between the glass and the liquid (1709). He concluded that only those particles of the glass which are very near the surface have any influence on the phenomenon.* ***James Jurin*** *(1718) showed that the height to which the same liquid rises in tubes, is inversely proportional to their radii. The concept of* surface-tension[430] *was first introduced in* 1751 *by* ***Johann Andreas von Segner*** *(1704–1777). He ascribed it to short range attractive forces. Segner also attempted to calculate the effect of surface tension in determining the form of a drop of liquid. His results had a most important effect on the subsequent development of the theory: first, they showed that* macroscopically, *the surface of a liquid is in a state of tension similar to that of a two-dimensional elastic membrane, stretched equally in all direction. Second, it gave hope for deducing this surface tension from a* microscopic *molecular theory.*

Indeed, the works of ***Thomas Young*** *(1804) and* ***P.S. Laplace*** *(1806) provided the natural quantitative aspect to Segner's ideas. Thomas Young founded the theory of capillary phenomena on the principle of surface tension. He also observed the constancy of the angle of contact of a liquid surface with a solid, and showed how to deduce from these two principles the phenomena of capillary action. He supposed particles to act on one another with two different kind of forces: one of which, the attractive force of cohesion, extends to particles at a greater distance than those to which a repulsive force is confined. The attractive force is constant throughout the small distance to*

[430] In liquids (and solids) the average distance between molecules is about the same as a molecular diameter, so molecules are essentially in contact with their nearest neighbors.

The molecules of a liquid experience strong attractive (*cohesive*) forces that resist attempts to separate molecules. These forces have short ranges. Consequently, the molecules in a liquid interact only with their nearest neighbors. But surface molecules have less neighbors than bulk ones. So the surface effectively has a positive potential energy proportional to the number of molecules, i.e. to the surface area. Any physical system will tend spontaneously toward a condition of minimum potential energy (apart from entropy effects). In a liquid, this entails a tendency toward minimizing the surface area. Because the surface area in equilibrium is minimal, work is required to increase the surface area. The amount of this work per unit area is called the *surface tension*. A fixed volume of free liquid will therefore assume the shape of a sphere, because this shape has the least surface area for a given volume.

which it extends, but the repulsive force increases rapidly as the distance diminishes.

The subject was taken up by Laplace, who furnished us with seminal quantitative results that have never been surpassed.

Consider an experimental set up in which air can be pumped into a spherical soap bubble. To increase the radius of the bubble by an amount dr, a certain amount dW of external work must be done to overcome the resistance of the surface to an increase of its area. Clearly $dW = pdV = p4\pi r^2 dr$, where p is the excess pressure inside the bubble above the outside pressure and dV is the change of volume due to expansion.

In a state of equilibrium, the bubble is held together by a surface tension $T\ \left(\frac{\text{force}}{\text{length}}\right)$. The work done by this force is TdS, where $dS = d(4\pi r^2)$ is the change of surface area due to the expansion. Equating the two expressions we arrive at Laplace's law: $p = \frac{2T}{r}$. When the same logic is applied to a cylinder of radius r, we obtain $p = \frac{T}{r}$.

Laplace (1806) generalized these results to arbitrary curvature radii R_1 and R_2 in two perpendicular directions: $p = T\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$. This equation equates the outward excess pressure p to the inward *pressure due to the surface tension. At equilibrium, p and T are fixed over the entire surface, that is*

$$\frac{1}{R_1} + \frac{1}{R_2} = \textit{mean curvature} = \textit{const},$$

at all points. This equation then represents a surface of minimal area *for the volume enclosed, otherwise known as a* minimal surface*. Planes, cylinders and spheres belong to the family of such surfaces.*

Note that for fixed T, p varies like $\frac{1}{r}$. This explains why very small spheres with only thin walls can withstand enormous internal pressures (plant cells often have internal pressures of 10 atmospheres with walls only a few microns thick!)

The rise, h, of a wetting liquid of density ρ in a capillary of radius r can be calculated by equating the total upward surface-tension force $2\pi r(T\cos\beta)$ [β = angle between tube wall and tangent to the liquid surface at the wall] to the weight of the liquid column $(\pi r^2 h)\rho g$, yielding

$$h = \frac{2T\cos\beta}{\rho g r}.$$

The same result can be obtained from energy considerations: the total energy is $E(h) = \frac{1}{2}\cdot(\pi r^2 h^2 \cdot \rho g) - (2\pi r h)T\cos\beta + c$. Solving the equation $\frac{\partial E}{\partial h} = 0$ for h, yields the same result.

The next step was taken by ***C.F. Gauss*** *(1830). Instead of calculating the direction and magnitude of the resulting force on each particle, arising from the action of neighboring particles, he formed a single expression for the sum of the potential energies of the system constituents: the first depending on the action of gravity, the second on the mutual action between the particles of the fluid, and the third on the action between particles of the fluid and the particles of a solid or fluid in contact with it. The condition of equilibrium is that this expression shall be a minimum. The condition, when worked out, gives not only the equation of the free surface in the form already established by Laplace, but the conditions of the angle of contact of this surface with the surface of a solid.*

During 1830–1869, ***J.A.F. Plateau*** *made an elaborate study of the phenomena of surface tension.* ***Lord Kelvin*** *(1887) calculated the effect of surface tension on the propagation of surface waves of a liquid.*[431]

1807 CE **Robert Fulton** (1765–1815, U.S.A.). An American inventor. Designed and built the first commercially successful *steamboat*. Also made important contributions to the development of the *submarine*.

Fulton was born on a farm in Lancaster County, Pennsylvania, and showed inventive talent at an early age. He went to Philadelphia at the age of 17 and was apprenticed to a jeweler. At the age of 21 he went to England and made a moderate living in London as an artist. After 1793 he gave his full attention to developments in science and engineering, and painted only for amusement. He began to travel, studied science and higher mathematics and learned French, Italian and German.

About 1797 Fulton turned his attention to submarines, a project which claimed his energies until 1806. The problem of submarine navigation received his practical attention during the time that he was making his experiments on steam propulsion. He constructed two submarine boats in France, and one in America. One of the former, the "*Nautilus*"[432] was built with the direct

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431 In the 1970's a Minkowski-space version of the Plateau minimal-surface-area problem was applied to studying the relativistic, quantum dynamics of *fundamental strings*, leading to major advances in elementary particle theory.

432 It seems that this name was adopted by **Jules Verne** (1828–1905, France) in his book "*Twenty Thousand Leagues Under the Sea*" (1870). It is a story about Captain Nemo, a mad sea captain who cruises beneath the oceans in an internationally-manned submarine.

encouragement of Napoleon in 1801. It was 6.4 m long, and supplied with compressed air for respiration. He descended in it to a depth of 8 m, remaining under water for fully 4 hours. Although Fulton's submarine ideas interested both Napoleon and the British Admirality, neither nation showed much interest in the craft, even though it sank several ships in demonstrations[433].

Fulton's submarine was propelled by manual power; two horizontal screws were employed for propulsion and two vertical screws for descending and ascending. It was built of wood with iron ribs, and was sheathed with copper.

In 1802 Fulton became interested in the steamboat. An experimental boat, launched on the Seine River in Paris in 1803, sank because the engine was too heavy. But a second boat, which was built in the same year, operated successfully[434].

Fulton returned to the United States in 1806, and in 1807 he directed the building of a steamboat in New York, which he named the *Clermont*. On Aug. 17, 1807, this vessel began its first successful trip up to Hudson River to Albany.

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433 Fulton failed to convince either the English, French or the United States governments of the adequacy of his submarine boats. Thus, in Brest harbor, he was able (1801) to blow up a small vessel with a torpedo sent from his 6.5 m long Paris-built *Nautilus*, before the watchful eyes of a commission appointed by Napoleon. Although it was still propelled manually, the *Nautilus* was equipped with a *reduction gearing system*; it could maintain 4 men under water for 4 hours. His *Nautilus II*, was launched at Brest harbor heading for the British fleet, but was unable to get close to any of the ships because the English (informed by spies) had rowing boats on constant patrol around their vessels. *Nautilus II* returned to port without having attached its explosive charge to anything, and France lost interest in the so-called *fish-boat*. Turning to the enemy (1805), Fulton tried to persuade England to adopt his submarine. Despite having the support of the Prime Minister Pitt, he came up against the opposition of the first Lord of the Admiralty, John Jervis, who saw in it a serious treat to the British supremacy of the seas. When Fulton succeeded, experimentally, in blowing up the schooner *Dorothy* by attaching to it an explosive charge which was detonated from a distance using an electric cable, the success of his demonstration only served to reinforce Jervis' hostility. Queen Victoria had her own peculiar reservation to "submarine" warfare on the ground that it was an unBritish (i.e. ungentlemanly!) way to win a war!

434 Allegedly, Fulton offered Napoleon his services in building for him a fleet of steam battleships for the invasion of Britain. Napoleon, however, rejected the idea. One wonders whether or not history could have changed its course, had the French warlord accepted the challenge.

1807–1816 CE **Humphry Davy** (1778–1829, England). Chemist. Isolated potassium and sodium through electrolysis (1807). Invented the arc lamp in 1809. Proposed *Chlorine*[435] as an element (1810) [Until then it was commonly believed that it was a *compound* which contained Oxygen and known by the name *Oxymuriatic acid*]. **Davy** (1816) was also able to finally prove that diamond is actually carbon.

Diamond (16–1955 CE)

Diamond is one of the most valuable precious stones. Its unequaled physical properties and intrinsic beauty place it in a unique position among other minerals. The discovery of diamonds dates back to ancient times[436]*, but the first undoubted application of the name to diamond is found in* ***Manilius*** *(16 CE) and* ***Pliny the Elder*** *(ca 75 CE), though Romans only knew diamonds of small dimensions. Most of the known large, valuable stones were*

[435] In 1811 **J.S.C. Schweigger** proposed the name *halogen* for Chlorine. **Berzelius** (1823) accepted Davy's theory that Chlorine is an element. Then (1825), Berzelius used the name *halogen* for the elements Fluorine, Chlorine and Iodine. Bromine was not discovered and added until 1826.

[436] The name '$\alpha\delta\alpha\mu\alpha\varsigma$' ("the invincible") was probably applied by the Greek to hard metals, and thence to corundum [(Al_2O_3), has some valuable transparent colored varieties, including *ruby* and *sapphire*]. The "diamond" (*Yahalom*) mentioned in the Old Testament [Ex **39**, 11], used in the breastplate of the high priest, was certainly some other stone, for it bore the name of a tribe, and methods of engraving the true diamond cannot have been known so early. The stone became familiar to the Romans only after being introduced from India, where it was probably mined at a very early period. Later Roman authors mentioned various rivers in India as yielding the *Adamas* among their sands. The name *Adamas* became corrupted into the forms *adamant*, *diamaunt*, *diamant*, *diamond*.

found only at the beginning of the 17th century. However, the nature of the diamond still remained a mystery.

Isaac Newton *(1675) conjectured that the diamond was combustible on account of its high refractive power; this was first established experimentally by the Italian Academicians,* ***Averani***[437] *and* ***Targioni***[438]*.*

Smithson Tennant *(1797) and* ***Humphry Davy*** *(1816) finally proved that diamond actually is* carbon*.*

Of all the solid elements (at S.T.P.), only sulphur*,* gold *and* diamond *(carbon) are found in nature in their pure state. Since ancient times, diamonds have been found in the form of grains, or small octahedrons, in alluvial deposits, whence they had been carried from dark, igneous rocks. Efforts to find diamonds in the original rock hardly ever met with success.*

There are only four known instances of diamonds being found in their parent rock. The oldest one is the mine in the Kimberley region of South Africa, where they occur in a decomposed olivine rock (Kimberlite). Another substantial deposit of diamonds in parent rock is in the Vilyuy river basin in Yakutsk (Russia). In 1961, diamonds in Kimberlite were discovered in Sierra Leone. The most well-known diamond alluvial deposits are in Zaïre, Minas Gerais (Brazil), Angola, Tanzania, Ghana and on the West African coast (Guinea, Ivory Coast, Liberia). In North America they are found in Arkansas, Ohio, Indiana and Wisconsin.

Diamonds are formed in nature at great depths (80 km or more) at $1100-1300\,^{\circ}C$ *under great pressure by eruption in the volcanic pipes. It is carried with the rising Kimberlite to the surface of the earth. Diamonds, together with other minerals and rocks, are carried away by water, along rivers, some of them into the sea. Therefore, secondary deposits are found in river beds, on old valley terraces and along the Atlantic coast on old beaches. Small diamonds were also discovered in certain* meteorites*, both stones and irons; [e.g. Novo-Urei, Penza, Russia 1886; Carcote, Chile; Canyon Diablo, Arizona].*

The history of many of the diamond discoveries is interesting and the fate which pursued the prospectors and the stone themselves is vivid and

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437 **Giuseppe Averani** (1662–1738, Italy). Professor at the University of Pisa. Performed various experiments in physics and botany. Member of the London Royal Society.

438 **Giovanni Targioni-Tozzetti** (1712–1783, Italy). Naturalist and physician. Curator of the Botanic Garden and professor of botany at the University of Florence. Director of the Magliabechiana Library (1739).

dramatic[439]. *Many large diamonds of rare quality are now the property of royalty or of governments. The largest stone ever discovered was the Cullinan (Kimberly, 1905, original weight* $= 3106$ *carats;* 1 *carat* $= 0.2054$ *gram.*

The commercial manufacture of diamonds was begun in 1955 by the General Electric Company in the US, after technique for obtaining very high pressure (over $70,000$ *atm.) at high temperature (*$2000\,^{\circ}C$*) had been developed. The crystallization of artificial diamonds is favored by the addition of a small amount of a metal such as nickel. Artificial diamonds therefore contain some nickel atoms replacing pairs of carbon atoms.*

Physical structure and properties:

Density $= 3.510$*; refractive index* $= 2.417$ *for the D line; high dispersive power.*

Diamond has a cubic unit of structure with side $a = 3.56\text{Å}$*; there are*

8 *atoms in the unit cell at coordinates:*

$$(0,0,0);\ (\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2});\ (\tfrac{1}{2},0,\tfrac{1}{2});\ (\tfrac{1}{2},\tfrac{1}{2},0);\ (\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{4});\ (\tfrac{1}{4},\tfrac{3}{4},\tfrac{3}{4});\ (\tfrac{3}{4},\tfrac{1}{4},\tfrac{3}{4});\ (\tfrac{3}{4},\tfrac{3}{4},\tfrac{1}{4}).$$

Carbon, with 4 *electrons missing from a complete octet, can form* 4 *covalent bonds. In the diamond, each atom is bonded strongly to* 4 *neighboring atoms which are held about it at the corners of a regular tetrahedron. These covalent bonds bind all of the atoms into a single giant molecule. Since the C-C bonds are very strong, this* network solid *crystal is very hard, in fact the hardest substance known.*

Recently (1999) it was conjectured that Saturn's atmosphere rains diamonds (!), the free fall of which converts gravitational energy into heat, thus driving that planet's weather system. The same may be true for Uranus and Neptune.

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439 Perhaps the most dramatic of them all is *the affair of the diamond necklace* (1784–1786) — a mysterious incident at the court of Louis 16^{th} of France, which involved the queen Marie Antoinette and contributed to render her very unpopular, probably sealing her fate. It involved: cardinal Louis de Rohan (formerly ambassador to Vienna), Comte and Comtesse de Lamotte, Boehmer and Bassenge (Paris jewelers), Marie Lejay, Reteaux de Villete (small time crook) and Cagliostro (famous charlatan) — a complex story of greed, passion and doom.

1807–1820 CE Gas lights first introduced in London (1807). By 1820 much of the cities of London and Paris were lighted.

1807–1821 CE **Georg (Wilhelm Friedrich) Hegel** (1770–1831, Germany). Philosopher. Perhaps the most abstruse of the Teutonic thinkers. Built a huge edifice to contain the whole of human knowledge. In his effort to reveal the implications of reality and reason, he employed the method of thesis, antithesis and synthesis, with analysis as the starting point. The examination of contradictions as the second step, and finally the arrival at unity by means of reason in a summation of ultimate truths. Hegel's system is applied to the whole experience, beginning with logic, going on to the philosophy of nature and then to the philosophy of mind and spirit. Within these categories, anthropology, philosophy, metaphysics, law, ethics, morality, government, property, the family, emotions, customs, art, religion, history and many other facets of thought and life were examined analytically, in their opposites and finally in synthesis.

His point of view, that *everything* is a logical process of thought obeying the laws of evolution from the simple to the more complex, held sway in Germany and influenced other countries until the middle of 19^{th} century; even though it lost some of its popularity after that date, it continued to influence world thought for many years thereafter to such a degree that it had profound effects on the ideas and *political events* of 20^{th} century.

His theories echo through the writings of **Marx**, **Kierkegaard**, **A. N. Whitehead**, **John Dewey** and a group of British and American thinkers known as *Neo-Hegelians*.

Among his important work are: *Phenomenology of mind* (1807); *Science and Logic* (1812–1816), *Encyclopedia of Philosophy* (1817); *Philosophy of Law* (1821).

Hegel developed the most systematic and comprehensive philosophy of modern times. He sought to synthesize the ontology of the ancient Greeks (Particularly the theories of **Aristotle**) with Kantianism[440]. Some of his views may be traced to the influence of Heraclitos, **Spinoza**, **Schelling** and **Fichte**. In fact, notwithstanding its revolutionary emphasis, *the Hegel system represents largely a synthesis of other philosophers*.

Kant had set a precedent for such synthesis when he combined the conceptual (rational) world of ideas with the phenomenal world of perception as the basis for valid knowledge. Similarly, Hegel set out to synthesize all opposites

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[440] Kant argued that one could *suppose* God's existence, but no system could *prove* it; Hegel instead seeks to justify the *idea* of God. Kant separated science from religion, Hegel wanted to make religion into a new science.

to arrive at truth which is considered as an organic unity of applied parts; according to his grammar of logical thinking, every condition of thought or of things, every idea or every situation in the world (*thesis*), leads irresistibly to its opposite (*anti-thesis*), and then unites with it from a higher or more complex whole (synthesis)[441]. This scheme constitute the formula and secret

[441] This '*dialectic movement*' runs through everything that Hegel wrote. It is an old thought, foreshadowed by **Empedocles**, and embodied in the 'golden mean' of **Aristotle**, who wrote that '*the knowledge of opposites is one*'. Moreover it comprises one of the 13 logical principles on which the Hebrew Talmudic exegetics are based; ["*Shnei ketuvim ha'makishim ze et ze...*"; Barai'ta d'Rabi Ishmael; ca 115 AD].

Hegel generalized this principle to embrace all things and thoughts; thus, a social system with free economy stimulating individualism is required in a period of economic adolescence and unexploited resources, but in a later age a cooperative commonwealth is preferable; the future will see neither the present reality nor the envisioned ideas, but a synthesis in which something of both will come together to beget a higher life. For Hegel, history too is a dialectical movement, almost a series of revolutions, in which people after people, and genius after genius, become the instrument of the spirit of the Age (*Zeitgeist*). What actually happens to a state or people represents the final judgment as to the worth of a national policy or course of action. For Hegel world history constitutes the world's court of justice. Reason is constantly evolving in history toward an absolute goal. God exists only as a 'world-spirit' which is real because it is rational. HISTORY ADVANCES AND PROGRESSES ONLY BECAUSE OF CONFLICTS, WARS, REVOLUTIONS, I.E. THROUGH RELIGIOUS STRUGGLES. PEACE AND HARMONY DO NOT MAKE FOR PROGRESS. HEGEL'S LOGIC LEADS TO THE CONCLUSION THAT WAR IS JUSTIFIED BECAUSE IT IS THE MEANS BY WHICH PROGRESS IS MADE. Moreover, it carries the unfortunate implication that whatever has been successful is thereby also somehow 'right' and superior to what had been unsuccessful. Whatever vanished from the memory of history (because it was destroyed or unsuccessful) was to Hegel '*unjustified existence*'. The Hegelian system was adopted by the Prussian state and many Prussian thinkers held that the Prussian state was destined to carry forward the realization of universal reason through its eventual conquest of the world.

The dialectical process makes change in the cardinal principle of life. No condition is permanent; in every stage of things there is a contradiction which only the strife of opposition can resolve.

After Hegel died, German philosophers gravitated around him, some approved him, and some other supported his theories. His followers '*Young Hegelians*' eventually splitted into '*right*', '*left*' and '*center*' over questions of theology: 'Right Hegelians' defended traditional Christianity; 'Center' sought to reinterpret religious dogma in Hegelian terms to give it a new, more scientific language;

of all development and all reality[442].

For not only do thoughts develop and evolve according to this '*dialectical movement*'[443], but things do equally; every condition of affairs contains a contradiction which evolution must resolve by a reconciling unity. The higher stage, if reached, too will divide into a productive contradiction, and rise still to loftier levels of organization, complexity and unity. The movement of thought, then, is the same as the movement of things; in each there is a dialectical progression from unity trough diversity to diversity in unity. Thought and being follow the same law; and logic and metaphysics are one.

Hegel, the native of Stuttgart, studied philosophy and theology at the national university of Tübingen (1788–1793). He took part in the walks,

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'Left Hegelians' criticized Christianity and developed Hegel's ideas toward radical conclusions, not only in theology. **Mosses Hess** (1812–1875), **Ludwig Feuerbach** (1804–1872) and **Karl Marx** (1818–1883) turned Hegel's philosophy of history into a theory of *class struggles* leading by Hegelian necessity to inevitable socialism. In place of the Absolute as determining history through the *Zeitgeist*, Marx offered mass movements and *economic forces* as the basic causes of every fundamental change, whether in the world of things or in the life of thoughts. He has argued a la Hegel, that since change is a road to better thing, a society based on private property would give way to one in which socialism was supreme via a synthesis of opposites. The collapse of Soviet Communism in 1989 demonstrated that at least Marx's extrapolation was wrong.
Since the second half of the 19^{th} century, *positivists* and *existentialists* questioned the role of Hegel's philosophical reasoning and sought to replace it [**Comte** (1798–1857); **Kierkegaard** (1813–1855); **Husserl** (1859–1938)].

442 In modern physics one could view the electron through 'Hegelian eyes' according to a triadic structure: thesis (particle), antithesis (wave) => synthesis (non-classical quantum-mechanical entity). Likewise, in modern biology, protein synthesis is achieved through the combination of two single ladders into a double-stranded DNA molecule. In general, the movement of evolution is a continuous development of oppositions, and their merging and reconciliation. **Darwin** (1809–1882), like Hegel, also starts from what has been empirically successful and argues back to the supposed necessity of it's appearance. In Darwin, however, there is no longer a rational dialectic of nature, but instead a principle of '*natural selection*'. Both Hegel and Darwin can be miss-construed to support a belief in the '*survival of the fittest*'. Seen in the light of such a 'Darwinian Hegelienism', world history presents a very ugly spectacle – at its most grotesque in the rise and fall of Nazism and Soviet communism.

443 Way back in ancient times, Greek philosophers applied this strategy to arrive at truth; a system of arguments, which bring out the *contradiction* in ones opponent's reasoning.

beer-drinks and love-making of his fellows and gained most from intellectual intercourse with his contemporaries **Hölderlin** and **Schelling**.

After leaving the university he became a private tutor at Bern and lived there in intellectual isolation (1793–1796). Schelling recommended his appointment to the faculty of Jena University (1797–1808). But in 1807, the wife of Hegel's landlord gave birth to Hegel's illegitimate son and the philosopher moved to Nuremberg, where he assumed the post of a teacher of philosophy in a classical high school for boys (1808–1816). During that period he married (1811) Marie von Tucker (the daughter of a respected Nuremberg family) scarcely half his age. When, Christiana Burckhardt, the mother of Hegel's illegitimate son, Ludwig, heard of the marriage and tried create a stir, Hegel had been paying money to support his son and appeared to have placated her.

At the age of 46 (1816) Hegel went to Heidelberg to take up his first secure full-time academic post. Finally, in 1818 he succeeded **Fichte** at the University of Berlin. From this point on, in accordance with his new status and public role in Berlin, Hegel's philosophizing took on the form of *lectures*.

In his last years, Hegel denounced the radicals and aligned himself with the Prussian Government and basked in the sun of its academic favors. His enemies called him '*the official philosopher*'[444]. He began to think of the Hegelian system as part of the natural laws of the world; he forgot that his own dialectic condemned his thought to impermanence and decay. Never did philosophy assume such a lofty tone, and never were its royal honors so fully recognized and secured as in 1830 in Berlin.

When the cholera epidemic came to Berlin in 1830, his weakened body was one of the first to succumb to the contagion. Just as the space of a year had seen the birth of Napoleon Beethoven and Hegel, so in the years from 1827 to 1832 Germany lost Goethe, Hegel and Beethoven. It was the end of an epoch, the last fine effort of Germany's greatest age.

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[444] E.g.: Hegel did not believe in the immorality of the soul. But being a respectable civil servant of the Prussian State he was forced to give in a bit and not let his ideas to be spread among people.

Worldview XVIII: Georg W.F. Hegel

* * *

"Philosophy always comes an the scene too late to give instructions to what the world ought to be. As a thought of the world, it appears only when it is already there, cut and dried, after the process of formation has been completed."

* * *

"What exists is reason; ... Reason is the substance of the universe, ... The design of the world is absolutely rational[445]*".*

* * *

"In a true tragedy, both parties must be right".

* * *

"The people are that part of the state which does not know what it wants".

* * *

"We learn from the history that we do not learn from history".

* * *

[445] In the light of the experience of the Holocaust and Stalin's totalitarianism, reason itself appears insane as the world acquires systematic totality.

"My task is to turn Kantian criticism into a true system, in other words, to overcome the divisions it still contains by deriving all its elements from a single fundamental principle".

* * *

"What is rational is actual, and what is actual is rational".

* * *

"As far as history goes, we must rather deal with what had been and what is. In philosophy, on the other hand, with what is *and is eternally".*

* * *

"Beauty is the mediation between the sensible (or sensuous) and the rational (or intellectual). My definition of beauty as 'pure appearance of the idea to sense' is true of beauty throughout the history of its embodiment in art. But art has the particular task of showing within the realm of the human, the essence of the divine".

* * *

"... It is science which had led you into this labyrinth of the soul, and science alone is capable of leading you out again and healing you".

* * *

"I believe that philosophy, like geometry, is teachable and must no less than geometry have a regular structure".

* * *

"Nothing great in the world has been accomplished without passion".

* * *

"Life is not made for happiness, but for achievement".

* * *

"The history of the world is not a theater of happiness; periods of happiness are blank pages in it, for they are periods of harmony".

* * *

"Great men had no consciousness of the general idea they were unfolding, ... but they possessed insight into the requirement of the time – what was ripe for development. This was the very Truth for their age, for their world; they merely placed another stone on the pile, as other have done; somehow he has the good fortune to come last, and when he placed his stone the arch stood self-supported".

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Philosophers on Hegel

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"Philosophers are doomed to find Hegel waiting patiently at the end of whatever road we travel".

Richard Rorty

* * *

"All the great philosophical ideas of the past century – the philosophies of Marx and Nietzsche, phenomenology, German existentialism, and psychoanalysis – had their beginnings in Hegel".

Maurice Merieau-Ponty (1908–1961)

* * *

"Whether through logic or epistemology, whether through Marx or Nietzsche, our entire epoch struggles to disentangle itself from Hegel".

Michel Foucault (1926–1984)

* * *

"It may well be that the future of the world, and thus the sense of the present and the significance of the past, will depend in the last analysis on contemporary interpretation of Hegel's work".

Alexandre Kojère (1900–1968)

* * *

"The highest of audacity in serving up pure nonsense, in stringing together senseless and extravagant mazes of words, such as had previously been known only in madhouses, was finally reached by Hegel, and become the instrument of the most bare-faced general mystification that has ever taken place, with a result which will appear fabulous to posterity, and will remain as a monument to German stupidity."

Arthur Schopenhauer (1788–1860)

* * *

"We surrealists, recognize Hegel as one of the first of our own mad company, willing to explore the furthest reaches of Unreason in order to win a new, expanded and higher form of Reason".

Andre Breton (1892–1966)

* * *

"One could write intellectual history of our century without mentioning Hegel. The 19th century thinkers whose spirits have dominated the 20th century have been Marx, Kirkegaard and Nietzsche. At the beginning of this century Sigmund Freud brought to light the unconscious and Ferdinand de Saussure the structure of Language. Meanwhile science had made explosive progress, more or less obvious to the continuing debates among philosophers of science. It is possible to leave Hegel out of the picture".

Lloyd Spencer (1996)

* * *

1807–1822 CE **Jean Baptiste Joseph Fourier** (1768–1830, France). Mathematical physicist and politician who exerted great influence on his field. Was first to assert (1807) that an *arbitrary* function (such as were understood in his time), given in the interval $(-\pi, \pi)$, could be expanded in a trigonometric series

$$a_0 + \frac{1}{\pi}\sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx],$$

if

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx;$$

$$a_n = \int_{-\pi}^{\pi} f(x)\cos nx dx;$$

$$b_n = \int_{-\pi}^{\pi} f(n)\sin nx dx, \qquad n > 1.$$

First to establish a mathematical theory of heat conduction in isotropic solids (1811).

He proved that the above expansion, known today as the *Fourier expansion*, or *Fourier series*, holds for certain simple functions which he needed in the problems of heat conduction. Since then, these series have been used extensively in the solution of the differential equations of mathematical physics[446].

Although **Euler** (1748), **d'Alembert** (1749, 1754), **Clairaut** (1757), **Lagrange** (1759) and again **Euler** in 1777, used the above coefficients (Fourier made no claim to its discovery!) the credit goes to Fourier[447] because he was the first to apply these coefficients to the representation of an entirely arbitrary function. He was also the first to allow that the arbitrary function might be given by different analytical expressions in different parts of the interval.

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[446] The Fourier coefficients a_n and b_n have the remarkable property that they give the best least-squares fit among all possible approximations when a function $f(x)$ is expanded in terms of an orthonormal set of functions. This implies that the *mean square error* $\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[f(x) - F_n(x)\right]^2 dx$ is minimized when $F_n(x)$ coincides with the partial n-th sum of the Fourier-series expansion for $f(x)$.

[447] Why did great mathematicians like **Euler** and **Lagrange** miss the final crucial step?

It was, no doubt, partially because **Fourier**'s disregard for rigor, that he was able to take *conceptual* steps which were inherently impossible to men of more critical genius. However, once the floodgate of the new idea opened, mathematicians hurried to exploit it and make it into one of the most efficient tools of modern linear mathematics.

Indeed, the theory of Fourier Series was further developed by **Poisson** (1820), **Cauchy** (1826), **Dirichlet** (1837), **Stokes** (1847), **Riemann** (1854), **Lipschitz** (1864), **H.E. Heine** (1870), **Cantor** (1872), **Du Bios-Reymond** (1875), **U. Dini** (1880), **Jordan** (1881), **Lebesgue** (1902), **Fejer** (1904), **Riesz** (1907) and **Fisher** (1907). The origin of the theory of the *Fourier Integrals* is found in Fourier's "Analytical Theory of Heat" (1822). *Fourier transforms* are due to **Cauchy** who pointed out the reciprocity of the Fourier integrals (1826).

Fourier was born at Auxerre. He was the son of a tailor, and was orphaned in his 8^{th} year. His admission into the military school of his native town was secured through the kindness of a friend. He soon distinguished himself in mathematics. Barred from entering the army on account of his poverty and low birth, he was appointed teacher of mathematics at the same school. In 1787 he became a novice at the abbey of St. Benoit-sur-Loire, but he left in 1789 and returned to his college. From 1789 to 1794 Fourier taught in secondary schools and also became actively involved in the French Revolution. As a result of this latter activity, Fourier spent some time in the prison of Auxerre in 1794. In 1795 he was on the faculty of the École Normale in Paris and thereafter occupied the chair of analysis at the École Polytechnique.

Fourier was one of the savants who accompanied Bonaparte to Egypt in 1798. During his expedition he was called on to discharge important political duties in addition to his scientific ones. He was for a time virtually governor of half of Egypt. He returned to France in 1801 and during the rest of his life combined scientific and political activities. As a politician Fourier achieved uncommon success, but his fame rests chiefly on his strikingly original contributions to science and mathematics.

The theory of heat conduction in solids engaged his attention quite early, and in 1812 he obtained a prize offered by the Academy of Sciences for his memoir: *Theorie des mouvements de la chaleur dans les corps solides* — an epochal paper in the history of mathematical and physical science. The works of Fourier have been collected and edited by **Gaston Darboux** in 1889–1890.

Scientists on Fourier Analysis

* * *

“Fourier’s book was of paramount importance in the history of mathematics and pure analysis perhaps owed it even more than applied mathematics.”

(Poincaré, 1895)

* * *

“Looking back, we can see Fourier’s memoir as heralding the surge of new mathematical methods and results which were to mark the new century. His ideas are built into the commonsense of our society.”

(T.W. Körner, 1988)

* * *

“It is difficult to say which of Fourier results is most to be praised: their uniquely original quality, their transcendently intense mathematical interest, or their perennially important instructiveness for physical science.”

(Lord Kelvin, 1880)

* * *

“Fourier’s book is a great mathematical poem.”

(J.C. Maxwell)

* * *

Fourier and his Integral [448]

When Joseph Fourier published his book "Téorie analytique de la chaleur" (1822), he certainly could not foresee that he had provided scientists of the 20th century with one of their most powerful research tools. Indeed, one can hardly find a better example for the metamorphosis of a successful idea, than the story of the Fourier integral and its applications.

Historical Perspectives

In the 17th century, ***Isaac Newton*** *showed that the way to understand the natural world is to use differential equations that govern the motion of objects under given forces.* ***Albert Einstein*** *merited it as "the greatest intellectual stride that has ever been granted to any man to make". Predictive science became possible, prompting* ***Laplace*** *to imagine a single formula that would describe the motion of every object, for all time.*

Some 150 years after Newton, ***Joseph Fourier*** *provided a practical way to extract the truth from a whole class of such equations: linear partial differential equations. He asserted that virtually any 2π-periodic function $f(x)$ can be represented as the infinite sum of sines and cosines, now known as a Fourier series:*

$$f(x) = a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx), \tag{1}$$

where the Fourier coefficients $\{a_0, a_k, b_k\}$ are calculated by

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx;$$

[448] For further reading, see:

- Wiener, N., *The Fourier Integral and Certain of its Applications*, Dover Publications: New York, 1958, 201 pp.
- Titchmarsh, E.C., *Introduction to the Theory of the Fourier Integrals*, Oxford University Press: Oxford, 1948, 391 pp.
- Sneddon, I.N., *Fourier Transforms*, McGraw-Hill Book Company: New York, 1951, 542 pp.

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx dx; \tag{2}$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx dx.$$

Roughly, what this means is that any curve that periodically repeats itself, no matter how jagged or irregular, can be expressed as the sum of perfectly smooth oscillations. Knowing $\{a_0, a_k, b_k\}$*, one may reconstruct the original function from its Fourier coefficients.*

The Fourier coefficients a_k *and* b_k *have the remarkable property that they give the best least-squares fit among all possible approximations when a function* $f(x)$ *is expanded in terms of an orthogonal set of functions. This implies that the mean square error* $\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x) - F_n(x)]^2 dx$ *is minimized when* $F_n(x)$ *coincides with the partial n-th sum of the Fourier-series expansion for* $f(x)$.

To prove this we want to minimize, for an arbitrary expansion of $f(x)$*, the expression:*

$$S(a_k, b_k) = \int_{-\pi}^{\pi}[f(x) - a_0 - \sum_{k=1}^{n}(a_k\cos kx + b_k\sin kx)]^2 dx.$$

This requirement is met when the partial derivatives of S *w.r.t.* a_0, a_r*, and* b_r *are set to zero, namely*

$$\int_{-\pi}^{\pi}[f(x) - a_0 - \sum_{k=1}^{n}(a_k\cos kx + b_k\sin kx)]dx = 0$$

$$\int_{-\pi}^{\pi}\cos rx[f(x) - a_0 - \sum_{k=1}^{n}(a_k\cos kx + b_k\sin kx)]dx = 0, \qquad r = 1, 2, ..., n$$

$$\int_{-\pi}^{\pi}\sin rx[f(x) - a_0 - \sum_{k=1}^{n}(a_k\cos kx + b_k\sin kx)]dx = 0$$

Using the orthogonality relations

$$\int_{-\pi}^{\pi}\cos rx\cos kx dx = \pi\delta_{kr} = \int_{-\pi}^{\pi}\sin rx\sin kx dx,$$

$$\int_{-\pi}^{\pi} \sin rx \cos kx dx = \int_{-\pi}^{\pi} \cos rx \sin kx dx = 0,$$

we regain the Fourier coefficients:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx;$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx; \qquad k \neq 0$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx.$$

Dirichlet *(1829, 1837) showed that when the function $f(x)$ is* bounded *in the interval $(-\pi, \pi)$, and this interval can be broken up into a finite number of partial intervals in each of which $f(x)$ is* monotonic*, the Fourier series converge at every point within the interval to $\frac{1}{2}[f(x+0) + f(x-0)]$, and at the end-points to $\frac{1}{2}[f(-\pi+0) + f(\pi-0)]$. These* sufficient conditions *(and their extensions to unbounded function) cover most of the cases that are likely to be required in the applications of Fourier series to the solution of the differential equations of mathematical physics and engineering.*

During 1850–1905, we pass into the domain of pure mathematics. ***Riemann*** *aimed at finding a* necessary and sufficient *conditions which an arbitrary function must satisfy so that, at a point x in the interval, the corresponding Fourier series shall converge to $f(x)$. The question Riemann set himself to answer has not yet been solved. But in the consideration of the problem he realized that the concept of the definite integral should be widened. And thus it transpires that we owe the* Riemann Integral *to the study of Fourier series.*

Riemann showed (1854) that for any bounded and integrable function $f(x)$, the integral

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \left[\begin{array}{c} \sin nx \\ \cos nx \end{array} \right] dx$$

tends to zero as n tends to infinity. This theorem implies that, if $f(x)$ is bounded and integrable in $(-\pi, \pi)$, the convergence of its Fourier series at a point in $(-\pi, \pi)$ depends only on the behavior of $f(x)$ in the neighborhood *of that point.*

The nature of the convergence of Fourier series received attention, especially after the introduction of the concept of uniform convergence *(****Stokes***

1847, ***Seidel*** *1848).* ***Jordan*** *(1881) simplified the treatment of Fourier series by introducing his functions of bounded variation. His criterion states that the Fourier series for the integrable function $f(x)$ converges to $\frac{1}{2}[f(x+0)+f(x-0)]$ at every point of which $f(x)$ is of bounded variation.*

If Fourier's Series for $f(x)$ is not convergent, it may converge when one or the other of the methods of 'summation' applied to divergent series is adopted. ***Fejer*** *(1904) proved that when the series is summed by the method of arithmetical means, its sum is $\frac{1}{2}[f(x+0)+f(x-0)]$ at every point in $(-\pi,\pi)$ at which $f(x\pm 0)$ exist. The condition attached to $f(x)$ is:*

- *If bounded, it shall be integrable in $(-\pi,\pi)$.*
- *If unbounded, $\int_{-\pi}^{\pi} f(x)dx$ shall be absolutely convergent.*

Although applied mathematicians were quite satisfied with the new limiting processes placed in their hand by ***Dirichlet***, ***Riemann***, ***Cantor*** *and* ***Jordan***, *pure mathematician were still unsatisfied because of the lack of unity, symmetry and completeness of the overall theory. Some advancement made during 1905–1920 greatly improved this situation. The most important contribution were made by* ***Lebesgue*** *(1902–1905),* ***Fejer*** *(1904),* ***de la Vallée Poussin*** *(1893) and* ***W.H. Young*** *(1912). One of the advantages of the Lebesgue integral is that a function which is integrable-L (Lebesque) need not be continuous 'almost everywhere' in the interval of integration, as is the case of a function integrable-R (Riemann). The Riemann-Lebesgue Lemma now guarantees that if $f(x)$ is integrable-L in (a,b), then*

$$\lim_{n\to\infty}\int_a^b f(x)\begin{bmatrix}\sin nx\\ \cos nx\end{bmatrix}dx=0, \tag{3}$$

whether $f(x)$ is bounded or not.

One of the most remarkable results which follow from the use of the Lebesgue integral in the theory of Fourier Series is the converse of Parseval's Theorem, known as the Riesz-Fisher Theorem: Any trigonometric series for which $\sum_1^\infty(a_n^2+b_n^2)$ converges is the Fourier Series of a function whose square is integrable-L^2 in $(-\pi,\pi)$.

From Fourier Series to the Fourier Integral

When applying the theory of Fourier Series to time signals it is convenient to change the name of the variable x to t and f to g. Thus, if $g(t)$ is a

continuous function with a finite number of bounded discontinuities, defined in the interval $T_1 \le t \le T_2, \quad T = T_2 - T_1$, one defines the Fourier expansion of $g(t)$ over T by

$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t}, \qquad c_m = \frac{1}{T}\int_{T_1}^{T_2} g(s) e^{-im\omega_0 s} ds, \qquad \omega_0 = \frac{2\pi}{T}. \tag{4}$$

Clearly, the Fourier series representation of $g(t)$ is periodic *with period T.*

If $g(t)$ is also periodic with period T, the Fourier series will render $g(t)$ for all values of t; If, however, $f(t)$ is not periodic, the sum will be equal to $g(t)$ only for $T_1 \le t \le T_2$. If $g(t)$ is real *$c_m = c^*_{-m}$. At points of discontinuity of $g(t)$ the sum will converge to the arithmetic mean of the values of the function on both sides of the discontinuity.*

For real $g(t)$ we may write

$$g(t) = \frac{a_0}{2} + \sum_1^{\infty}[a_n \cos n\omega_0 t + b_n \sin n\omega_0 t] = \frac{a_0}{2} + \sum_1^{\infty} B_n \cos(\omega_0 n t - \varphi_n), \tag{5}$$

where

$$\begin{aligned} a_n &= \frac{2}{T}\int_{T_1}^{T_2} g(s)\cos(n\omega_0 s) ds; \qquad b_n = \frac{2}{T}\int_{T_1}^{T_2} g(s)\sin(n\omega_0 s) ds; \\ \varphi_n &= \tan^{-1}\frac{b_n}{a_n}; \qquad B_n = \sqrt{a_n^2 + b_n^2} \\ a_0 &= \frac{2}{T}\int_{T_1}^{T_2} g(s) ds; \qquad b_0 = 0; \quad c_0 = \frac{a_0}{2}; \\ c_n &= \frac{a_n - ib_n}{2}; \qquad c_{-n} = \frac{a_n + ib_n}{2}; \\ B_n &= 2\sqrt{c_n c_{-n}} \end{aligned} \tag{6}$$

Inserting the integral expression of c_m from (4) into the infinite-sum representation of $g(t)$ with the provisions

$$T_1 = -\frac{T}{2}, \quad T_2 = \frac{T}{2}, \quad c_m = c^*_{-m} \quad [\text{real } g(t)] \tag{7}$$

one obtains

$$g(t) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\frac{T}{2}}^{\frac{T}{2}} g(s) e^{-in\omega_0 s} ds \right] \omega_0 e^{in\omega_0 t}. \tag{8}$$

In the limit $T \to \infty$

$$\omega_0 = \Delta\omega \ (say) \to 0; \qquad n\omega_0 = n\Delta\omega \to \omega \quad as \quad n \to \infty, \tag{9}$$

where any 'harmonic' $n\omega_0$ must now correspond to the general frequency variable which describes a continuous spectrum.

Consequently the summation in (8) becomes an integration over ω, and the function $g(t)$ has the representation

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} g(s) e^{-i\omega s} ds \right] e^{i\omega t} d\omega. \tag{10}$$

If we define

$$G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-i\omega t} dt, \tag{11}$$

then (10) becomes

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) e^{i\omega t} d\omega. \tag{12}$$

Eqs. (11) and (12) comprise the Fourier representation of the nonperiodic function, and are known together as the Fourier Integral Theorem.

Note the fundamental difference between the representation of a $g(t)$ through a Fourier Series and a Fourier integral; the Fourier Series of a periodic function concerns only those sines and cosines whose frequencies are integer multiples of the base frequency. If a function is not periodic but decreases sufficiently fast at infinity, it is still possible to describe it as a superposition of sines and cosines i.e. to analyze it in term of its frequencies. But now we must compute the coefficients for all possible frequencies. To this end, one may recast (11) and (12) in the symmetric form (for a variable x which does not necessarily represent time)

$$f(x) = \int_{-\infty}^{\infty} F(\xi) e^{2\pi i \xi x} d\xi; \qquad F(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} dx \tag{13}$$

and call $F(\xi)$ the Fourier transform of $f(x)$. The Fourier transform is essentially a mathematical prism, breaking up the function $f(x)$ into the frequency components that compose it, as a prism disperses white light into colors.

With the substitution $\omega = 2\pi\xi$, (13) can be recast in the less symmetric form

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega x}d\omega; \qquad F(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx, \tag{14}$$

which agrees with (11)–(12).

Mathematicians spent much of the 19^{th} century coming to terms with the ideas of Fourier. This process yielded a surge of new mathematical methods and results. Thus, questions of convergence and divergence have provided a great deal of work for mathematicians[449]. The need to formulate conditions on Fourier- decomposeable or -transformable functions, sum-shorted (together with advances in DE) the discipline of function spaces (topological and vector); measure theory; functional analysis, as well as special and generalized functions.

The Fourier Integral Theorem and its immediate consequences

Michaël Plancherel *(1885–1967, Switzerland) formulated (1910) the Fourier Integral Theorem in a form which is completely symmetrical with the aid of a new concept known as mean convergence. If $f_n(x)$, $\quad n = 1, 2, 3...$ are absolutely square integrable functions over (a, b) and if*

$$\lim_{n\to\infty}\int_a^b |f_n(x) - f(x)|^2\, dx = 0, \tag{15}$$

where $f(x)$ is square integrable over (a, b), then we say that $f_n(x)$ converges in the mean to $f(x)$ with index 2, and write it as

$$f(x) = \underset{n\to\infty}{\mathrm{l.i.m}}\, f_n(x). \tag{16}$$

(limit in the mean)

449 Virtually every periodic function can be represented as a series, or sum, of sines and cosines, but not every series of sines and cosines represent a function. If a series can be proved to *converge*, one can work with a *finite number of terms*, confident that adding more terms will not significantly change the results. If, however, the coefficients of the series do not become small fast enough, the series diverges and does not represent the function.

A similar notation holds if the parameter n tending to infinity is replaced by a variable tending continuously or discretely to some other limit.

Note that pointwise convergence does not imply convergence in the mean [e.g. $f_n(x) = n^{3/2}xe^{-n^2x^2} \to 0$ for every x while $\int_{-1}^{1} |f_n(x)|^2 dx \to \frac{1}{2}\sqrt{\pi}$ and $f_n(x)$ does not converge in the mean]. However, if can be shown that if $f_n(x)$ converges to a limit $f(x)$ almost everywhere on (a, b) and at the same time converges in the mean to a limit $g(x)$, then $f(x) = g(x)$ almost everywhere.

Plancherel's theorem: If the complex function $f(t)$ is absolutely square integrable on $(-\infty, \infty)$ than

$$F(\omega) = \underset{n\to\infty}{\text{l.i.m}} \int_{-\lambda}^{\lambda} f(t)e^{-i\omega t} dt, \tag{17}$$

known as the Fourier transform of $f(t)$, exists and is absolutely square integrable, and we have

$$f(t) = \underset{\lambda\to\infty}{\text{l.i.m}} \frac{1}{2\pi} \int_{-\lambda}^{\lambda} F(\omega)e^{i\omega t} d\omega \tag{18}$$

where

$$\int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega = 2\pi \int_{-\infty}^{\infty} |f(t)|^2 \, dt. \tag{19}$$

The conditions of this theorem are only sufficient. This means that the Fourier Integral Theorem may be valid for many functions which do not obey these conditions.

It is easy to prove the Fourier Integral Theorem for the more restrictive class of piecewise smooth functions $f(t)$ for $-\infty < t < \infty$ where t is a real variable, and $\int_{-\infty}^{\infty} |f(t)|dt$ converges. This is done with the aid of the δ-function concept. We wish to show that

$$f(t) = \lim_{\lambda\to\infty} \frac{1}{2\pi} \int_{-\lambda}^{\lambda} e^{i\omega t} d\omega \int_{-\infty}^{\infty} f(\xi)e^{-i\omega\xi} d\xi. \tag{20}$$

To this end define the Fourier transform of $f(t)$

$$F(\omega) = \int_{-\infty}^{\infty} f(\xi)e^{-i\omega\xi} d\xi, \tag{21}$$

multiply both sides by $e^{i\omega t}$ *and integrate over the* ω*-range* $(-\lambda, \lambda)$*, to obtain*

$$\int_{-\lambda}^{\lambda} F(\omega)e^{i\omega t}d\omega = \int_{-\infty}^{\infty} f(\xi)d\xi \int_{-\lambda}^{\lambda} e^{i\omega(t-\xi)}d\omega. \tag{22}$$

However

$$\lim_{\lambda\to\infty} \int_{-\lambda}^{\lambda} e^{i\omega(t-\xi)}d\omega = 2\pi\delta(t-\xi), \tag{23}$$

where δ *is the 'Dirac delta function'. Eqs. (22)–(23) then yields* $f(t) = \int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega$*, by use of the "sifting" property of* $\delta(\cdot)$*.*

In the 20th *century the concept of l.i.m. has been extended to 'stochastic processes' in which all functions are random variables and the l.h.s. of (15) is replaced by bits 'expectation'.*

A number of important special cases and consequences can be drawn from the Fourier Integral Theorem:

Sine and cosine transforms

If $f(t)$ *is real and* $F(\omega) = R(\omega) + iX(\omega)$*, then*

$$R(\omega) = Re\ F(\omega) = \int_{-\infty}^{\infty} f(t)\cos(\omega t)dt \equiv \int_{0}^{\infty} [f(t) + f(-t)]\cos\omega t dt$$

$$X(\omega) = Im\ F(\omega) = -\int_{-\infty}^{\infty} f(t)\sin\omega t dt \equiv -\int_{0}^{\infty} [f(t) - f(-t)]\sin\omega t dt$$

So, if $f(t)$ *is real and even*

$$R(\omega) = R(-\omega) = 2\int_{0}^{\infty} f(t)\cos\omega t dt, \qquad X(\omega) = 0$$

and hence (by the Fourier Integral Theorem)

$$f(t) = \frac{1}{\pi}\int_{0}^{\infty} R(\omega)\cos(\omega t)d\omega. \tag{24}$$

On the other hand, if $f(t)$ is real and odd

$$R(\omega) = 0, \qquad X(\omega) = -X(-\omega) = -2\int_0^\infty f(t)\sin(\omega t)dt,$$

$$f(t) = -\frac{1}{\pi}\int_0^\infty X(\omega)\sin(\omega t)d\omega. \tag{25}$$

Since an arbitrary function $f(t)$ can always be decomposed into a sum of an even and an odd function

$$f(t) = \frac{f(t) + f(-t)}{2} + \frac{f(t) - f(-t)}{2} = f_{even} + f_{odd},$$

we can write,

$$f_{even}(t) = \frac{1}{\pi}\int_0^\infty R(\omega)\cos(\omega t)d\omega, \qquad f_{odd}(t) = -\frac{1}{\pi}\int_0^\infty X(\omega)\sin(\omega t)d\omega. \tag{26}$$

Next, consider the so-called causal function *$f(t) = 0$ for $t < 0$. Then since for this case*

$$f(t) = 2f_{even}(t) = 2f_{odd}(t), \qquad t > 0 \tag{27}$$

we find that

$$\begin{gathered} f(t) = \frac{2}{\pi}\int_0^\infty R(\omega)\cos\omega t\omega = -\frac{2}{\pi}\int_9^\infty X(\omega)\sin\omega t d\omega, \quad t > 0 \\ f(0) = \frac{1}{\pi}\int_0^\infty R(\omega)d\omega = \frac{1}{2}f(0^+). \end{gathered} \tag{28}$$

In this case, the functions $R(\omega)$ and $X(\omega)$ are not independent. In fact they are Hilbert transforms of each other.

***Cauchy** (1826) pointed out that the sine and cosine transforms lead to reciprocal relations between pairs of functions. If we write*

$$F_c(u) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos utdt,$$

then,

$$f(t) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(u)\cos tudu, \tag{29}$$

and the relation between $f(x)$ and $F_c(x)$ is thus reciprocal. For example e^{-x}, $\sqrt{\frac{2}{\pi}}\frac{1}{1+x^2}$ are a pair of Fourier cosine transforms. Likewise, from Fourier's sine formula, we obtain

$$F_s(u) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin utdt,$$

$$f(t) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(u)\sin tudu, \tag{30}$$

which are sine transforms of each other. Thus e^{-x}, $\sqrt{\frac{2}{\pi}}\frac{x}{1+x^2}$ belong to this family.

The Poisson Summation formula

Consider the function

$$S(x) = \sum_{n=-\infty}^{\infty} f(x+nx_0), \tag{31}$$

where f is some function. $S(x)$ is periodic with period x_0 because with $n+1 = m$

$$S(x+x_0) = \sum_{n=-\infty}^{\infty} f[x+(n+1)x_0] = \sum_{m=-\infty}^{\infty} f(x+mx_0) = S(x), \tag{32}$$

and so it may be expressed as a Fourier Series

$$S(x) = \sum_{l=-\infty}^{\infty} C_l e^{\frac{2\pi ilx}{x_0}}; \qquad C_l = \frac{1}{x_0}\int_{-\frac{x_0}{2}}^{\frac{x_0}{2}} S(x)e^{-\frac{2\pi ilx}{x_0}}\,dx. \tag{33}$$

Substituting $S(x)$ from (31) into (33) and changing variables via $y = x+nx_0$, we have

$$C_l = \frac{1}{x_0}\sum_{n=-\infty}^{\infty}\int_{(n-1/2)x_0}^{(n+1/2)x_0} f(y)e^{-\frac{2\pi ily}{x_0}}\,dy =$$

$$= \frac{1}{x_0}\int_{-\infty}^{\infty} f(y)e^{-\frac{2\pi ily}{x_0}}\,dy = \frac{1}{x_0}F\left(\frac{2\pi l}{x_0}\right)$$

where $F(k)$ is the Fourier transform of $f(x)$ (Eq. (21)). Therefore we arrive at the Poisson sum formula:

$$\boxed{S(x) = \sum_{n=-\infty}^{\infty} f(x + nx_0) = \frac{1}{x_0} \sum_{l=-\infty}^{\infty} F\left(\frac{2\pi l}{x_0}\right) e^{\frac{2\pi i l x}{x_0}}.} \tag{34}$$

In many cases the summation over Fourier-transformed functions is easier then the original sum. For the special case $x = 0$, $x_0 = 1$

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{l=-\infty}^{\infty} F(2\pi l) \equiv \sum_{l=-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) e^{-2\pi i l u} du. \tag{35}$$

Example 1

To sum the series

$$S = \sum_{n=0}^{\infty} \frac{1}{a^2 + n^2} = \frac{1}{2a^2} + \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{1}{a^2 + n^2}, \tag{36}$$

we put $f(u) = \frac{1}{a^2+u^2}$, $v = 2\pi l$. It remains to calculate the integral

$$\int_{-\infty}^{\infty} \frac{e^{-iuv}}{a^2 + u^2} du = \int_{-\infty}^{\infty} \frac{\cos uv}{a^2 + u^2} du + i \int_{-\infty}^{\infty} \frac{\sin uv}{a^2 + u^2} du = \int_{-\infty}^{\infty} \frac{\cos uv}{a^2 + u^2} du.$$

For $v > 0$, the residue theorem is applied to a closed contour completed in the lower v half-plane, yielding $\frac{\pi}{a} e^{-av}$ for the integral. For $v < 0$, the contour is deformed around the upper half-plane and the result is similarly $\frac{\pi}{a} e^{av}$. Combining the two results

$$S = \frac{1}{2a^2} - \frac{\pi}{2a} + \frac{\pi}{a} \frac{1}{1 - e^{-2\pi a}} \equiv \frac{\pi}{2a} \left[\coth(\pi a) + \frac{1}{\pi a}\right]. \tag{37}$$

When $a \to 0$, $S(a) - \frac{1}{a^2} \to \frac{\pi^2}{6}$, so $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. By the same method one can prove the more general result

$$\sum_{n=0}^{\infty} \frac{\cos nx}{n^2 + a^2} = \frac{1}{2a^2} + \frac{\pi \cosh[a(\pi - x)]}{2a \sinh(\pi a)}$$

Example 2

Let

$$f(u) = e^{-2u^2+2\pi i a u\lambda}.$$

Then Eq. (34) yields

$$1 + 2\sum_{m=1}^{\infty} e^{-m^2\lambda^2}\cos(2m\pi\lambda a) =$$

$$\frac{\sqrt{\pi}}{\lambda}e^{-\pi^2 a^2}\left[1 + 2\sum_{n=1}^{\infty} e^{-\frac{\pi^2 n^2}{\lambda^2}}\cosh\frac{2\pi^2 na}{\lambda}\right]. \tag{38}$$

Taking $\lambda = \pi t$, $v = \lambda a$, we obtain

$$1 + 2\sum_{m=1}^{\infty} e^{-m^2\pi^2 t^2}\cos(2m\pi v) = \frac{1}{t\sqrt{\pi}}\sum_{n=-\infty}^{\infty} e^{-\left[\frac{(n+v)}{t}\right]^2}. \tag{39}$$

If $f(x)$ is an even function of x which can be expanded in a Fourier Series of cosines, in the open interval $(-a, a)$, (34) takes the form ($x_0 = 2a$)

$$\begin{aligned}\sum_{n=-\infty}^{\infty} f(x+2na) &= \frac{1}{a}\int_0^{\infty} f(x')dx' \\ &+ \frac{2}{a}\sum_{n=1}^{\infty}\cos\frac{n\pi x}{a}\int_0^{\infty} f(x')\cos\frac{\pi n x'}{a}dx'.\end{aligned} \tag{40}$$

If in (34) we set $x = 0$, $x_0 = \beta$, $\alpha\beta = 2\pi$, $n = l$ and define the symmetric cosine transforms

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F(t)\cos xt dt, \qquad F(t) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos xt dx \tag{41}$$

then, for $\alpha > 0$, $\beta > 0$, the summation formula for even $f(x)$ takes the form

$$\boxed{\sqrt{\beta}\sum_{n=-\infty}^{\infty} f(n\beta) = \sqrt{\alpha}\sum_{n=-\infty}^{\infty} F(n\alpha)}. \tag{42}$$

*This formula may have been known earlier to **Gauss**. When it was first observed by **A.L. Cauchy**, his admiration was very great. He said that it was a discovery worthy of the genius of **Laplace**. It has much in common with the Euler-Maclaurin summation formula and in some applications is superior to it.*

The techniques that Fourier invented have had an impact well beyond studies of heat, or even solutions of differential equations. Real data tend to be very irregular. Consider electrocardiogram arabesques – tantalizing curves that contain all the information of the signal but hide it from our comprehension. Fourier analysis translates these signals into a form that makes sense, transforming a signal that varies with time (or in some cases, with spatial dimensions) into a new function, the Fourier transform *of the signal, which tells how much of each frequency the signal contains.*

In many cases these frequencies correspond to the frequencies of the actual physical waves making up the signal. This applies to sound waves (e.g. speech, music) and all kinds of electromagnetic waves (radio waves, microwaves, infrared, visible light and x-rays), not even known in Fourier's time.

Being able to break down such waves into frequencies has myriad uses, including tuning your radio to your favorite station, interpreting radiation from distant galaxies, using ultrasound for medical diagnosis, and making cheap long-distance telephone calls.

With the discovery of quantum mechanics, it became clear that Fourier analysis is directly relevant to any dynamical system. On the "position space" side of the Fourier transform, one can talk about an elementary particle's position; on the other side, in "Fourier space," one can talk about its momentum or think of it as a wave; and similarly for time *vs.* energy*; electric vs. magnetic field;* intensity *vs.* phase *of an EM wave; and other such "conjugate pairs" of dynamical variables. The modern realization that matter and energy at very small scales behave differently from matter and energy on a human scale – that (for example) an elementary particle does not simultaneously have a precise position and a precise momentum – is naturally expressed in the language of Fourier analysis and transform.*

While irregular functions, defined on compact domain, can be expressed as sums of sines and cosines, usually those sums are infinite. Why translate a complex signal into an endless arithmetic problem in which one must calculate an infinite number of coefficients and sum an infinite number of waves?

We seem to be jumping from the pot into the frying pan. Fortunately a small number of coefficients is often adequate. In the case of the heat diffusion equation, for example, Fourier showed that the coefficients of high-frequency sines and cosines rapidly approach zero, so all but the first few frequencies can safely be ignored. In other cases engineers may assume that a limited number of calculations gives a sufficient approximation, until proved otherwise.

In addition, engineers and scientists using Fourier analysis often don't bother to add up the sines and cosines to reconstruct the signal; instead they "read" Fourier coefficients (or at least the amplitudes; phases are more

difficult) to get the information they want, the way some musicians can hear music silently by reading the notes.

They may spend hours on end working happily in this "Fourier space," rarely emerging into "physical space." (For one-dimensional signals, "physical space" generally corresponds to time, but Fourier analysis can also be applied to pictures. In this case, "physical space" corresponds to position.)

Fourier decomposition and transforms have been extended, in both classical and quantum physics and engineering, to other sets of base functions besides sinusoids. These may be orthonormal or not. Examples are: spatial vibration eigenmodes in non-rectangular geometries; wavelets and Gabor transforms in space an/or time; and coherent states in quantum optics.

But the time it takes to calculate Fourier coefficients is a problem: without computers and fast algorithms, Fourier analysis would have remained a theoretical tool, and digital signal-processing technology would not pervade modern life.

1808 CE **Christian Kramp** (1760–1826, France). Mathematician at Strasbourg. Introduced the factorial symbol $n!$

1808–1810 CE **Etienne-Louis Malus** (1775–1812, France). Engineer and physicist. Discovered polarization of light by reflection and presented a theory explaining double refraction of light in crystals (1810).

Malus was educated at the Ecole Polytechnique and remained associated with it all his life as an examiner, but his main career was in the army. As an engineer he accompanied Napoleon's expedition to Egypt and Syria (1798–1801).

1808–1837 CE **Simeon Dennis Poisson** (1781–1840, France). Notable mathematician. A principal successor to **Laplace**, both in interests and position. There are few branches of mathematics to which he did not contribute something, but it was in the application of mathematics to physical subjects that his greatest services to science were performed. He considered such matters as physical astronomy, stability of planetary orbits (1808), heat conduction (1811), analytical mechanics (1833), the attraction of ellipsoids, probability theory, definite integrals, Fourier series and theory of elasticity. One encounters *Poisson Brackets*, *Poisson's Constant*, *Poisson Integral*, *Poisson*

Equation, *Poisson Summation Formula* (1827) and the *Poisson Distribution* (1837).

He was first to predict the existence of longitudinal and transverse elastic waves[450] (1828) and deduced, in an alternative way to that of Navier, the basic equations of a viscous fluid. He also studied the propagation of waves in anisotropic media (crystals).

He derived the equation satisfied by the gravitational potential *within* a distribution of matter, which now bears his name ($\nabla^2\psi = 4\pi G\rho$, 1813).

Poisson was born in Pithiviers. He was educated by his father who had served as private soldier in the Hanoverian wars but deserted, disgusted by the ill-treatment he received from his patrician officers. Poisson entered the École Polytechnique to study mathematics (1798) and immediately began to attract the notice of Lagrange and Laplace, the latter regarding him almost as his son. In 1806 he became a full professor, in succession to **J. Fourier**. In 1808 he became astronomer to the Bureau des Longitudes and in 1809, when the Faculté des Sciences was instituted, he was appointed professor of rational mechanics.

His father, whose early experience let him to hate aristocrats, bred him in the stern creed of the First Republic. Throughout the Empire period, Poisson faithfully adhered to the family principles and refused to worship Napoleon. After the Second Restoration, his fidelity was recognized by his elevation to the dignity of Baron in 1825, but he never used the title. The revolution of 1830 threatened him with the loss of all his honors, but his disgrace was averted with the help of his friend **Francois Arago** (1786–1853) and in 1837 he was made a peer of France — not for political reasons but as a representative of French science.

In all his work, his role was that of an insightful extender rather than that of a bold originator. As a scientist, however, his activity has rarely, if ever, been equaled. Notwithstanding his many official duties, he found time to publish more than 300 works, several of them extensive treatises, and many of them memoirs dealing with the most abstruse branches of pure and applied mathematics. There is a remark of his that explains how he accomplished so much: "*La vie c'est le travail*".

1809–1810 CE **George Cayley** (1773–1857, England). Father of modern aeronautics. Contributed many ideas to early aviation. Clearly defined for the first time the idea that sustentation can be accomplished by moving

[450] His findings, however, created at that time a new difficulty in the wave theory of light: for if the luminiferous ether behaved like an elastic solid, his analysis showed that *two* waves, instead of one, should be visible!

an inclined surface in the flight direction, provided one has mechanical power to compensate for the air resistance which hinders this motion.

He belonged to a group of enthusiasts who tried to empirically solve the problem of flight by building models and studying bird flight. In his papers, published in 1809–1810, he clearly defined and separated the problem of sustentation, or in modern scientific language — the problem of lift, from that of drag i.e. the component of total resistance that works against the flight direction, and must be compensated by propulsion in order to maintain level flight.

Cayley understood the effect of streamlining on drag, and advocated borrowing from nature in the design of low-drag cross sections (e.g. spindles of the trout and woodcock). The shape of his profiles almost exactly coincided with certain modern airfoil sections. Cayley wrote about helicopters and parachutes. He conceived the biplane and built a glider that carried a coachman for 270 meters. Cayley was born in Brompton, England.

1809–1822 CE **Jean-Baptiste (Pierre Antoine de Monet, le Chevalier) de Lamarck** (1744–1829, France). Naturalist. A thinker who played an important part in preparing the way for universal acceptance of the doctrine of evolution . He also propounded a theory, known by his name as *Lamarckism* that evolutionary change might have occurred by the inheritance of 'acquired characteristics', i.e. he believed that changes that came about during an organism's lifetime, as a result of active adaptation to circumstances, would become impressed upon its genome, or chromosomes, and thus be reproduced in succeeding generations. His ideas were outlined in his book *Philosophie Zoologique* (1809).

Lamarck was born in Bazentin, Picardy. He studied medicine, meteorology and botany, and traveled across Europe as botanist to King Louis XVI from 1781. In 1793 he was made professor of zoology at the Museum of Natural History in Paris.

Lamarck was the first to distinguish vertebrate from invertebrate animals by the presence of a bony spinal column. He was also the first to establish the *crustaceans*, *arachnids*, and *annelids* among the invertebrates[451]. It was Lamarck who coined the word '*biology*'. His studies of both living and fossil invertebrates were described in his book *Natural History of Invertebrate Animals* (1815–1822).

Unfortunately, all attempts to demonstrate a Lamarckian scenario in real-life heredity have failed, either because the demonstration itself has been

[451] So little was known about invertebrates at this time that some scientists grouped snakes and crocodiles with insects.

unconvincing or because the phenomenon said to have been observed were open to an alternative, Darwinian interpretation. The incentive to support Lamarck lay partly in the fact that his theory seems only fair: the skills that human beings obtain by their own endeavors and exertions should surely become part of their children's heritage. This, after all, is what happens regularly in the kind of inheritance that takes place extragenetically through culture[452].

1810–1820 CE **Franz Joseph Gall** (1758–1828, Germany and France). Physician and anatomist. Pioneer in ascribing cerebral functions to various areas of the brain; first to identify grey matter of brain with neurons and white matter with ganglia; sought to establish relationship between faculties and shape of skull (*phrenology*)[453]. Wrote *Anatomy and Physiology of the Nervous System* (1810–1820).

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[452] The Lamarckian idea was not new: the first theory of evolution came from the Book of *Genesis* (**25** – **30**), where the story about breeding suggests that environmental influences can affect heredity.

According to the story, Jacob came to his father-in-law, Laban, and to Laban's daughters to claim his reward for 20 years of service to them. After some discussion it was agreed that his reward should be to take for his own from Laban's flocks all the brown sheep and all the spotted, speckled and banded goats.

Jacob accordingly set about increasing the proportion of such goats in the flock. He did this by causing them, as the famous depiction by the seventeenth-century Spanish painter Bartolomé Murillo shows, to mate in the presence of rods, or wands, of green poplar, hazel and chest-nut stripped of bark in such a way that they were emblazoned with alternating bands of white and dark wood. This is alleged to have done the trick. The offspring of goats influenced by the stripes now had among them, according to the tale, an increased proportion of "ring-straked (banded) speckled, and spotted" goats!

The story of Jacob and the sheep of Laban is by no means uplifting, but it does illustrate the great antiquity of Lamarckian ideas and helps to explain why those ideas enjoy a popular revival every few decades.

[453] A pseudo scientific theory based on the idea that certain mental faculties and character traits are indicated by the configuration of a person's skull, i.e. that *mental* qualities are associated with *physical* characteristics. The sciences of physiology and psychology have shown that different portions of the brain do have certain functions, but these usually merely receive sensory stimuli and relate them to action. However, the knowledge we have accumulated so far tends to disprove phrenology.

Gall was born in Tiefenbrunn, Germany; Physician in Vienna (1785); took up residence in Paris (1807). His lectures on phrenology were popular, but suppressed (1802) as being subversive to religion.

1810–1836 CE **Jöns Jacob Berzelius** (1779–1848, Sweden). Distinguished chemist. Made valuable contributions to the development of atomic theory. Discovered the elements selenium and thorium (1829). He was the first to isolate the elements *calcium*, *silicon* (1823) and *tantalum*, and to note and describe *allotropy*, *isomorphism* and *chemical catalysis* (1835). He originated the system of writing chemical symbols[454] and formulae (1813), and undertook an extensive program to determine the relative atomic weights of the known elements (1818–1826).

Berzelius was born at Väfversunda Sorgard, near Linköping, Sweden. He went to Uppsala University, where he studied chemistry and medicine, and graduated as M.D. in 1802. He served as a professor of chemistry at Stockholm from 1807 until 1832. About 1807 he began to devote himself to what he made the chief object of his life — the elucidation of the composition of chemical compounds through atomic theory. During 1810–1820 he analyzed over 2000 inorganic compounds to determine the weight ratios of the various constituent elements, using *oxygen* as the basis of reference. This resulted in the first table of atomic weights which he published in 1818 and 1826: most elements are presented with atomic weights very close to those accepted in the 20^{th} century.

Another service of the utmost importance which he rendered to the study of chemistry was in continuing and extending the efforts of **Lavoisier** and his associates to establish a convenient system of chemical nomenclature, later

[454] Some of the elements retained their old names: *copper* = the metal from Cyprus (301 AD); *gold* = gelb = yellow (teutonic; the Latin *aurum* is akin to Aurora, goddess of the dawn).

Many elements have names derived from Greek roots: *chlorine*, from its color, chlorus = yellowish green (**Davy**, 1811); *chromium*, from chroma = color. Other elements have been named after mythological deities or personages: *vanadium* from Vanadis, one of the names of the Norse goddess Freya; *thorium* from Thor, the Scandinavian war-god; *tantalum* and *niobium* from Tantalus and Niobe, of Greek mythology. Names of places where compounds of elements were first discovered have sometimes formed the bases of other names: *strontium*, from Strontian, in Scotland; *ruthenium*, from Ruthenia (Russia); *ytterbium*, from Ytterby (Sweden); *hafnium*, named after Copenhagen, formerly called Hafnia; *masurium*, after a lake in East Prussia; *rhenium*, from the Rhine; *palladium* and *uranium* after Pallas and Uranus, discovered about the same time; *selenium* and *tellurium* are named after the Moon (selene) and the Earth (tellus).

to become accepted in the scientific literature: an element is generally represented by the first letter of its Latin name, or, in the event of elements with the same first letter, by the first two letters. Compounds are symbolized by juxtaposing the element symbols, superscribed[455] with the number of atoms involved if greater than one; e.g., carbon dioxide is symbolized as CO^2. Berzelius was first to classify minerals on a chemical basis.

1811 CE **Amadeo Avogadro** (1776–1856, Italy). Physicist. Postulated the *Avogadro hypothesis* which states that for a given temperature and pressure, equal volumes of gas have the same number of molecules (moles)[456]. This provides an explanation for the law of integral volume ratio [asserts that when two gases combine chemically, they do so such that the two volumes involved are in the ratio of whole numbers]. It was discovered by **Joseph Louis Gay-Lussac** (1778–1850, France) in 1809.

Avogadro's hypothesis was ignored until 1865, when **Joseph Loschmidt** (1821–1895, Austria) used the new kinetic theory of gases to obtain the number of molecules of an ideal gas in a cubic centimeter as 2.69×10^{19} under standard conditions. [*Loschmidt-Avogadro number*. Also given as 6.022×10^{23} molecules/mole, since $\frac{6.022}{2.69} 10^4 \cong 22400$, the number of cm^3 in 22.4 liters.] This number is one of the fundamental constants of nature. Unlike the dimensionless constants, this one belongs to the category of constants whose numerical values depend on conventions and system units. Here specifically,

455 In 1834, **Justus von Liebig** revised this by replacing superscript by subscript, e.g., $CO^2 \to CO_2$.

456 The ideal gas equation [which combines *Boyle's law* (1660) with *Charles law* (1787)], is: $PV = \alpha T$, where $\alpha = \frac{P_0 m}{\rho_0 T_0}$ is a constant which depends on the selected mass m of a particular gas. There are two options for making this a useful law: either to agree on a *fixed mass* of gas (say, one gram) or choose a *variable mass*, but one that always has the same number of molecules (molecular mass, mole). In the first case, α will differ from one gas to another. In the second case, α (denoted by R) will have the same value for 1 mole of all gases. It is known as the *gas constant*. Thus, the ideal gas equation, referring to one mole of any gas, is $PV = RT$, and $PV = nRT$ for n moles. Here $n = \frac{m}{M}$, m being the mass in grams and M is the molecular weight (mass). It has been found experimentally that one mole of any gas under standard pressure and temperature occupies approximately 22.4 liters.

Avogadro's hypothesis (which later became a *law* in the framework of kinetic theory of gases) removed a serious obstruction to progress in chemistry since it provided a simple way of comparing masses of molecules by weighing equal volumes of two gases. The results agreed with other evidence, leading chemists to trust the hypothesis.

on the centimeter unit and the values of standard temperature and pressure (for Loschmidt's number), and of the numerical scale of atomic and molecular weights (for Avogadro's number[457]). Once these definitions are made, the value of the constants is immutable. All that remains is to measure it as accurately as possible.

Avogadro was born at Turin. He was for many years professor of physics at Turin University. He published numerous physical memoirs but is chiefly remembered for his "*Essai d'une maniére de déterminer le masses relatives des molécules elémentaires des corps, et les proportions selon lesquelles elles entrent dans les combinaisons*", in which he enunciated his hypothesis. He coined the term *molecule*.

1811 CE Birth of the Siamese twins **Eng and Chang** (1811–1874), identical twins joined together at the hip. They ended up as American citizens, taking the name Bunker, and before the Civil War they were shareholders in North Carolina. Eng and Chang had seven daughters and three sons for Chang, seven sons and five daughters for Eng. (The birth of Siamese twins is very rare, about 1:50,000).

1811 CE **Charles Bell** (1774–1842, Scotland). Surgeon and anatomist. Discovered distinct functions of sensory and motor nerves and the dual nature of spinal nerves (1811).

Asserted in his *Idea of a New Anatomy of the Brain* that different parts of the brain undertake different functions and that the specific functions of each of the various divisions of the peripheral nerves derive from the part of the brain connected to that division.

1812 CE Volcanic eruption in the Azores (1811) led to a bitter winter in 1812, and was a major factor in the defeat of Napoleon's army in Russia.

457 The precise value of Avogadro's number for the ^{12}C atomic weight scale is 6.022169×10^{23} atoms per gram atomic weight. One gram atomic weight of any element (i.e., the atomic weight of the element, expressed in grams) is called a *mole*. One mole of *any* pure substance — whether it is composed of atoms, molecules, ions, electrons, or any other kind of particle — contains (by definition) Avogadro's number of particles. For this reason, Avogadro's number is given by $N_A = 6.022169 \times 10^{23}$ mole^{-1} with the numerator "particle" being understood. Thus, 1 mole of H atoms weighs precisely 1.00797 g; 1 mole of N atoms weighs 14.0067 g. Similarly, 1 mole of water molecules H_2O weighs 15.9994 g+2(1.00797 g) = 18.0153 g. Avogadro's number of *photons* (a mole of photons) is called 1 *einstein*. The energy of one einstein at wavelength λ is $\frac{E_A hc}{\lambda}$.

1811–1816 CE Organized bands of English rioters called *Luddites*[458] destroyed labor-saving machines as a protest against their low wages and terrible working conditions, and because of the widespread prejudice that its use produced unemployment.

The riots arose out of severe distress caused by the war with France. Apart from this prejudice, it was inevitable that the economic and social revolution implied in a change from manual labor to work by machinery should give rise to great misery. The riots began (1811) with the destruction of stocking and lace frames in Nottingham. Continuing through the winter and the following spring, it spread into Yorkshire, Lancashire, Derbyshire and Leicestshire. They were met with severe repressive legislation. In 1816 the rioting was resumed (caused by depression which followed the peace of 1815 and aggravated by one of the worst recorded harvests) and extended over the whole kingdom. Vigorous repressive measures, and, especially, reviving prosperity, brought the movement to an end.

1811–1848 CE **Dominique-Francois-Jean Arago** (1786–1853, France). Physicist and statesman. Contributed to the discovery of laws of *light polarization*, ruling out the previously assumed longitudinal nature of light. He thus lead **Young** (1817) to the correct transverse nature of light's vibrations.

Arago devised an experiment (1816) by which the nature of light was demonstrated via its reduced speed through dense media. Working with Biot, he made measurements of arc length on the earth which led to the standardization of the metric system of lengths. Encouraged his student Le Verrier to investigate irregularities in *Uranus'* orbit, which led to the discovery of *Neptune*.

Arago was educated at the Ecole Polytechnique in Paris and became professor of geometry there at the age of 23. Later (from 1830) he became director of the Paris Observatory. He was minister of war and marine in the provisional government (1848); responsible for the abolition of slavery in the colonies.

1812 CE, June Napoleon invaded Russia with a grand army of ca 600,000 men. Of these, only some 90,000 reached Moscow. The rest succumbed to the common campaign diseases of *dysentery* and *typhus*. Typhus had been endemic in Poland and Russia for many years. Lack of water and insufficient changes of clothing made bodily cleanliness impossible. Fear of Russian attack and Polish reprisals caused the men to sleep close together in large groups.

[458] Named after **Ned Ludd**, who in 1779 destroyed frames used in stocking machines in a village in Leicestershire.

The lice of infested hovels crept everywhere, clung to the seams of clothing, to the hair and bore with them the bacteria of typhus. Disease alone had rubbed Napoleon's central force of some 265,000 men of its effective strength by the end of the first month!

By June 1813, less than 3000 of his grand army were alive. By the autumn of 1813 some 470,000 new troops were mobilized for the final battle.

1812–1823 CE **Jacque-Philippe-Marie Binet** (1786–1856, France). Mathematician and astronomer. Discovered the rule for *matrix multiplication*. He continued to investigate the foundation of matrix theory, thus setting the scene for later work by Cayley and others. Derived the laws of motion of a particle in a field of a central force (*Binet's formulas*[459]).

Binet was a student at the Ecole Polytechnique in Paris and after graduating worked for the Department of Bridges and Roads of the French government. Appointed to the chair of astronomy at the College de France (1823).

1813 CE **Simon-Antoine-Jean Lhuilier** (1750–1840, Switzerland). Mathematician. Noticed that Euler's formula $v-e+f=2$ was wrong for solids with holes in them and derived instead the more general formula $v-e+f=2-2g$ where g is the number of holes. This was the first known result on what we call today *topological invariant*.

1813–1822 CE **Pierre-Charles-Francois Dupin** (1784–1873, France). Differential geometer. Invented the *Dupin indicatrix*[460] which gives an indication of the local behavior of a surface. Dupin was a pupil of Monge at the Paris Ecole Polytechnique. Entering the Napoleonic Navy as an engineer, Dupin lived to be a promoter of science and industry, a peer of France and a senator under Napoleon III. He was a professor at Conservatoire des Artes (1819–1864).

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459 *Binet's formulas*: Given the equation of the particle's path $r=r(\theta)$ under a central force in the xy-plane, the first formula is $v^2=h^2\left[\frac{1}{r^2}+\left(\frac{d\,\frac{1}{r}}{d\,\theta}\right)^2\right]$ where v is the particles's velocity; the second formula is $F=-\frac{mh^2}{r^2}\left[\frac{1}{r}+\frac{d^2}{d\,\theta^2}\left(\frac{1}{r}\right)\right]$, where F is the centripetal force acting in the central motion, m is the particle's mass, and h is twice the areal velocity.

460 *Dupin's indicatrix* is the ellipse $\kappa_1x^2+\kappa_2y^2=1$ with principal semi-axes $\left\{\frac{1}{\sqrt{\kappa_1}},\frac{1}{\sqrt{\kappa_2}}\right\}$ where $\{\kappa_1,\kappa_2\}$ are the principal curvatures of a surface. The *normal curvature* $\kappa=\kappa_1\cos^2\alpha+\kappa_2\sin^2\alpha$ is obtained graphically by intersecting the ellipse with a line $y=x\tan\alpha$ through the center; the distance intercepted by the ellipse in the α direction is then equal to $\frac{1}{\sqrt{\kappa}}$.

1814–1823 CE **Joseph von Fraunhofer** (1787–1826, Germany). Optician and physicist. The true founder of astrophysics; laid the foundations of solar and stellar spectroscopy. Born at Straubing in Bavaria to a very poor family, he was apprenticed to an optician at the age of 11 and lived in a half-ruined house in Munich. One day this slum collapsed, killing all its occupants except the young boy, who was pulled out of the ruins seriously injured. The Elector of Bavaria showed his compassion to the survivor by granting him the sum of 18 ducats, which Joseph spent on books and optical instruments. Through solitary and dogged labor he became an expert optician, and at 19 he went to work for a large glassware and scientific instrument factory. Three years later he became one of its directors!

In an attempt to improve telescope objectives, he embarked on a study of prisms and refraction. He repeated Newton's experiments, but added to the prism a small telescope which received the colored beams and gave a particularly clear image of the spectrum. Having far better prisms at his disposal than had Newton, he discovered (1814) that the continuous spectrum of the sun is interrupted by a large number of dark lines: certain wave-lengths were lacking, or at least were greatly suppressed, in the light of the sun.

These dark lines were subsequently called *Fraunhofer lines*[461]. He not only discovered them, but undertook the measurement of their relative positions, making up a map of the spectrum. By a series of ingenious experiments, he proved that these dark lines really are a property of sunlight and are not due to instrumental shortcomings.

Fraunhofer then turned to the spectra of other celestial bodies. In the spectrum of Sirius he could see lines which were completely different from those he found in sunlight. He noticed that different stars had different spectra, thus showing the way to *stellar spectroscopy*. At the same time this tireless optician was studying artificial light sources; he noticed in particular that the same bright yellow line, or rather a pair of lines, was to be found in nearly all flames. This double line is in exactly the same position as a very strong line in the solar spectrum, called the *D line* by Fraunhofer.

He invented the *diffraction grating* and established the fundamental law which makes it possible to find the wavelength of monochromatic radiation from the position of the corresponding line[462] in the grating spectrum of order

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461 **Whollaston** had detected 4 of the strongest lines in 1802, but he thought that they were the *natural* separations between the colors!

462 A diffracted wave, obtained by a grating from a plane wave at normal incidence, has local maxima in directions making angles θ with that of the white central image such that $\sin\theta = n\frac{\lambda}{d}$, where d is the distance between 2 successive slits in the grating and n are integers. Thus, the grating disperses light composed of

n. With the aid of his gratings Fraunhofer determined the first accurate value of the wavelength of the sodium[463] D lines in 1822.

In his studies he used a beam of light coming practically from infinity (plane-wave), which is the one used in observing the stars through a telescope. This kind of diffraction has since been named *Fraunhofer diffraction* in his honor [in contradistinction to the *Fresnel diffraction* from *close* point-sources]. In 1823 he was appointed conservator of the physical cabinet at Munich, and in the following year he received from the elector of Bavaria the civil order of merit.

All the scientific achievements of Joseph von Fraunhofer were carried out in his spare time, and one sometimes wonders how such an enormous amount of work could have been done in so short a life. He died at Munich and was buried near Reichenbach. On his tomb is the inscription "*Approximativ sidera*".

Fraunhofer started his apprenticeship at a tender age, produced a large number of inventions, both large and small, and died before he was 40. One might say that he was, in a way, the Mozart of physics.

1814–1825 CE **George Stephenson** (1781–1848, England). Engineer and inventor. Known as the '*Founder of Railways*'. Completed the adaptation of the steam engine to the railroad.

He was born in Wylam, near Newcastle, the son of a coal-ship fireman. In boyhood he was employed as a cowherd, in his 14^{th} year he became assistant fireman to his father at a shilling a day, and in his 17^{th} year was yet unable to read. In his 18^{th} year he began to attend a night school and made remarkably rapid progress. In 1804 he moved to Killingworth and there devised his miner's safety lamp (1815), independently of **Humphrey Davy** who was producing his lamp at about the same time.

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various wavelengths more, the closer the spacing of the slits. Hence the efforts of this ingenious physicist to engrave gratings with closer and closer lines. He constructed a machine that could engrave 3000 lines in one centimeter of glass, with a diamond point.

463 Sodium (Na) was first isolated by **Humphry Davy** (1807) through *electrolysis* of Na_2CO_3 (caustic soda), using a *voltaic battery* (1800). Previous to this discovery, caustic alkalies were regarded as elements, although **Lavoisier** (1789) hinted that alkaline earths might be oxides of unknown metals.
Fraunhofer did not know in 1822 that the lines that he tabbed "D" were due to sodium, but **Kirchhoff** (1859–1861) showed that Fraunhofer's D-lines were produced by the cool *sodium vapor* of the solar atmosphere.

In 1814 he completed his construction of a traveling engine for the tramroads between the coal-ship and the shipping port, 15 km distant. The engine, which he named '*My Lord*', ran a successful trial on 25^{th} of July, 1814 at a speed of 10 km/h. His second engine, '*Puffing Billy*', embodied his invention, the steam blast. This device increased the draft in the boiler. In turn, the fire became hotter and made steam of higher pressure.

He was instrumental in opening the world's first railway, the *Stockton and Darlington Railroad* on Sept. 27, 1825. His locomotive *The Rocket* (1829) traveled at the then unheard of speed of 48 km/h, and became a model for later locomotives.

With the wealth he amassed from his inventions he became a philanthropist for the miners cause, establishing night schools for miners and educational and recreational facilities for their children.

1815 CE, April 5–10 *The greatest volcanic explosion of recent centuries*. The eruption of *Mount Tambora* ($8\,^{\circ}15'$S, $118\,^{\circ}00'$E) on the Island of Sumbawa in Indonesia killed ca 100,000 persons. About 150 km^3 of tephra $(1.7 \times 10^6$ tons$)$ were ejected into the atmosphere, giving rise to remarkable sunsets and luminous twilights in England for 6 months after the eruption[464]. The total energy released in the two series of eruptions (April 5 and April 10), is estimated at 8.4×10^{26} erg, 80 times bigger than that of *Krakatoa* (1883).

The year that followed has sometimes been called the *year without a summer*, there being only 3 or 4 days without rain between May and October 1816 in Wales with subsequent poor harvests and food shortage.

The explosion affected climate on a world-wide scale: temperatures dropped by about $2\,^{\circ}$–$4\,^{\circ}$C in Paris, Geneva, Milan, and some North American locations[465], resulting in considerable famine and extremely cold winters in many parts of the world. This lasted for about 3 years. It was not until 1847 that the first scientific expedition went to Sumbawa to study Tambora.

1815 CE, June 16-18 *Battle of Waterloo* (Belgium): An allied army under the command of Wellington (mixed British-Dutch-German-Belgian force of 100,000 men) and Blücher (Prussian force, ca 120,000 strong) defeated Napoleon's French army (ca 124,000 men); Austrian and Prussian monarchies

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[464] These may have inspired some of the best works of the English painter J.M.W. Turner and the novel *Frankenstein* by **Mary Shelley** who lived at that time in Geneve with **Shelley** and **Byron**.

[465] A freezing cold was reported in New England on the night of June 10, 1816, and on July 04, 1816 during daytime!

restored; German confederation replaced Confederation of the Rhine; Kingdom of Netherlands formally united Belgium and Holland. France's boundaries restored to those of 1790.

Napoleon very nearly defeated Wellington at Waterloo; Napoleon's ill-health may have provided the necessary weight to tilt the balance (migraine, hemorrhoids, gall stone colic, peptic ulcer and thyroid deficiency).

1815–1820 CE **John Loudon McAdam** (1756–1836, Scotland). Inventor. Originated the paving of roads with crushed rock, known as the *macadam* type of road surface. He was the first man to recognize that dry soil supports the weight of traffic, and that pavement is useful only for forming a smooth surface and keeping the soil dry. His macadam pavements consist of crushed rock packed into thin layers. McAdam methods of road building spread to all nations.

He was born at Ayr, Scotland, being descended on his father side from the McGregors. In 1770 he went to New York and returned with a considerable fortune (1783). The highways of Great Britain were at this time in a very bad condition and McAdam at once began to consider how to effect reforms. In pursuing his investigations he had traveled over 50,000 km of roads. In 1819 he published a *Practical Essay on the Scientific Repair and Preservation of Roads*, followed, in 1820, by the *Present State of Road Making*. As a result of a parliamentary inquiry in 1823 into the whole question of road-making, his views were adopted, and in 1827 he was appointed general surveyor of roads.

History of Roads[466] *and Highways*

Early roads were built in the Near East soon after the wheel was invented (ca 3500 BCE). As travel developed between villages, towns, and cities, trade routes were made. One such early system of roads was the Old Silk Trade

466 The word *road* came from the Middle English word *rode*, meaning a *mounted journey*.This, in turn, was derived from the Old English *rad*, from the word *ridan*, meaning *to ride*.

Route which extended over 10,000 km connecting China and Rome and pre-Christian Europe across Turkestan, India, and Persia. The first road markers were piles of stones at intervals. Trails through forests were marked by blazing trees.

The Egyptians, Carthaginians and Etruscans all built roads. But the first really great road-builders were the Romans. They knew how to lay a solid base, paved with flat stones and recognized that the road must slope slightly from the center toward both sides to drain off water. They also dug ditches along the sides of the road to carry water away. Roman roads were intended mainly to transport soldiers across their empire. These roads ran in almost straight lines and passed over hills instead of cutting around them. The Roman built more than 80,000 km of roads and some are still in use.

In the Middle Ages there was little reason to build good roads, because most of the travel was on horseback. In South America from the 1200's to the 1500's the Inca Indians built a network of 16,000 km of roads connecting their cities. In England, certain main roads were higher than the surrounding ground because earth was thrown from the side ditches toward the center. Hence they were called highways. *These roads were under the protection of the king's men and were open to all travelers. In North America, early roads were surfaced with hand-broken stone and gravel. Some roads were covered with logs or planks, laid crosswise, and were therefore very bumpy.*

When the steam locomotive arrived in 1830, the rapid development of railroads began and people became convinced that the railroad was the best means of travel over long distances. From 1830 to 1900, there was little change in the surfacing materials for roads and highways. Even in the cities, only wood blocks, brick, and cobblestones were used. By 1900, because of the rapid development of the United States, there was a growing demand for good roads. It was mainly for roads extending a short distance from the railroad so farmers could get their produce to the rails. But with the ever growing use of the automobile after 1900, the demand arose for good roads to all places.

The first concrete road was laid in Detroit in 1908.

1815–1827 CE **William Prout** (1785–1850, England). Chemist and physician. Practiced in London. Suggested that *hydrogen* is the fundamental unit from which all elements are built (1815–1816). Among the first to classify food components into fats, carbohydrates, and proteins (1827). Made significant determinations of the density of air (1822–1823).

1815–1840 CE **Olinde Rodrigues** (1794–1851, France). Mathematician, economist and reformer with a brief career in mathematics. Born to Jewish parents of Portuguese ancestry. A student of **Monge** at the École Polytechnique. In 1815 he contributed to the differential geometry of surfaces (*Rodrigues formula* and *Rodrigues theorem*) and in 1816 his name became attached to a theorem in the theory of Legendre functions (also *Rodrigues formula*). Soon thereafter he became interested in the scientific organization of society, but made his living off the family banking business.

In 1840 he found some spare time to prove that every displacement of a rigid body is the resultant of a rotation and a translation. Described a rotation by *four* parameters, the first three determining the direction of the axis. He then developed explicit formulae for the resultant of two rotations and stressed the fact that the product is *not* commutative. He came to the aid of *Saint-Simon*[467] (founder of Socialism) in his destitute old age, supported him during the last years of his life and became one of his earliest adherents.

1816 CE **Renè Theophile Hyacinth Laënnec** (1781–1826, France). Physician. Invented the *stethoscope*: a device physicians use to hear the sounds produced by certain organs of the body, such as the heart, lungs, veins, and arteries.

Laënnec was a pupil of Napoleon's personal physician, Corvisart. He made the first stethoscope from a hollow wooden tube.

1816–1822 CE **Francois Magendie** (1783–1855, France). Physiologist. Showed for the first time that *nitrogenous* foods were needed for life. Professor at College de France (from 1831).

He fed dogs on diets composed of distilled water and one specific food, such as sugar, olive oil, or butter. The dogs in every case died after about a month.

Extended the work of Charles Bell (1811) on the functions of the dorsal and ventral roots of spinal nerves (1822). Formulated and demonstrated the *Bell-Magendie Law* that the anterior roots of the spinal cord control movements while the posterior roots control sensation.

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467 **Claude-Henri de Rouvroy**, **Comte de Saint-Simon** (1760–1825, France). Volunteer with the French troops fighting with Americans in the American Revolution (1777–1783); on his return to France (1783) made a fortune in land speculation but lost it (by 1805), and lived thereafter in poverty. Founded a 'religion of socialism', combining the teaching of Jesus with ideas of science and industrialism. His disciples spread his system, known as *Saint-Simonianism* throughout Europe.

History of Biology and Medicine, III – The 'Age of Reason'

During the Renaissance and Age of Discovery, renewed interest in empiricism as well as the rapidly increasing number of known organisms led to significant developments in biological thought; ***Vesalius*** *inaugurated the rise of experimentation and careful observation in physiology, and a series of naturalists culminating with* ***Linnaeus*** *and* ***Buffon*** *began to create a conceptual framework for analyzing the diversity of life and the fossil record, as well as the development and behavior of plants and animals. The growing importance of natural theology — partly a response to the rise of mechanical philosophy was also an important impetus for the growth of natural history (though it also further entrenched the argument from design).*

In the 18^{th} *century many fields of science — including botany, zoology, and geology — began to professionalize, forming the precursors of scientific disciplines in the modern sense (though the process would not be complete until the late 1800s).* ***Lavoisier*** *and other physical scientists began to connect the animate and inanimate worlds through the techniques and theory of physics and chemistry.*

In 1665, using an early microscope, ***Robert Hooke*** *discovered cells in cork, and a short time later in living plant tissue. The German* ***Leonhart Fuchs****, the Swiss* ***Conrad von Gesner****, and the British authors* ***Nicholas Culpeper*** *and* ***John Gerard*** *published herbals that gave information on the medicinal uses of plants.*

In 1628 ***William Harvey*** *explained that blood circulates throughout the body, and is pumped by the heart.* ***Antony van Leeuwenhoek****'s use and improvement of the microscope in about 1650 opened up the micro-world of biology. The History of Plants was greatly extended, almost into an encyclopedia, by* ***Giovanni Bodeo da Stapel*** *in 1644 CE.* ***Jan Swammerdam*** *(1658) and* ***Marcello Malpighi*** *(1660) were the first to observe and describe red blood cells, while Leeuwenhoek was the first to describe spermatozoa, bacteria and infusoria in the 1670's and 1680's. By the 1690's plants were, like animals, known to be sexual, having stamens and pistils.*

Systematizing, naming and classifying dominated biology throughout much of the 17th and 18th centuries. ***Carolus Linnaeus*** *published a basic taxonomy for the natural world in 1735, and in the 1750's introduced scientific names for all his species. The discovery and description of new species, and collecting specimens became a widespread passion of biologists.*

This work of classification was led by the Frenchmen ***Antoine de Jussieu*** *(1789),* ***Geoffray St. Hilaire*** *(1796),* ***Georges Cuvier*** *(1812) and* ***August de Condolle*** *(1819).*

One of the major evolutional trends during 1530–1750 CE was the passage from alchemy to medical chemistry. The ancient Greek biologists and medical writers had never considered the physiology of the human body in specifically chemical terms. Since ancient the Greek philosophers and the medieval scholars were not greatly interested in chemical substances and their properties, most medicines were not derived from mineral sources. However, some alchemists *became interested in the application of alchemy to medicine.*

Such a movement culminated in the work of ***Paracelsus*** *(1531), who endeavored to bring into being a new science of medical chemistry (iatrochemistry), by uniting medicine with alchemy.*

He put forward a theory that the human body was essentially a chemical system composed of mercury, sulphur and salt. Illness, according to Paracelsus, could arise from a lack of balance between these three elements, and the balance was to be restored by mineral medicines, not organic remedies. Iatrochemistry was developed further by ***van Helmont*** *(1648).*

One of the earliest chemists to put forward a mechanical theory of chemical change was ***John Ray*** *(1630). His line of thinking was extended by* ***Robert Boyle*** *(1684). Boyle was interested in the work of iatrochemists, particularly in their empirical observations, but he was of the view that those observations should be explained in terms of the mechanical philosophy, namely — that matter consists of particles of corpuscles in motion.*

The English school of medical chemists of the 17th century, including ***Robert Hooke*** *(1635–1703),* ***Richard Lower*** *(1631–1691), and* ***John Mayow*** *(1645–1679) — did not survive, and modern chemistry was founded in France at the end of the 18th century. Boyle had arrived at a reasonable definition of a chemical element and at a promising conception of method in chemistry.*

The discovery of the circulation of the blood by ***Harvey*** *(1628) established its primacy and many 18th century authors attributed to it alone all the properties formerly associated with the other humours. From the end of the 18th century physiological investigations concentrated more upon its constituent parts and assigned properties to them and, although modern science reliance on blood tests and transfusions has emphasized its role in diagnosis and therapy, it is now viewed primarily as a carrier and transmitter of other, more important, chemical substances round the body (e.g. hormones).*

The concept of the brain *as an anatomical entity emerged quite early in Western thought. The brain is first mentioned in Egyptian papyri and much of its detailed gross anatomy was described by the Greeks especially after the establishment of dissection as a valid method of inquiry.* ***Galen****, for example, was able to produce a classification of the* cranial nerves*.*

With the revival of anatomy during the Renaissance, more features of the brain were described, notably its arterial supply by ***Thomas Willis*** *(1664).* ***Marcello Malpighi*** *(1660) first investigated cerebral structure microscopically. By 1800, this anatomical tradition has elucidated most of the major visible features of the organ except for the regularity of the cerebral convulsions.*

Harvey's discovery of the blood's circulation (1628) also helped to elucidate the mechanics of breathing (respiration). First, ***Robert Boyle****,* ***John Mayow*** *and others (1645–1679) showed air necessary for life, Mayow recognizing a component of air indispensable for combustion, respiration and converting venous into arterial blood.*

Then ***Malpighi*** *(1679) microscopically identified the pulmonary capillaries and* ***Albrecht von Haller*** *(1752) expounded the mechanics of breathing. The identification of Mayow's* aerial nitre *with oxygen by* ***A. Lavoisier*** *(1780) and others, sealed the analogy between combustion and respiration.*

Finally, ***E.F.W. Pflüger*** *showed that the essential chemical changes of respirations occur in the tissues and cells rather than the lungs (metabolism).*

Table 3.9: LEADING BIOLOGISTS AND MEN OF MEDICINE (1600–1820)

Key:

B = Biology	ZO = Zoology	CL = Chemistry of Life
A = Anatomy	M = Medicine	EP = Epidemics
P = Physiology	EM = Embriology	T = Taxonomy
BO = Botany	MB = Microbiology	IM = Immunology
S = Surgery	MR = Medical Research	PA = Pathology
		EB = Evolutionary Biology

Name	*fl.*	*Specialization*
Gaspard Bauhin	*1588–1623*	*(BO), (A)*
Eliyahu de Luna Montalto	*1596–1616*	*(M), (MR)*
John Gerard	*1597–1607*	*(BO)*
Andreas Libau	*1597–1613*	*(P), (CL)*
John Tradescant	*1600–1638*	*(BO)*
Adrian van der Spiegel (Spigelius)	*1603–1625*	*(BO), (A)*
Santorio Santorio	*1603–1614*	*(P), (M)*
Theodore Turquet de Mayerne	*1603–1644*	*(P), (CL), (MR)*
Hieronimus Fabricius	*1604–1619*	*(EM), (A), (M)*
Joseph Solomon Delmedigo	*1616–1629*	*(M)*
Johann Baptista van Helmont	*1620–1648*	*(BO), (M), (CL)*
Zacutus Lusitanus	*1625–1642*	*(M), (MR)*
William Harvey	*1628–1651*	*(P), (M)*
Franciscus de la Boë	*1641–1672*	*(M), (A), (CL)*
Thomas Browne	*1645–1680*	*(BO), (M)*
Georg Rumpf van Hanau	*1655–1698*	*(BO)*
Thomas Wharton	*1656–1673*	*(M), (A)*
Jan Swammerdam	*1658–1673*	*(P), (A), (EM)*
Marcello Malpighi	*1660–1679*	*(P), (A)*
Lorenzo Bellini	*1664–1704*	*(M), (A)*
William Petty	*1664–1687*	*(A), (M)*
Thomas Willis	*1664–1672*	*(A), (P)*
Richard Lower	*1665–1691*	*(P)*
Robert Hooke	*1665–1703*	*(P)*
Thomas Sydenham	*1666–1686*	*(EP), (M)*

Table 3.9: (Cont.)

Name	*fl.*	*Specialization*
Francis Willughby	*1667–1704*	*(T)*
John Ray	*1667–1705*	*(T), (CL)*
Francesco Redi	*1668–1697*	*(B)*
Anton van Leeuwenhoek	*1668–1692*	*(MB)*
Regnier de Graaf	*1668–1673*	*(M), (A)*
Nehemiah Grew	*1672–1682*	*(M), (BO)*
John Mayow	*1674–1679*	*(P), (M)*
Robert Boyle	*1684–1691*	*(P)*
Rudolph Camerarius	*1694–1721*	*(M), (BO)*
Giacomo Pylarini of Smyrna	*1701*	*(M), (IM)*
Stephen Hales	*1705–1730*	*(P)*
Hermann Boerhaave	*1707–1732*	*(M), (BO)*
Antoine de Jussieu	*1719–1758*	*(BO)*
Pierre Fauchard	*1728–1761*	*(M)*
Jean Astruc	*1729–1753*	*(M)*
Carolus Linnaeus	*1735–1763*	*(BO), (T)*
Georges Louis Leclerc (de Buffon)	*1739–1788*	*(BO)*
Julien de la Mettrie	*1740–1751*	*(M), (S)*
Percival Pott	*1740–1780*	*(M), (S)*
Giovanni Battista Morgagni	*1740–1760*	*(M), (A)*
James Lind	*1747–1794*	*(S), (M)*
Frederik Hasselquist	*1749–1752*	*(BO)*
Victor Albrecht von Haller	*1752–1773*	*(M), (A), (P), (BO)*
John Hunter	*1760–1790*	*(M), (P)*
Joseph Gottlieb Kölreuter	*1761–1766*	*(BO)*
Lazzaro Spallanzani	*1765–1785*	*(P), (B)*
Peter Simon Pallas	*1766–1794*	*(BO), (ZO)*
William Hewson	*1769–1774*	*(P), (S), (A)*
Otto Frederik Müller	*1773–1778*	*(B)*
Jan Ingenhousz	*1779–1799*	*(M), (CL)*
Jiri Prochaska	*1784–1820*	*(A), (P), (MB)*
Christian Sprengel	*1793–1816*	*(BO)*
Edward Jenner	*1796–1823*	*(IM)*

Table 3.9: (Cont.)

Name	*fl.*	*Specialization*
Georg Cuvier	*1796–1825*	*(B)*
Étienne Saint Hilaire	*1798–1801*	*(ZO)*
Samuel Hahnemann	*1793–1843*	*(M)*
Jules de Savigny	*1798–1801*	*(ZO)*
Aimé Jacques Bonpland	*1799–1813*	*(M), (BO)*
Augustin de Candolle	*1799–1839*	*(BO)*
Karl Friedrich Burdach	*1800–1802*	*(B), (P)*
Gottfried R. Treviranus	*1802–1837*	*(B), (EB)*
Charles-Francois de Mirbel	*1802–1854*	*(BO)*
Nicolas Théodore de Saussure	*1804–1845*	*(CL), (BO)*
Friedrich Sertürner	*1805–1841*	*(CL)*
Jean-Baptiste de Lamarck	*1801–1822*	*(B), (T)*
Franz Joseph Gall	*1810–1820*	*(A), (M)*
Charles Bell	*1811–1842*	*(A), (M)*
William Prout	*1815–1827*	*(P)*
Renè Theophile Laënnec	*1816–1826*	*(M)*
Francois Magendie	*1816–1822*	*(P)*
Pierre Joseph Pelletier	*1817–1842*	*(CL)*
Joseph-Bienaime Caventou	*1817–1877*	*(CL)*
Alfred Donné	*1829–1878*	*(P), (BM), (M)*
Thomas Addison	*1837–1860*	*(P), (M)*
J.H. Bennett	*1845*	*(P)*
E.F.W. Pflüger	*1868*	*(P), (EB)*
Claudius Aymand		*(S), (M)*

1816 CE **John Farey** (1766–1826, England). Geologist and surveyor. In an article '*On a curious property of vulgar functions*' published in the *Philosophical Magazine* (1816) he constructed a sequence of common fractions (now called the 'Farey Sequence') defined as follows: For a fixed number n, one observes all rationals between 0 and 1 which, when expressed in their lowest terms, have denominator not exceeding n. The sequence is then written in

ascending order of magnitude beginning with the smallest. For example[468]

$$F_5 = \left\{ \frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1} \right\} .$$

It is then found that this sequence has the following 'curious property': each member of the sequence is equal to the rational whose numerator is the sum of the numerators of the fractions on either side, and whose denominator is the sum of the denominators of the fractions on either side. Thus

$$\frac{2}{5} = \frac{1+1}{3+2}; \quad \frac{1}{3} = \frac{1+2}{4+5}; \quad \frac{2}{3} = \frac{3+3}{5+4} .$$

In the final paragraph of his article, Farey wrote:

> *I am not acquainted, whether this curious property of vulgar fractions has been before pointed out?; or whether it may admit of some easy or general demonstration?; which are points on which I should be glad to learn the sentiments of some of your mathematical readers . . .*

One mathematical reader (at least of a French translation) was **Cauchy**, and he gave the necessary proof in his *Exercises de mathématique* which was published in the same year as Farey's article. This might have been the end of the story but there is more to tell.

Farey was not the first to notice the property. **Haros**[469], in 1802, wrote a paper on the approximation of decimal fractions by common fractions. He

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468 The question may arise as to how long is the Farey Sequence? It can be shown that the n^{th} sequence has the length

$$L(n) = 1 + \Phi(1) + \Phi(2) + \cdots + \Phi(n-1) + \Phi(n)$$

where $\Phi(n)$ is the Euler totient function, equal to the number of numbers smaller and relatively prime to n [e.g. $\Phi(1) = 1$, $\Phi(2) = 1$, $\Phi(3) = \Phi(4) = \Phi(6) = 2$, $\Phi(5) = \Phi(8) = \Phi(10) = 4$, $\Phi(7) = \Phi(9) = 6$, $\Phi(100) = 40$]. There is no simple formula for the above sum of totient numbers, but it is known that *asymptotically* for large n the sum is $L' = \left(\frac{3}{\pi^2} n^2\right)$. For example $L'(10) = 30.4$ compared to $L(10) = 30$.

469 Haros, C.: "Tables pour evaluer une fraction ordinaire avec autant de decimales qu'on voudra; et pour trouver la fraction ordinaire la plus simple, et qui approche sensiblement d'une fraction decimale", in Journal de L'Ecole Royale Polytechnique, Tome IV (cahier 11), pp. 364–368, Paris, 1802.

explains how to construct what is in fact the Farey sequence for $n = 99$ and Farey's "curious property" is built into his construction. However, this is certainly not a proof, nor for that matter a general statements of the "curious property".

Farey himself gave no proof, and it is unlikely that he had found one, since he seems to have been at the best an indifferent mathematician. As a geologist he is forgotten. However, the one thing in his life which survives is just his sequence.

The Farey sequence F_n for the first few values of n are

$$F_1 = \left\{\tfrac{0}{1}\,,\ \tfrac{1}{1}\right\}$$

$$F_2 = \left\{\tfrac{0}{1}\,,\ \tfrac{1}{2}\,,\ \tfrac{1}{1}\right\}$$

$$F_3 = \left\{\tfrac{0}{1}\,,\ \tfrac{1}{3}\,,\ \tfrac{1}{2}\,,\ \tfrac{2}{3}\,,\ \tfrac{1}{1}\right\}$$

$$F_4 = \left\{\tfrac{0}{1}\,,\ \tfrac{1}{4}\,,\ \tfrac{1}{3}\,,\ \tfrac{1}{2}\,,\ \tfrac{2}{3}\,,\ \tfrac{3}{4}\,,\ \tfrac{1}{1}\right\}$$

Except for F_1, each F_n has an *odd* number of terms and the middle term is always $\frac{1}{2}$. Let $\frac{p}{q}$, $\frac{p'}{q'}$ and $\frac{p''}{q''}$ be three successive terms in a Farey Sequence. Then

$$qp' - pq' = 1; \qquad \frac{p'}{q'} = \frac{p+p''}{q+q''}\,.$$

These two statements are actually equivalent.

A method of computing a Farey Sequence of order $n+1$ from a sequence of order n is as follows: Let $\frac{a}{c}$ be directly followed by $\frac{b}{d}$ in the n sequence. Then, the fraction $\frac{a+b}{c+d}$, $\quad c+d \geq n+1$ is the *mediant fraction* between $\frac{a}{c}$ and $\frac{b}{d}$ in the $(n+1)^{th}$ sequence. Interpolating the Farey Sequence of order n with such mediant $\frac{a+b}{c+d}$, satisfying $c+d \geq n+1$, we obtain the Farey Sequence of order $n+1$.

1816–1823 CE **Francis Ronalds** (1788–1873, England). Pioneer of telegraphy whose ideas were largely ignored. Experimented with sending messages of words and numbers over few hundred meters by single wires.

On the ground of his estate in Hammersmith, London, he erected an experimental telegraph system. It used a clockwork-driven rotating dials, engraved with letters of the alphabet and numbers, synchronized with each other, at both ends of the circuit. They were connected with an iron wire hung on two strong wooden frames.

Ronalds successfully transmitted and received letters. The British Admiralty was informed of his success but rejected his invention; they felt that

the telegraph was not needed in peacetime and that the existing semaphore system was adequate. In their own words: "*Telegraphs of any kind are wholly unnecessary*" (1823).

1816–1844 CE **Friedrich Wilhelm August Froebel** (1782–1852, Germany). Educator and founder of the *Kindergarten* system.[470] Created (1837) an environment for young children that nurtured self-education, spontaneous play, and intimacy with nature – the kindergarten.

Born in Oberweissbach, Turingia, Froebel was the fifth son of the village Lutheran pastor. His mother died when he was an infant and the boy would spend most of his time in the gardens and forests surrounding his home. Perhaps because of its botanical heritage, the village became the place where Froebel would begin to feel a deep, mystical connection with nature, which later influenced his ideas on education.

Having taken up the education of the sons of his deceased brother, Friedrich marveled at the auto-didactic nature of their play. Each was different, yet each led himself to new understandings and discoveries through individual role playing and adventures. Perhaps the prevailing idea that children should be "seen and not heard" was not correct. Perhaps children carried in them the seed of self-development, which should be encouraged, guided and nurtured by adults. "Kommt, lasst uns unsern Kindern leben" **Come, let us live with our children** became the cornerstone for a new approach to early childhood education. Children are like the tiny flowers; they are varied and need care, but each is beautiful alone and glorious when seen in the community of peers. "My school shall be called Kindergarten – the garden of children," he reasoned.

Froebel labeled his approach to education as "self-activity." This idea allows the children to be led by their own interests and to freely explore them. The teacher's role, therefore, is to be a guide rather than lecturer.

Froebel studied and worked under **Pestalozzi** at Yverdon, Switzerland (1808–10); served during the anti-French campaign (1813–14); assistant in mineralogical museum, Berlin (1814–16). Founded school at Griesheim (1816); moved to Keilhau (1817); founded a kindergarten at Blankenburg, Thuringia (1837); established training courses for kindergarten teachers and introduced kindergartens throughout Germany. Author of *Die Menschenerziehung* (1826), *Mutter- und Koselieder* (1844), etc.

[470] Prior to Froebel's kindergarten, children under the age of 7 did not attend school, since it was held that young children did not have the ability to focus or to develop cognitive emotional skills before this age. Froebel's ideas seem correct enough to us today, yet were radical in his day.

Froebel's ideas were promoted in Germany through the intervention of Baroness Bertha von Marenholtz-Buelow. Through her connections to the more liberal Weimar court and Thuringian nobility, as well as liberal urban educators and intellectuals in Dresden, Leipzig, Frankfurt and Berlin, Madame von Marenholtz-Buelow convinced skeptics and adherents alike that there was worth in his ideas. She took Froebel's philosophy to Switzerland, Holland, Belgium and England. In London, **Charles Dickens** attended her lectures and wrote that he was favorably impressed. Other liberal educators and followers of Froebel transplanted the educational system to the United States, Canada, and even Japan.

The influence of Froebel's system was not to end in the 19^{th} century. Although modified, child-centered kindergartens are now found throughout the world. There is a Froebel College on Roehampton Lane in London and another in Dublin, Ireland, which together with the Pestalozzi Froebelhaus in Berlin to this day further the child-friendly ideas started by the Thuringian educator over 165 years ago.[471]

1817 CE **David Ricardo** (1772–1823, England). Economist. Founder of the *classical school*[472] of economics. One of the leading economists of the 19^{th} century. In "*On the Principles of Political Economy and Taxation*" (1817) developed his theory of rent, profit and wages, and presented clear statements on the quantitative theory of money.

Ricardo was born in London to a religious Jewish family. His father (descended from Portuguese marano's) emigrated from Holland and became a successful member of the London Stock Exchange. At the age of 14, Ricardo

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471 Norman Brosterman, in his recent book *Inventing Kindergarten* (New York: Harry Abrams, 1997), theorizes that Froebel was the impetus for the creations of a number of renowned modern architects and artists, all who had attended Froebelian kindergartens where abstraction of natural forms through geometric shapes was explored. Hence, one finds commonalities in the work of such figures as **Georges Braque**, **Piet Mondrian**, **Paul Klee**, **Wassily Kandinsky**, **Frank Lloyd Wright**, and **Le Corbusier** amongst others.

472 *Classical economy* is based on the assumption that people behave *rationally*; they desire to maximize gain and have freedom of choice (of goods, occupations, etc.). The theory then *deduces* from these premises how people act individually and how their actions collectively determine prices and quantities of goods in the market. Deductive theory based on the assumption of *rational action* to maximize gain is still adhered to by many economists who see no possibility of any other theory.

The basic elements of classical economics were already contained in the writings of **Adam Smith** (1776) and **Robert Malthus** (1798).

entered his father's office, where he showed much aptitude for business. But in 1793 he married physician's daughter of a Quaker family and converted to the Anglican Church, severing his ties with his family and his faith-sakes altogether. He then entered a successful career in the profession to which he had been brought up and at the age of 25 was already rich. Ricardo retired from business (1819), became a land proprietor, and entered parliament (1819–1823).

Ricardo's work was the real first textbook on economics; he defined the conditions that would enable a nation's economy to reach its greatest potential. He believed that the accumulation of capital was the key to rapid economic growth[473], and argued that allowing businessmen to seek high profits would bring about a rapid accumulation of capital. He considered labor to be the most important source of wealth.

Labor, to Ricardo, was very much like any other commodity. When it was plentiful, it was cheap; when it was scarce, it was expensive. As long as there is an ample supply of workers, wages will inevitably sink to the lowest possible level of subsistence, just above starvation. To try to remedy this situation by lowering profits and raising wages would be futile, since it would merely increase the number of worker's children and, by limiting the supply of capital, cut down production. He therefore advocated that wages should be left to the fair and free competition of the market, and should never be controlled by the interference of the legislation.

Ricardo's theories influenced other thinkers; his theory of comparative advantage is still the basis for the modern theory of *international trade*. **Karl Marx** was influenced by Ricardo's *labor theory of value*, which held that the value of a commodity is determined by the amount of labor needed in its production. **John Stuart Mill** used Ricardo's ideas as the basis for a philosophy of social reform. In general, the tenets of Ricardo's theory were enthusiastically adopted by a rising manufacturing class which sought low wages and freedom from governmental interference.

1817 CE **Johann Wolfgang Döbereiner** (1780–1849, Germany). Chemist. Recognized (52 years ahead of Mendeleev) relationships between the properties of the chemical elements and their *atomic weights*, upon which the periodic table of the elements is based; classed closely related elements in group of three (known as *Döbereiner's triads*).

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473 The entire economic activity of Ricardo took place in the shadow of the *Napoleon Wars* and the following periods of Restoration. His theory offered solutions to concrete problems that bugged the economy of Britain at that time. Ricardo's economics enjoyed an immense practical success, culminating in the adoption of *free trade* in England (1846).

He was a professor at Jena from 1810.

1817–1848 CE **Richard Roberts** (1789–1864, England). Engineer and mechanical inventor. Machine-tools pioneer. Made a long series of inventions of machines in the cotton, railway and steam-engine industries. His inventions include a screw-cutting lathe and a planing machine. In 1848 he invented a machine for punching holes in steel plates. Incorporating the Jacquard method, he devised a machine for punching holes of any pitch in bridge plates and boiler plates. He later invented a machine for simultaneously shearing iron and punching both webs of angle iron to any pitch.

Roberts was born at Carreghova, Montomeryshire, Wales, — a son of a shoemaker. He had very little formal education. Starting as a toolmaker at Manchester he became one of the greatest mechanical engineers of the 19^{th} century. But with all his inventive genius, his lack of business acumen led him eventually to die in poverty.

1817–1820 CE **Bernhard (Bernhardus Placidus Johann Nepomuk) Bolzano** (1781–1848, Prague). Czech priest, mathematician and philosopher. Made many important contributions to mathematics in the first half of the 19^{th} century. Freed calculus from the concept of the infinitesimal. Was one of the first to recognize that many "obvious" statements about continuous functions[474] require proof. His observations concerning continuity were published posthumously in 1850.

In 1834 Bolzano devised a function which is continuous throughout an interval but has no derivative at any point on that interval. This work was overlooked for almost 30 years and credit for this function is given to **Weierstrass**, who rediscovered it in 1861. In 1840, Bolzano introduced the concept of denumerable and nondenumerable sets, 32 years ahead of **Cantor** (1872).

In 1865 Karl Weierstrass proved that if S is a bounded infinite set of points, then there exists a point P such that every neighborhood of P contains points of S. This is known as the *Bolzano-Weierstrass Theorem* in recognition of the earlier contribution of Bolzano.

Bolzano was born in Prague. His father, an Italian emigrant, was an art dealer and his mother was the daughter of a hardware tradesman. Bolzano studied philosophy, physics, mathematics and theology at the University of

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[474] *Bolzano's theorem*: Let $f(x)$ be continuous at each point of a closed interval $[a, b]$ and assume that $f(a)$ and $f(b)$ have opposite signs. Then there is at least one point c in the open interval (a, b) such that $f(c) = 0$.

Bolzano's definition of continuity: $f(x)$ is continuous for $x = \xi$ if, given $\delta > 0$, we can choose $\epsilon(\delta) > 0$ so that $|f(x) - f(\xi)| < \delta$ if $0 < |x - \xi| \leq \epsilon(\delta)$.

Prague (1796–1804). He distinguished himself at an early age, and after his ordination to the priesthood was appointed professor of the philosophy of religion at Prague University (1807). In 1816 he was accused of being connected with some of the student's revolutionary societies. He was compelled to resign and was also suspended from his priestly functions, spending the rest of his life in literary work. He was influenced by **Leibniz** and **Kant**.

1818–1827 CE **Augustin Jean Fresnel** (1788–1827, France). Prominent physicist. Derived the equation of wave-surfaces of purely transverse plane waves in anisotropic media (crystals). He discovered that the structure of anisotropic media permits two plane waves with different linear polarizations and distinct velocities of propagation, in any given direction.

In 1818 he was first to give a correct explanation to the phenomena of *diffraction* of light as the mutual interference of secondary waves from an aperture. In a series of calculations he demonstrated the ability of a transverse wave theory of light to account for the details of the observed phenomena of reflection, refraction, interference, polarization, and diffraction patterns that appear as light spreads around objects. His theory led to such fundamental concepts[475] as *Fresnel-diffraction*, *Fresnel-Huygens principle*, *Fresnel-zones*, *Fresnel integrals*, and *Fresnel equations*.

Fresnel, the son of an architect, was born at Broglie (Eure). His early progress in learning was slow, and when 8 years old he was still unable to read. At the age of 13 he entered the École Centrale in Caen, and at 16 he entered the École Polytechnique. Then he went to the École des Ponts et Chaussées and started his career as a civil engineer, engaged in the construction of roads in Southern France. Fresnel spoke openly against Napoleon, and as a consequence he had to resign his government position. This freed him to

[475] *Diffraction* — the deviation of light from rectilinear propagation. It occurs whenever the waves encounter an obstacle (either transparent or opaque), and results in alteration of the amplitude and phase of the incident radiation. The various segments of the wavefront propagate beyond the obstacle and *interfere* to cause the particular energy-density distribution referred to as a *diffraction pattern*. The hypothesis that each point on a wavefront is a source of secondary waves [Huygens' principle] was supplemented by Fresnel with the statement that these secondary waves are mutually coherent, and the waves emitted by them interfere. Thus, while analyzing the propagation of waves we must take into consideration their amplitudes *and* their phases. If we use *plane waves* to begin with and look at the interference pattern *far away* from the obstacle, the phenomenon is called *Fraunhofer diffraction*. If the original wavefront is not plane and if we study the interference pattern just past the obstacle, the phenomenon is called *Fresnel diffraction*.

devote himself entirely to optics. He was elected a member of the Académie des Sciences at Paris in 1823, and in 1825 he became a member of the Royal Society of London. In 1819 he was nominated a commissioner of lighthouses. He died of consumption at Ville-d'Avray, near Paris.

His work in optical science received only scant public recognition during his lifetime, and some of his papers were not printed by the Academie till many years after his death. But, as he wrote to **Young** in 1824: "*In me, that sensibility, or that vanity, which people call love of glory, had been blunted. All the compliments that I have received from* ***Arago****,* ***Laplace*** *and* ***Biot****, never gave me so much pleasure as the discovery of a theoretic truth, or the confirmation of a calculation by experiment*".

The Diffraction of Light – Fresnel vs. Poisson

In the year 1678 ***Christiaan Huygens*** *expressed the intuitive conviction that if each point on the wavefront of light signal were considered to be the source of a 'secondary' spherical disturbance, then the wave front of any later instant could be found by constructing the 'envelope' of these secondary wavelets. With this construction he could explain the wave phenomena of reflection. However, the phenomenon of diffraction, observed by* ***F.M. Grimaldi*** *(1660), could not be accounted for, neither by him nor by any of his contemporaries.*

In 1801, ***Thomas Young*** *discovered the interference of light waves and thus paved the way for* ***Augustin Jean Fresnel*** *(1818) to establish the real cause of diffraction: by making some rather arbitrary assumptions about the effective amplitudes and phases of Huygens' secondary sources, and by allowing the various wavelets to mutually interfere, Fresnel was able to calculate the distribution of light in diffraction patterns with excellent accuracy.*

The transversally of light motion was recognized by Young (1817) and the polarization of light was discovered by ***Malus*** *(1809). Thus, in a span of little more than one decade, all major difficulties in the wave theory of light were resolved. The centuries – old question of the nature of light was answered by stating that light was a transverse motion of waves in the elastic ether.*

In the meantime, the corpuscular theory had been developed further by ***P.S. de Laplace*** *and* ***J.B. Biot****, and under their influence the Paris Academy proposed the subject of diffraction for the prize question of 1818, in the expectation that a treatment of this subject would lead to the crowning triumph of the corpuscular theory.*

To their dismay, and in spite of strong opposition, the prize was awarded to ***A.J. Fresnel****, whose treatment was based on the wave theory. His work was the first of a succession of investigations, which, in the course of a few years, were to discredit the corpuscular theory completely.*

In his memoir Fresnel effected a synthesis of Huygens' envelope construction with Young's principle of interference. This was sufficient to explain diffraction phenomena. Fresnel calculated the diffraction caused by straight edges, small apertures, and screens. [He was advised by ***Francois Jean Arago*** *(1786–1853, France) to read the publications of Grimaldi and Young, but could not follow this advice because he could read neither English nor Latin.]*

Fresnel's theory predicted that in the center of the shadow of a small aperture there should appear a bright spot. This counter-intuitive fact caused ***S.D. Poisson*** *to refute the theory. Fresnel was saved by Arago, who performed the experiment by himself and verified that Fresnel's theory was indeed correct. Poisson acquired his share of fame in the event: the spot became known as Poisson's spot!*

The ideas of Huygens and Fresnel were put on a firmer mathematical foundation by ***Helmholtz*** *(1860),* ***Rayleigh*** *(1871) and* ***Kirchhoff*** *(1882). Helmholtz developed the mathematical theory of the Huygens principle for monochromatic steady-state scalar waves and Kirchhoff generalized the results of Helmholtz for a source with an arbitrary time-dependence. Both employed a mathematical vehicle formulated earlier by* ***Green*** *(1828), but had went unnoticed until 1845, when it was publicized by* ***Kelvin****. Both succeeded in showing that the amplitudes and phases ascribed to the secondary Huygens sources by Fresnel were indeed logical consequences of the wave nature of light.*

Kirchhoff based his mathematical formulation on two assumption about the boundary values of light incident on the surface of an obstacle placed in the way of the propagating light. These assumption were later proved inconsistent by ***Poincaré*** *(1892).*

As a consequence of these criticism, Kirchhoff's formulation of the so-called Huygens-Fresnel principle must be regarded as a first approximation although under most conditions it yield results that agree amazingly well with experiments.

*The advent of the Maxwell electromagnetic theory [***Maxwell** *(1864),* **Hertz** *(1888)], left no doubt that light was not an elastic wave, and in 1881* **Rayleigh** *analyzed the* scattering of light *by small particles based on the electromagnetic theory of light. It is interesting to note that in the beginning Kirchhoff and Rayleigh based their diffraction theories upon the elastic theory of light,* ignoring Maxwell's equations. *However, as far as their approximations were concerned, light could be treated either way.*

The first truly rigorous solution of a diffraction problem was given in 1896 by **Arnold Sommerfeld** *and in 1897 by* **Rayleigh** *and is known as the Rayleigh-Sommerfeld diffraction theory. In it, Sommerfeld treated the 2-dimensional case of a plane-wave incident on an infinitesimally thin, perfectly conducting half-plane.*

In general, rigorous diffraction theory involves solving the Maxwell equations subject to boundary conditions assumed for the aperture screen. The geometrical limitations inherent in the Fresnel-Kirchhoff theory do not exist in the rigorous theory, which therefore renders a complete description of the field in the vicinity of the aperture boundary, as well as at great distances. Unfortunately, the number of cases that can be treated rigorously is very limited, and even the simplest case, (that treated by Sommerfeld) involves complicated mathematical analysis.

However, even so, the Rayleigh-Sommerfeld theory employs certain simplifications and approximations ab initio. Central to these is the treatment of light as a scalar *phenomenon. i.e., only the scalar amplitude of a single transverse component of either the electric or the magnetic field vector is considered, it being assumed that any other component of interest can be treated* independently *in a similar fashion. Such on approach entirely neglects the fact that the various component of the electric and magnetic field vectors are* coupled through Maxwell's equations *and cannot be treated independently.*

Experiments have shown that the scalar theory yields accurate results if two conditions are met[476]*:*

- *The diffracting aperture must be large compared with the wave-length.*
- *The diffracted field must not be observed too close to the aperture.*

[476] There exist important problems for which these conditions are *not* met, e.g., in the theory of high-resolution *diffraction gratings*. There, the vectorial nature of the fields *must* be taken into account if reasonably accurate result are to be obtained.

We next give a succinct account on scalar diffraction theory, encompassing the main results of **Fresnel**, **Helmholtz**, **Kirchhoff** *and* **Rayleigh**. *For the sake of both clarity and brevity we do not follow the historical sequence of evolution, and use modern notation.*

Scalar diffraction theory[477]

Although from a wave point of view, we find it quite reasonable that waves can "bend around corners", it is still to be seen how to handle the phenomenon quantitatively. Historically, most of the development of the theory of diffraction has been in the context of visible light, but it must be recognized that the theory is applicable to any physical process that can be described by the ordinary wave equation in two or three spatial dimensions (e.g. electromagnetic, acoustic, seismic, etc.)

A point-source at P_s emits an outgoing monochromatic spherical wave which interacts with an opaque screen S, having a small aperture ΔS. According to **Huygens**' *construction, every point of the wavefront may be considered as a center and source of a secondary disturbance which gives rise to spherical wavelets, and the wavefront at any later instant may be regarded as an envelope of these wavelets.* **Fresnel** *supplemented the Huygens construction with the postulate that the secondary wavelets mutually interfere. This combination of Huygens' construction with the principle of interference is known as the Huygens-Fresnel principle. Let us invoke it in the above setup: let the wave reach an aperture point Q centered on an infinitesimal area element ds at distance r_s from P_s. There it excites secondary spherical wavelets which reach an observation point P_0 at distance r_0 from Q.*

Now, the wave-amplitude reaching Q from P_s is a solution of the wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi = 0,$$

namely $\Psi = \frac{A}{r}e^{i(kr-\omega t)}$, where A is the amplitude at unit distance from the source, λ is the wavelength, $k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$ is the wave-number, ω is angular frequency and c is the wave velocity. A unit wavelet with zero phase, emerging

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477 To dig deeper, see:

- Born, M. and E. Wolf, *Principles of Optics*, Macmillan Co.: New York, 1964, 808 pp.
- Stone, J.M., *Radiation and Optics*, McGraw-Hill, New York, 1963, 544 pp.

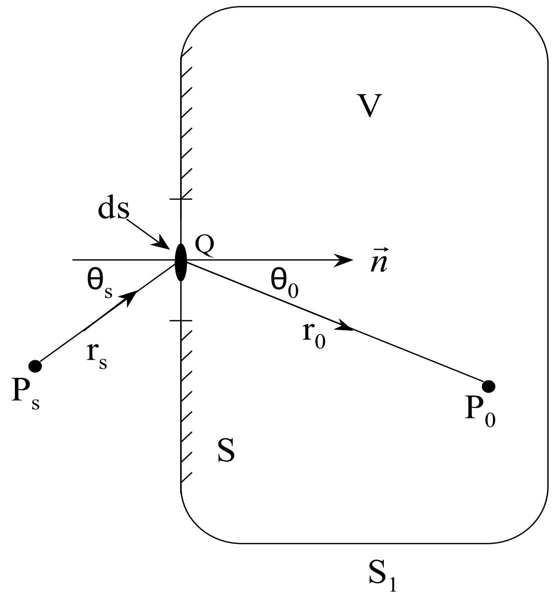

Fig. 3.3: Scalar diffraction by a small aperture

from Q, reaches P_0 with amplitude $\frac{1}{r_0}e^{ikr_0}$. Accordingly, the wavelet that arrives at P_0 from the secondary source element at Q may be written as

$$d\Psi = \Psi_0 f(\theta_s, \theta_0)\left[\frac{A}{r_s}e^{i(kr_s-\omega t)}\right]\left[\frac{1}{r_0}e^{ikr_0}\right],$$

where Ψ_0 is a constant complex source-amplitude and $f(\theta_s, \theta_0)$ represents the dependence of the amplitude of the secondary wavelet on its angular position in the aperture relative to P and P_0 (the angle θ_s is between the normal $\mathbf{n}$ at Q and the vector $\mathbf{r_s}$, and the angle θ_0 is between the same normal at Q and the vector $\mathbf{r_0}$. The entire disturbance at P_0 is found by summing up all contributions across the aperture

$$\Psi(P_0) = \Psi_0 A e^{-i\omega t}\int\limits_{\Delta S} f(\theta_s, \theta_0)\frac{e^{ik(r_s+r_0)}}{r_s r_0}ds. \qquad (1)$$

The results is known as the Fresnel-Kirchhoff formula. **Fresnel** *rendered a convenient contrivance to calculate the above integral, known as the Fresnel zone construction.*

Helmholtz and ***Kirchhoff*** *realized that in order to obtain explicit values for Ψ_0 and $f(\theta_s, \theta_0)$ in (1), the above heuristic derivation must be properly represented as a boundary value problem. To this end they used the 1824 Green's theorem which states that for surface Σ enclosing a volume V*

$$\int_V \left(\Phi\nabla^2\Psi - \Psi\nabla^2\Phi\right) d^3x = \int_\Sigma \left[\Phi\frac{\partial\Psi}{\partial n} - \Psi\frac{\partial\Phi}{\partial n}\right] d\Sigma. \tag{2}$$

Let the field Ψ be assumed to satisfy the homogeneous scalar Helmholtz wave equation

$$(\nabla^2 + k^2)\Psi(\mathbf{x}) = 0$$

and let $\Phi = G$ be the Green's function for the Helmholtz wave equation

$$(\nabla^2 + k^2)G(\mathbf{x}, \mathbf{x}') = -\delta(\mathbf{x} - \mathbf{x}').$$

Eq. (2) then yields for points $\mathbf{x}$ inside V and points $\mathbf{x}'$ on Σ

$$\Psi(\mathbf{x}) = \int_\Sigma \left[\Psi(\mathbf{x}')\mathbf{n}'\cdot\nabla' G(\mathbf{x}, \mathbf{x}') - G(\mathbf{x}, \mathbf{x}')\mathbf{n}'\cdot\nabla'\Psi(\mathbf{x})\right] d\Sigma, \tag{3}$$

where $\mathbf{n}'$ is the inward directed normal to Σ.

Choosing $G(\mathbf{x}, \mathbf{x}') = \frac{e^{ikr_0}}{4\pi r_0}$, $r_0 = |\mathbf{x} - \mathbf{x}'|$, we divide the integral over Σ into two parts, one over the screen and its aperture (ΔS), the other over a surface S_∞ which is made to recede to infinity. It can be shown that the contribution from S_∞ vanishes under a requirement on the behavior of Ψ at infinity, known as the Sommerfeld radiation condition. Then (3) becomes

$$\Psi(\mathbf{x}) = -\frac{1}{4\pi}\int_{\Delta S + Screen} \frac{e^{ikr_0}}{r_0}\mathbf{n}'\cdot\left[\nabla'\Psi + ik\left(1 + \frac{i}{kr_0}\right)\mathbf{e}_{r_0}\Psi\right] ds \tag{4}$$

where $\mathbf{e}_{r_0}$ is a unit vector in the direction of $\mathbf{r}_0$. In order to apply (4), it is necessary to know the values of Ψ and $\frac{\partial\Psi}{\partial n}$ on ΔS and the screen. But these values are not known, unless the problem has been solved. Kirchhoff's approach was to approximate the values of Ψ and $\frac{\partial\Psi}{\partial n}$ over the aperture and the screen by assuming:

- *Ψ and $\frac{\partial\Psi}{\partial n}$ vanish everywhere on the screen.*

- *The values of* Ψ *and* $\frac{\partial \Psi}{\partial n}$ *on the aperture are equal to the values of the incident wave in the absence of the obstacle, ergo* $\Psi \simeq A\frac{e^{ikr_s}}{r_s}\, e^{-i\omega t}$.
- $kr_0 >> 1$ *('far-field' approximation).*

With these assumptions, the Sommerfeld-Kirchhoff integral becomes

$$\Psi(P_0, t) = -\frac{iA}{\lambda}e^{-i\omega t}\int_{\Delta S}\left[\frac{\cos\theta_s + \cos\theta_0}{2}\right]\frac{e^{ik(r_s+r_0)}}{r_s r_0}ds, \tag{5}$$

where we have used the ancillary relations $(\mathbf{n}' \cdot \mathbf{e}_{r_s}) = \cos\theta_s$ *and* $(\mathbf{n}' \cdot \mathbf{e}_{r_0}) = \cos\theta_0$*. Upon comparison with the Fresnel-Kirchhoff formula we find*

$$\Psi_0 = -\frac{i}{\lambda}, \qquad f(\theta_s, \theta_0) = \frac{1}{2}(\cos\theta_s + \cos\theta_0).$$

The factor $(-i)$ *signifies that Huygens wavelet is radiated with a phase advance of* $90\,^\circ$*, a feature not anticipated in the phenomenological treatment of Huygens and Fresnel. The factor* $f(\theta_s, \theta_0)$ *has a maximum value of unity in the forward direction and goes to zero for the portion of the Huygens wavelet returning toward the source.*

It can be shown that there are serious mathematical inconsistencies in the first two assumptions of Kirchhoff. **Rayleigh** *showed that these can be removed by an alternative choice of the Green's function. Indeed, choosing*

$$G_1(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi}\left[\frac{e^{ikR}}{R} - \frac{e^{ikR'}}{R'}\right], \tag{6}$$

where $R' = |\mathbf{x} - \mathbf{x}''|$, $\mathbf{x}''$ *being the mirror-image of* $\mathbf{x}'$ *[i.e.*

$$R^2 = (x - x')^2 + (y - y')^2 + (z - z')^2;$$
$$R'^2 = (x + x')^2 + (y + y')^2 + (z + z')^2.$$

With this choice G_1 *will vanish on* Σ *and consequently*

$$\Psi(\mathbf{x}) = \int_{\Sigma}\Psi(\mathbf{x}')\mathbf{n}' \cdot \nabla' G_1(\mathbf{x}, \mathbf{x}')d\Sigma,$$

leading to the Rayleigh integral

$$\Psi(P_0, t) = -\frac{iA}{\lambda}e^{-i\omega t}\int_{\Delta S}\cos\theta_0\frac{e^{ik(r_s+r_0)}}{r_0 r_s}. \tag{7}$$

Similarly, the choice

$$G_2(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi}\left[\frac{e^{ikR}}{R} + \frac{e^{ikR'}}{R}\right]$$

will end with

$$\Psi(P_0, t) = -\frac{iA}{\lambda} e^{-i\omega t} \int_{\Delta S} \cos\theta_s \frac{e^{ik(r_s+r_0)}}{r_0 r_s}. \tag{8}$$

Note that in (7), Ψ *is approximated on* ΔS*, while in (8) it is* $\frac{\partial\Psi}{\partial n}$ *which is approximated on* ΔS*. Hence the difference. But how can we have three different approximation to the Helmholtz-Kirchhoff integral? The answer is simple: if the source point* P_s *and the observation point* P_0 *are far from the screen in terms of aperture dimensions, the function* $f(\theta_s, \theta_0)$ *can be treated as constant. For normal incidence all values of* f *are approximately unity.*

Rayleigh diffraction formulas (1897)

In the analysis of diffraction by an aperture in a plane screen one must determine the solution of Helmholtz wave equation $(\nabla^2 + k^2)\Psi(\mathbf{r}) = 0$ *for the wavefield* $\Psi(x, y, z)$ *valid throughout the Half-space* $z > 0$ *from (approximate) knowledge of the boundary values* $\Psi(x, y, z)$ *on the half-space boundary* $z = 0$*.*

An outgoing monochromatic wave in the half-space has the Fourier-integral representation

$$\Psi(x, y, z; t) = e^{-i\omega t} \iint_{-\infty}^{\infty} a(p, q) e^{ik(px+qy+mz)} dp dq \tag{9}$$

where the support of $a(p, q)$ *is assumed contained in the disk* $p^2 + q^2 \leq 1$*,* $m^2 = 1 - p^2 - q^2$*, and*

$$a(p, q) = \left(\frac{k}{2\pi}\right)^2 \iint_{-\infty}^{\infty} \Psi(x', y', 0; 0) e^{-ik(px'+qy')} dx' dy'. \tag{10}$$

Substituting (10) into (9) and interchanging the orders of integrations, one obtains the expression of the field in terms of the boundary values

$$\Psi(x, y, z; t) = e^{-i\omega t} \iint_{-\infty}^{\infty} \Psi(x', y', 0; 0) G(x - x', y - y', z) dx' dy' \tag{11}$$

with

$$G(x-x',y-y',z) = \left(\frac{k}{2\pi}\right)^2 \iint\limits_{-\infty}^{\infty} e^{ik[p(x-x')+q(y-y')+mz]} dpdq$$

where $p^2+q^2+m^2=1$.

We next use Weyl's integral (Weyl, 1919) with $\mathbf{r} = (x,y,z>0)$, $\mathbf{r} = (x_0,y_0,0)$ *which represents an outgoing spherical wave as a superposition of plane waves*

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} = \frac{ik}{2\pi} \iint\limits_{-\infty}^{\infty} \frac{1}{m} e^{ik[p(x-x')+q(y-y')+mz]} dpdq, \tag{12}$$

from which we deduce that

$$G(x-x',y-y',z) = -\frac{1}{2\pi}\frac{\partial}{\partial z}\left[\frac{e^{ikR}}{R}\right], \qquad R^2 = (x-x')^2+(y-y')^2+z^2. \tag{13}$$

Inserting this in (11), we arrive at the first Rayleigh formula

$$\Psi(x,y,z;t) = -\frac{e^{-i\omega t}}{2\pi} \iint\limits_{-\infty}^{\infty} \Psi(x',y';0)\frac{\partial}{\partial z}\left[\frac{e^{ikR}}{R}\right] dx'dy', \tag{14}$$

since

$$\frac{\partial}{\partial z}\left[\frac{e^{ikR}}{R}\right] \simeq \frac{ikz}{R}\frac{e^{ikR}}{R}$$

for $kR \to \infty$ *(the 'far field').*

Then since $\frac{z}{R} = \cos\theta_0$, *one can write*

$$\Psi(x,y,z;t) = -\frac{i}{\lambda}e^{-i\omega t} \iint\limits_{-\infty}^{\infty} \cos\theta_0 \frac{e^{ikR}}{R}\Psi(x',y',0)dx'dy', \tag{15}$$

which a more general form of the Rayleigh integral in (7).

Summary

The Fresnel-Kirchhoff diffraction theory is intrinsically a high-frequency approximation; it gives incorrect results when the aperture dimensions are much smaller than a wavelength. Furthermore, even if such dimensions are

large and one uses theory to predict fields at only those distances which are large compared with a wavelength, the predictions may be in substantial disagreement at large angular deviations from the direction **n**.

Nevertheless, the theory is satisfactory for explaining small-angle, high-frequency diffraction phenomena and has an advantage in simplicity compared with rigorous theories of diffraction.

It is extensively used in optics; applications to acoustics are limited (except for ultrasonics) because many of the diffraction phenomena of interest either involve dimensions small compared with a wavelength or require on understanding of diffraction through large angles.

The Fresnel-zone construction

According to ***Huygens****' construction, every point of a wavefront may be considered as a center of a secondary disturbance which gives rise to spherical wavelets and the wavefront at any later instant may be regarded as the envelope of these wavelets.* ***Fresnel*** *was able to account for diffraction by supplementing Huygens' construction with the postulate that the secondary wavelets mutually interfere. This combination of Huygens' construction with the principle of interference is called the Huygens-Fresnel principle.*

Before applying it to the study of diffraction effects one is tempted to verify that, with certain simple additional assumptions, the principle correctly describes the propagation of light in free space.

To see this we consider the instantaneous position of a spherical monochromatic wavefront of radius r_s *which proceeds from a point source* P_s *to another point* P_0 *where the light disturbance is to be determined.*

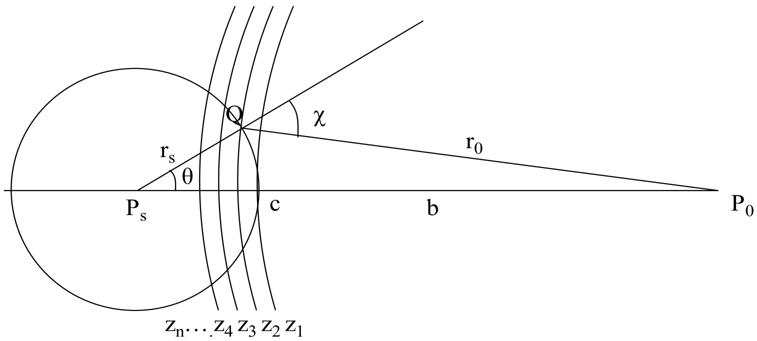

Fig. 3.4: Fresnel's zone construction

Let the time factor be $e^{-i\omega t}$ (omitted) and let the disturbance at Q on the wavefront be Ae^{ikr_s}/r_s, where A is the amplitude at unit distance from the source. In accordance with the Huygens-Fresnel principle we regard each element of the wavefront as the center of a secondary disturbance which is propagated in the form of spherical wavelets, and obtain for the contribution $d\Psi(P_0)$ due to the element dS at Q the expression

$$d\Psi = k(\chi)A\frac{e^{ikr_s}}{r_s}\frac{e^{ikr_0}}{r_0}dS,$$

where $r_0 = QP_0$ and $k(\chi)$ is the inclination factor, describing the variation with direction of the amplitude of the secondary waves, χ being the angle between the normal at Q and the direction QP_0.

We assume that k is maximum for $\chi = 0$ and zero for $\chi = \pi/2$. Hence the total disturbance at P_0 is given by

$$\Psi(P_0) = A\frac{e^{ikr_s}}{r_s}\iint_{S'}\frac{e^{ikr_0}}{r_0}k(\chi)dS \tag{16}$$

where S' is that part of S which is not obstructed by obstacles situated between P_s and P_0.

To evaluate (16) we draw spheres from P_0 of radii

$$r_j = b + j\left(\frac{\lambda}{2}\right) \quad j = 0, 1, 2, ..., n, \qquad r_n^2 = (b + r_s)^2 - r_s^2$$

(P_0Q tangent to the primary wavefront at $r = r_n$).

These spheres divide S into a number of zones $z_1,\ z_2,\ z_3, ..., z_n$. Assuming $r_s >> \lambda, \quad r_0 >> \lambda$, then k may be assumed to have the same value K_j, for points on one and the same zone.

Now by the law of cosines,

$$r_0^2 = r_s^2 + (r_s + b)^2 - 2r_s(r_s + b)\cos\theta$$

so that

$$r_0 dr_0 = r_s(r_s + b)\sin\theta d\theta \qquad b, r_s \text{ fixed}$$

and therefore

$$dS = r_s^2 \sin\theta d\theta d\phi = \frac{r_s}{r_s + b} r_0 dr_0 d\phi \qquad \phi = \text{azimuth angle}$$

Hence, the contribution of the j^{th} zone (spherical ring) to $\Psi(P_0)$ is:

$$\begin{aligned} \Psi_j(P_0) &= 2\pi \frac{Ae^{ikr_s}}{r_s + b} K_j \int\limits_{b+(j-1)\frac{\lambda}{2}}^{b+j\frac{\lambda}{2}} e^{ikr_0} dr_0 \\ &= 2i\lambda(-1)^{j+1} K_j \frac{Ae^{ik(r_s+b)}}{r_s + b} \end{aligned}$$

The total wave at P_0 is

$$\Psi(P_0) = 2i\lambda A \frac{e^{ik(r_s+b)}}{r_s + b} \sum_{j=1}^{n} (-1)^{j+1} K_j. \tag{17}$$

The contributions of the successive zones are alternately positive and negative

$$\Sigma = K_1 - K_2 + K_3 - ... + (-1)^{n+1} K_n.$$

It can be shown that (approximately)

$$\Sigma \approx \frac{K_1}{2} + \frac{K_n}{2} \qquad (n \text{ odd})$$

$$\Sigma \approx \frac{K_1}{2} - \frac{K_n}{2} \qquad (n \text{ even}).$$

Since $K_n = 0 \quad [\chi = \frac{\pi}{2}]$ *for the last zone,* $\Sigma \approx \frac{K_1}{2}$ *and we have approximately:*

$$\Psi(P_0) = i\lambda K_1 A \frac{e^{ik(r_s+b)}}{r_s+b} = \frac{1}{2}\Psi_1(P_0), \tag{18}$$

showing that the total disturbance at P_0 *is approximately equal to half of the disturbance due to the first zone. This last result is in agreement with the field at* P_0 *obtained by simply assuming a spherical wave beginning at* P_s *and ending at* P_0. *i.e*

$$\Psi(P_0) \approx A\frac{e^{ik(r_s+b)}}{r_s+b}$$

if $i\lambda K_1 = 1$, *namely*

$$K_1 = -\frac{i}{\lambda} = \frac{1}{\lambda}e^{-\frac{\pi i}{2}}.$$

The factor $e^{-\frac{\pi i}{2}}$ *may be accounted for by assuming that the secondary waves have an initial phase retardation of a quarter of a period, relative to the primary wave. There is also the amplitude factor of* $\frac{1}{\lambda}$.

By means of the above method, Fresnel was able to "calibrate" his approximate integrating scheme and thus test the validity of his principle. To enable additional experimental tests *he introduced a* plane screen *with circular opening, perpendicular to the optical axis* P_sP_0, *with its center on this line. The total disturbance at* P_0 *must now be regarded as due to wavelets from only those zones that are not obstructed by the screen. Four experiments could be performed:*

- *The screen covers all but half the first zone; according to Eq. (18) with* $j = 1$

  $$\Psi(P_0) = \frac{1}{2}\Psi_1(P_0) = i\lambda K_1 \frac{Ae^{ik(r_s+b)}}{r_s+b} = \frac{Ae^{ik(r_s+b)}}{r_s+b}.$$

  This is the same disturbance as would be obtained if no *screen were present.*

- *All zones are covered except the first one; then (17)–(18) yields*

  $$\Psi(P_0) = 2i\lambda K_1 \frac{Ae^{ik(r_s+b)}}{r_s+b} = 2\frac{Ae^{ik(r_s+b)}}{r_s+b}.$$

  The intensity $I(P_0) = |\Psi(P_0)|^2$ *is four times larger than if the screen were absent. [Conservation of energy clearly demands that there be other points where the intensity has decreased.]*

- *Only the first two zones are left open; since K_1 and K_2 are nearly equal (which can be shown), there will be almost complete darkness. In general, when the size of the opening is varied, there is a periodic fluctuation in intensity at P_0. Similar results are obtained when the size of the opening and the source's position are fixed but P_0 gradually moved along the axis.*

- *When only the first zone is obstructed by a small circular disk placed at right angles to P_sP_0, the field is*

$$\Psi(P_0) = 2i\lambda \frac{Ae^{ik(r_s+b)}}{r_s+b} \underbrace{[-K_2 + K_3 - K_4 + \cdots]}_{\approx -\frac{K_2}{2}},$$

but since $K_1 \approx K_2$ it follows that there is light in the geometrical shadow of the disk! and (even more remarkably) the intensity there is the same as if no disk were present.

This prediction[478] of Fresnel's theory made a strong impression on his contemporaries, and was one of the decisive factors which temporarily ended the long battle between the corpuscular and the wave theories of light in favor of the latter.

We next integrate the expression

$$dS = \frac{r_s}{r_s+b} r_0 dr_0 d\phi$$

over the j-th zone to get the area of that zone:

$$A_j = \frac{2\pi r_s}{r_s+b} \int_{b+(j-1)\frac{\lambda}{2}}^{b+j\frac{\lambda}{2}} r_0 dr_0 = \frac{\lambda\pi r_s}{r_s+b}\left[b + \frac{(2j-1)\lambda}{4}\right]$$
$$= \frac{\lambda\pi r_s b}{r_s+b}\left[1 + \frac{(2j-1)}{4}\frac{\lambda}{b}\right].$$

478 That a bright spot should appear at the center of the shadow of a small disk was deduced from Fresnel's theory by **S.D. Poisson** (1818). Poisson, who was a member of the committee of the French Academy which reviewed Fresnel's prize memoir, considered this conclusion contrary to experiment and so rejected Fresnel's theory. However, **Arago**, another member of the committee, performed the experiment and found that the surprising prediction was correct. A similar observation was made in 1723 by **Jacques Philippe Maraldi** (1665–1729), a nephew of G.D. Cassini, but was forgotten.

It is also found that the mean distance from the field point P_0 to the j-th zone is $r_j = b + \left(\frac{2j-1}{4}\right)\lambda$ *so that A_j/r_j is constant.*

When $\lambda << b$ and for small j values, we have approximately

$$A \approx \frac{r_s}{r_s + b}\pi b\lambda \qquad \text{(independent of } j\text{)}$$

If the aperture has a radius R, a good approximation for the number of zones within it is thus simply

$$\frac{\pi R^2}{A} = \frac{(r_s + b)R^2}{r_s b\lambda}.$$

If the point source has been moved so far from the aperture (diffraction screen) that the incoming wave can be regarded as a plane wave ($r_s \to \infty$), two facts emerge:

- $A \simeq \pi b\lambda$ *independent of j*
- *Since* $r_j = b + j\frac{\lambda}{2}$, *we have* $\pi R_j^2 \approx \pi(b + j\frac{\lambda}{2})^2 - \pi b^2$ *or*

  $$R_j^2 \approx jb\lambda + j^2\frac{\lambda^2}{4} \simeq jb\lambda \qquad \therefore \qquad R_j \cong \sqrt{jb\lambda}$$

  as long as j is not extremely large. So the radii are proportional to the square roots of the integers.

It should be borne in mind that, the sensor at P_0 merely records the light amplitude (or intensity), the zones having no reality. It is just a convenient contrivance for the evaluation of the field.

1818–1844 CE **Arthur Schopenhauer** (1788–1860, Germany). Philosopher. One of the first Western thinkers to concern himself with the dilemmas and tragedies of real modern life, not just with abstract philosophical problems. Espoused pessimism that saw life as being essentially evil and futile. Under influence of Eastern thought, he saw hope in aesthetics, sympathy for others and ascetic living. His ideas influenced the fields of music, psychology, literature and physics (through **Einstein and Schrödinger**). Accorded the arts a more important place in the overall scheme of things than any other major philosopher.

While still in his twenties he wrote his masterpiece: "*The World as Will and Representation*" (1818), and then "*On the Will as Nature*" (1836), showing that the ongoing progress of science was supporting the arguments of his main work. Finally he produced two books on ethics: "*The Freedom of the Will*" (1841), and "*The Foundations of Morality*" (1841).

Schopenhauer was born near Danzig, the son of a rich Hanseatic merchant of Dutch heritage. As his parents had strong feeling against any kind of nationalism, the name Arthur was selected especially because it was the same in English, German and French.

After the city fell to Prussia during the second partition of Poland (1793), the family fled to Hamburg. In 1805 Schopenhauer's father died (possibly by suicide) and his mother, Johanna, moved to Weimar, where she kept a literary salon at which she entertained such figures as Goethe and the Brothers Grimm.[479]

Schopenhauer studied at the University of Göttingen and was awarded a PhD from the University of Jena. As a youth, he traveled widely, becoming fluent in English and French, so that his prose style acquired a lightness and clarity quite unlikely the murky philosophic German of his times. His first education was that of a man of the world, only later did he obtain the usual academic credentials.

He became a friend of Goethe, and in 1816 wrote a small book on color theory[480] inspired by the ideas of the older man.

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479 She herself achieved fame as a romantic novelist, and one of her poems was set to music by **Schubert**. When **Goethe** told her that he thought her son was destined for great things, Johanna objected: she had never heard there could be *two* geniuses in a single family.

480 Schopenhauer's philosophy of science has its embarrassing aspects: Schopenhauer did not understand the new physics of light and electricity that had been developed by **Thomas Young** (1773–1829) and **Michael Faraday** (1791–1867). He disparaged the wave theory of light, which Young had definitively established, as a "crude materialism", and "mechanical, Democritean, ponderous, and truly clumsy." Unfortunately, Schopenhauer does not seem to have understood the evidence for Young's discoveries about light, or even for Newton's — he still clung to Goethe's clever but clueless theory of colors. Nevertheless, Schopenhauer would have been happy to learn how his beloved *qualitates occultae* would return in force with quantum mechanics: Things like strangeness, charm, baryon number, lepton number, etc., are exactly the kinds of irreducible types he demanded.

Because of a large inheritance from his father, Schopenhauer was able to retire early, and, as a private scholar, was able to devote his life to the study of philosophy.

Schopenhauer was influenced by **Friedrich Schelling**, regarded himself as the true spiritual descendant of **Kant**, and despised **Hegel**. He thought that Hegel's belief in a happy ending to human history was the ramblings of a "stupid and clumsy charlatan." He maintained that Hegel, and other university philosophers had perverted the Kantian gospels.

The first edition of his major work, *The World as Will and Representation* appeared in 1818. He followed Kant in the belief that the mind is not merely a passive recipient of sense impressions, but takes an active role in fitting the phenomena into the categories of space and time, the principle of causality being the necessary method for creating this representation of the world. Kant taught that the real world, the *noumenon*, the thing-in-itself [*Ding an sich*], can never be accessible to human thought or experience. Schopenhauer did not agree: he believed that the thing-in-itself can be identified as *will*. Every person experiences himself in two different ways, as an object like any other, and through self-consciousness as a *will*. The will is neither a phenomenon nor a representation, it is a directly experienced reality.

What is true of the microcosm of man, is also true of the world: its thing-in-itself is will. On the foundation of this primary intuition, which of course can be neither proved nor disproved, Schopenhauer constructed a philosophy that has continued to fascinate and influence thinkers of all kinds: **Nietzsche**, **Tolstoy**, **Chekhov**, **Zola**, **Maupassant**, **Bernard Shaw**, **R.M. Rilke**, **Thomas Mann**, **Freud**, **Klimt**, and **Schrödinger**, to mention a few. The reason for this wide range of influence must be sought in a combination of an unparalleled depth of insight into the human condition with a literary style of exceptional quality.

This use of the term "will" has led to much misunderstanding, because people find it difficult to think of a will that has no personality, no kind of mind or intelligence, and no aims or goals: but this is what Schopenhauer says quite clearly that he means. He would have regarded the discovery by physics in the 20^{th} century that the entire contents of the empirical world, including all material objects, are reducible to energy and fields of force, operating in a space-time framework, as fitting in perfectly with his philosophy.

Schopenhauer agreed with Kant that human beings can only ever live in the phenomenal world. But for Schopenhauer, the phenomenal world is an illusory one, always controlled by the *Will*. The Will directs every living being, including humans.

Human beings like to believe that their own individual lives have some kind of higher meaning, but there is no more to their lives than the urge

to satisfy their desires. Different individual wills then inevitably come into conflict, and this is what produces human suffering.

According to Schopenhauer, the *Will to Live* (Wille zum Leben), is defined as an inherent drive within human beings, and indeed all creatures, to stay alive and to *reproduce*. He refused to conceive lore as either trifling or accidental, but rather understood it to be an immensely powerful force lying unseen within man's *psyche* and dramatically shaping the world. He saw '*falling in love*' as the process whereby the noumenon, Kant's 'thing in itself,' enters the world of phenomena.

He had more to say about sexual love than any previous philosopher, since 'the sexual relation in the world of mankind ... is really the invisible central point of all action and conduct ... The ultimate aim of all love affairs, whether played in sock or in buskin, is actually more important than all other aims in man's life; and therefore it is quite worthy of the profound seriousness with which everyone pursues it. What is decided by it is nothing less than the *composition of the next generation*.' The new individual who will arise from the love affair is like a new Platonic idea, and 'just as all the Ideas strive to enter into the phenomenal with the greatest vehemence, avidly seizing for this purpose the matter which the law of causality divides among them all, so does this particular Idea of a human individuality strive with the greatest eagerness and vehemence for its realization in the phenomenon. This eagerness and vehemence is identical with the passion for each other of the two future parents.'

Schopenhauer's view of the human condition is that of a world of violence and injustice ending in death. He took the blackest view of our existence that is possible to take and still remain sane.

He believed, however, that a momentary release from our imprisonment in the dark dungeon of this world could be achieved through the *arts*: painting, sculpture, poetry, drama, and above all – music.

Through these media we are in touch with something *outside* the empirical realm. We are taken out of time and space and also out of ourselves (body included). In particular, he regarded music as a sort of super-art transcending all the others in metaphysical significance.

Worldview XIX: Arthur Schopenhauer

* * *

All truth passes through three stages. First, it is ridiculed. Second, it is violently opposed. Third, it is accepted as being self-evident.

* * *

Each day is a little life; every waking and rising a little birth; every fresh morning a little youth; every going to rest and sleep a little death.

* * *

After your death you will be what you were before your birth.

* * *

Every possession and every happiness is but lent by chance for an uncertain time, and may therefore be demanded back the next hour.

* * *

Human life must be some form of mistake.

* * *

If a man sets out to hate all the miserable creatures he meets, he will not have much energy left for anything else; whereas he can despise them, one and all, with the greatest ease.

* * *

The discovery of truth is prevented more effectively, not by the false appearance things present and which mislead into error, not directly by weakness of the reasoning powers, but by preconceived opinion, by prejudice.

* * *

The first forty years of life give us the text; the next thirty supply the commentary on it.

* * *

The fundament upon which all our knowledge and learning rests is the inexplicable.

* * *

Physics is unable to stand on its own feet, but needs a metaphysics on which to support itself.

* * *

The closing years of life are like the end of a masquerade party, when the masks are dropped.

* * *

A man must have grown old and lived long in order to see how short life is.

* * *

Fate gives us the hand, and we play the cards.

* * *

Wealth is like sea-water; the more we drink, the thirstier we become; and the same is true of fame.

* * *

The alchemists in their search for gold discovered many other things of greater value.

* * *

A man can do what he wants, but not want what he wants.

* * *

The wise have always said the same things, and fools, who are the majority have always done just the opposite.

* * *

Money is human happiness in the abstract; he, then, who is no longer capable of enjoying human happiness in the concrete devotes himself utterly to money.

* * *

The present is the only reality and the only certainty.

* * *

Martyrdom is the only way a man can become famous without ability.

* * *

Sleep is the interest we have to pay on the capital which is called in at death; and the higher the rate of interest and the more regularly it is paid, the further the date of redemption is postponed.

* * *

The deep pain that is felt at the death of every friendly soul arise from the feeling that there is in every individual something which is inexpressible, peculiar to him alone, and is, therefore, absolutely and irretrievably lost.

* * *

The highest, most varied and lasting pleasures are those of the mind.

* * *

Life swings like a pendulum backward and forward between pain and boredom.

* * *

Natural ability can almost compensate for the want of every kind of cultivation; but no cultivation of the mind can make up for the want of natural ability.

* * *

To go to the theater is like making one's toilet with a mirror.

* * *

Will power is to the mind like a strong blind man who carries on his shoulders a lame man who can see.

* * *

The difficulty is to try and teach the multitude that something can be true and untrue at the same time.

* * *

The more unintelligent a man is, the less mysterious existence seems to him.

* * *

Without books the development of civilization would have been impossible. They are the engines of change, windows on the world, "Lighthouses" as the poet said "erected in the sea of time." They are companions, teachers, magicians, bankers of the treasures of the mind, Books are humanity in print.

* * *

Books are like a mirror. If an ass looks in, you can't expect an angel to look out.

* * *

Authors may be divided into falling stars, planets, and fixed stars: the first have a momentary effect; the second have a much longer duration; but the third are unchangeable, possess their own light, and work for all time.

* * *

Any book, which is at all important, should be reread immediately.

* * *

Almost all of our sorrows spring out of our relations with other people.

* * *

With people of limited ability modesty is merely honesty. But with those who possess great talent it is hypocrisy.

* * *

Every man takes the limits of his own field of vision for the limits of the world.

* * *

Boredom is just the reverse side of fascination: both depend on being outside rather than inside a situation, and one leads to the other.

* * *

Do not shorten the morning by getting up late; look upon it as the quintessence of life, as to a certain extent sacred.

* * *

Journalists are like dogs, whenever anything moves they begin to bark.

* * *

Exaggeration of every kind is as essential to journalism as it is to dramatic art, for the object of journalism is to make events go as far as possible.

* * *

To find out your real opinion of someone, judge the impression you have when you first see a letter from them.

* * *

Religion is the masterpiece of the art of animal training, for it trains people as to how they shall think.

* * *

Buying books would be a good thing if one could also buy the time to read them in: but as a rule the purchase of books is mistaken for the appropriation of their contents.

* * *

Great men are like eagles, and build their nest on some lofty solitude.

* * *

A man's face as a rule says more, and more interesting things, than his mouth, for it is a compendium of everything his mouth will ever say, in that it is the monogram of all this man's thoughts and aspirations.

* * *

Change alone is eternal, perpetual, immortal.

* * *

A word too much always defeats its purpose.

* * *

We forfeit three-fourths of ourselves to be like other people.

* * *

Honor has not to be won; it must only not be lost.

* * *

It is a clear gain to sacrifice pleasure in order to avoid pain.

* * *

Ignorance is degrading only when found in company with great riches.

* * *

In action a great heart is the chief qualification. In work, a great head.

* * *

Compassion is the basis of morality.

*
* *

So long as we are given up to the throng of desires with its constant hopes and fears, we never obtain lasting happiness or peace.

* *
*

Actions are transitory while works remain: The most noble action *still has only a temporary effect; the work of genius, on the other hand, lives and has beneficial and uplifting effect through all times.*

* *
*

1819–1851 CE **Leopold (Yom-Tov Lipmann) Zunz** (1794–1886, Germany). The first modern Jewish thinker. Devoted his long life to the refurbishment of the old-style Jewish learning and its presentation in a modern scientific spirit.

Zunz and his friends of the immediate post-Napoleonic period, called their work the *Wissenschaft des Judentums* (The science of Judaism). In 1819, immediately after the *Hep-Hep riots*, they realized the fragility of the acceptance of the Jews even in modern-minded Germany, and set up the *Society for the promotion of Jewish Culture and Science*[481]. Its object was to investigate the nature of Judaism by modern scientific methods and demonstrate the universal value of Jewish knowledge.

Zunz then embarked on a grand project: an encyclopedia of Jewish intellectual history: He translated an enormous amount of Jewish literature and elaborated a philosophy of Jewish history. He visited the great libraries in search of material, in an overall effort to emancipate Jewish writing from the theologians and rise to the historical viewpoint. In practice it involved accepting that the history of the Jews was merely an element of world history. In that Zunz was influenced by Hegelian ideas of progression from lower to higher forms, and inevitably applied this dialectic to Judaism.

There had been only one period in Jewish history, he said, when their inner spirit and their external form had matched, and they had become the center of world history, and that was under the ancient commonwealth. Thereafter they were delivered into the hands of other nations. Their internal history became a history of ideas, their external history a long tale of suffering. A day will come, he believed, that the distilled legacy of Jewish ideas became part of the common property of enlightened mankind.

While Zunz' interpretation of Jewish history and learning as a contribution to the world stock made some impression on gentile society, it involved almost by definition a severance from a great part of Judaism.[482]

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481 The society soon dissolved (1824), because most of its members and officers converted to Christianity in the most opportunistic manner. Nonetheless, under the guidance of Zunz it had successfully initiated the scientific method in the study of Jewish religion, history and culture. Among the initiators was **Heinrich Heine**, then still a fledgling poet.

482 The Jewish orthodoxy in his time, rejected this dualism. To them it was not Judaism: there are *no* two kinds of knowledge, sacred and secular; there was only *one*. Moreover, there was only one legitimate purpose in acquiring it: to discover the exact will of God, in order to obey it. Hence the 'Science of Judaism', as a dislocated academic discipline, was contrary to Jewish belief. They believed that, without Israel, there would have been no world and therefore no history.

Zunz was born in Detmold. Studied at Berlin University (1815–1821) and received his Ph.D. at the University at Halle. Eventually he became the principal of a teachers' seminary established by the Jews in Berlin (1840). His foremost work is "*The Religious Discourses of the Jews*". It proved that Judaism never stood still, but underwent changes in accordance with requirements of time and place; it changed itself even when there was no reformers advocating change.

1819 CE **William George Horner** (1786–1837, England). Mathematician. Rediscovered the ancient Chinese computational scheme for the evaluation of a polynomial and hence solving for the real roots of algebraic equations.

Horner was born in Bristol. He began his humble career as an assistant schoolmaster at Kingston (1802), worked his way to become a Headmaster there (1806), and founded his own school at Bath (1809). Although not a man of great ability as a mathematician, he succeeded in making for himself a name that is well known to students of Algebra.

While a schoolteacher at Bath, he came independently upon a method known to Chu Shih-Chieh (ca 1300 CE) for the approximation of the real roots of a numerical algebraic equation. This method, which has been practically forgotten in China, was made known in a paper read by Horner before the London Royal Society (1819), and since that time has become familiar in all parts of the English speaking world.

It was recently discovered that **Ruffini** (1765–1822) had described a similar method a few years earlier (1813).

An Age of Transition[483]

"Greek civilization depended essentially on slave-labor but could not progress without the harnessing of natural forces to labor-saving machines. Only the free man, not a slave, has a disposition and interest to improve implements or to invent them. Accordingly, in the devising of a complicated machine, the workmen employed upon it are generally co-inventors. The eccentric *and the* governor*, most important part of the steam-engine, were devised by laborers. The improvement of established industrial methods by slaves, themselves industrial machines, is out of question.*

Justus von Liebig (1803–1873)

"Prior to 1890*, the steam engine did more for science than science for the steam engine".*

L.J. Henderson

The industrial age started in England at about 1740*, continued to France in ca* 1810*, and arrived in Germany and the U.S.A. in about* 1830*. It became pronounced after* 1815*. By* 1850 *industrialization had become widespread in Western Europe as well as the northeastern United States. This process eventually took manufacturing out of the home and workshop. Power-driven machines replaced handwork, and factories developed as the best way of bringing together the machines and the workers to operate them.*

As industrialization grew, private investors and financial institutions were needed to provide money for its further expansion. Financiers and banks thus became as important as industrialists and factories in the growth of the industry. For the first time in European history, wealthy businessmen called capitalists *took over the control and organization of manufacturing.*

This was a great turning point in the history of mankind. It changed the Western world from a rural and agricultural society to a basically urban

483 For further reading, see:

- Craig, G.A., *Europe Since 1815*, The Dryden Press: Hinsdale, IL, 1974, 620 pp.
- Talmon, J.L., *Romanticism and Revolt* (*Europe 1815–1848*), Harcourt, Brace and World: New York, 1970.

society. Industrialization brought many material benefits, but also created a large number of problems that still remain critical in the modern world. Some of the most acute of these problems today are air and water pollution, the depletion of natural resources and other man-made alterations of the biosphere.

The transformation of Europe's economy from agriculture to industry affected science in both direct and indirect ways: the invention of the steam engine stimulated interest in thermodynamics and the concepts of power, work and energy began to be formalized. The advent of chemical industry based on chemical processes accelerated the renaissance of the atomistic theory of matter. The kinetic theory of gases and the discovery of electromagnetism are also associated with the industrial age.

In addition, industrialization stimulated the rise of mechanical inventions, especially those associated with transportation on land and sea.

*The advent of the efficient steam engine (**Watt**, 1769), the steam locomotive (**Trevithick**, 1804), the steamboat (**Fulton**, 1807) and magnetic telegraphy (**Morse**, 1838) revolutionized transport, travel and communication.*

The first commercial steam railroad was opened between Liverpool and Manchester in 1830. *By* 1840 *the first transatlantic steamer line was established. In* 1844 *the first telegraph line was connected between Washington and Baltimore and in* 1850, *the first submarine cable was laid under the English Channel between Dover and Calais.*

*The negative reaction to Newtonian science and mathematics, and the industrial revolution that followed in its wake, found its powerful expression in the best-known lyric of **William Blake** (1810):*

And did those feet in ancient time
Walk upon England's mountains green?
And was the holy Lamb of God
On England's pleasant pastures seen?

And did the Countenance Divine
Shine forth upon our clouded hills?
And was Jerusalem builded here
Among these dark Satanic mills?

Bring me my Bow of burning gold:
Bring me my Arrows of desire:
Bring me my Spear: O clouds unfold!
Bring me my Chariot of fire.

I will not cease from Mental Fight,
Nor shall my Sword sleep in my hand

Till we have built Jerusalem
In England's green & pleasant Land.

Set to music, and usually misnamed Jerusalem, it is frequently sung as a hymn today, even though its allusions are obscure. Blake believed in the legend that England had once been part of Atlantis, the home of mankind during the Golden Age. To Blake mills are both the ugly factories and symbols of the chains of Newtonian science. The weapons of gold and fire are weapons of the imagination to be used in the Mental Fight to restore the harmony of reason and vision in man's thinking.

The Agricultural revolution of the 10^{th} millennium BCE and the Industrial revolution of the 18^{th} century AD, on the other hand, created deep breaches in the continuity of the historical process. With each one of these two Revolutions, a new story begins, dramatically and completely alien to the previous one. Continuity is broken between the cave-man and builders of the pyramids, just as continuity is broken between the ancient ploughman and the modern operator of a nuclear power station. Clearly, each Revolution had its roots in the past, but each created a deep break with the very same past; the first "Revolution" transformed hunters and food-gatherers into farmers and shepherds, while the second "Revolution" transformed farmers and shepherds into operators of machines fed with inanimate energy.

The ten millennia or so that separate the two "Revolutions" witnessed a great number of discoveries and innovations that increased man's control over energy sources, but until the Industrial Revolution man continued to rely mainly on plants, animals and other men for energy – plants for food and fuel, animals for food and mechanical energy, other men for mechanical energy. The use of other available sources – mainly wind and water power – remained limited.

If the Agricultural Revolution is the process whereby man came to control and increase the supply of biological converters (plants and animals), the Industrial Revolution can be regarded as the process whereby the large scale exploitation of new sources of energy by means of inanimate converters was set on foot.

Looking at things from this point of view, one easily understands the key role played by the cultural revolution of the 16^{th} and 17^{th} centuries is the shaping of the destiny of mankind. It was in fact the cultural revolution that gave to man the conceptual tools which enabled him to master new sources of energy. The conscious systematic investigation of phenomena revealed in man's environment became a fundamental cultural trait of early modern Europe since the days of the Renaissance.

In the north-west part of Europe the 16^{th} *and* 17^{th} *centuries witnessed also a most remarkable mercantile development which favored the accumulation of physical wealth and of entrepreneurial skills. In England these cultural and economic developments happened to coincide with a shortage of a traditional form of energy (timber) and the presence of large supplies of coal. It was the union of certain happy mental qualities with material resources of an altogether peculiar character that provided the explosive formula.*

In the second half of the 18^{th} *century,* ***James Watt*** *perfected previous discoveries and constructed a steam engine (1765), the commercial use of which mounted through 1800–1820. Steam engines were used in metallurgical and textile activities as well as in mining coal and in surface transportation. As more machine power made it possible to produce more coal and to transport it at an enormously accelerated rate, more coal in its turn meant more machine power. Coal became a strategic element in the emergence and diffusion of the industrial civilization. It meant a rapidly expanding supply of energy that could be used for heating and lighting and for power in sea and land transportation and in almost all the various forms of industry*[484].

A cumulative interaction was soon set in motion; the extraordinary growth in the supply of energy stimulated economic growth, *which in turn stimulated* education and scientific research *leading to the discovery of new sources of energy*[485]*! Under the impact of these discoveries, the process quickened: the more energy was produced, the more energy was sought. Man turned to the sun, the tides, earth-heat, tropical waters, and atmospheric electricity. Then, toward the middle of the* 20^{th} *century, man discovered that energy could be obtained from atoms through the process of fusion or fission.*

Man needs capital to trap energy, and still more capital to exploit this energy for productive purposes. Capital accumulation is a necessary condition for any society's survival and progress. There is a definite correlation between capital and output. In a hunting economy, *the capital needs are very limited: a few bones (used as tools or weapons), and in more developed cultures: bows, arrows and stone implements. In an* agricultural economy *the capital needed is: stocks of seeds, fertilizers, ploughs, draught animals, silos, mills,*

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484 Around 1800 the world production of coal amounted to $15 \times 10^6 \frac{ton}{year}$. By 1860 it rose to $132 \times 10^6 \frac{ton}{year}$ ($= 1057 \times 10^6$ *megawatt-hours*) and by 1900 to $702 \times 10^6 \frac{ton}{year}$. By 1950 the corresponding figures were $1454 \times 10^6 \frac{ton}{year}$.

485 The year 1860 marks the advent of the American oil-well industry. The *gas-engine* was patented in the same year by **Lenoir**. The *electric* industry was born with Faraday's discoveries (1822–1831). By 1870 practical types of generators were already available to produce either direct or alternating current. The great consumption of electricity followed the evolution of the incandescent lamp.

boats, wagons, etc. In an industrial economy capital needs are still more complex and much larger: machinery, railways, chemical and atomic plants, dams, research laboratories, etc. The greater the production, the greater the volume of capital needed.

Capital is made possible by saving. If resources are consumed they are obviously not available for capital accumulation (if you eat your cow today, you cannot hope to have your milk tomorrow!). Only by forgoing present consumption can a society cumulate capital. In any agricultural society, given the low per capita income, saving per capita is rather low. Temples, pyramids, mansions, jewelery, warfare etc absorb a large quota of resources squeezed out of current income.

Furthermore, pre-industrial societies are typically characterized by inadequate transport facilities. Mass transportation was generally non-existent and communications were costly and insecure. Consequently, any pre-industrial society must have kept inventories of all commodities in much larger proportion to current production than any industrial society does.

To accomplish the transition from agricultural to industrial society the active population must acquire new skills and adopt new pattern of living to change the patterns of capital formation; further capital is needed for investment in education.

In all agricultural societies of our past we find that, mainly because of limitations of energy sources known and exploited, the great mass of people could hardly afford to satisfy anything but the more elementary needs: food, clothing and housing, and even these at rather unsatisfactory levels. Correspondingly, the most of the available resources were employed in agriculture, textile manufacture, and building. On the fringe, there was always some trade.

All historical records seem to show that where trade flourished, demographic and economic levels were the highest attainable within the range of agricultural possibilities. Actually, almost all the great agricultural civilizations of the pre-industrial past were founded on the expansion of the mercantile sector. Indeed, it was an exaggerated expansion of this sector in the 17^{th} *– and* 18^{th}*–century England that created the preconditions of the Industrial Revolution.*

Under this regime, new sources of energy, larger amounts of capital, and more efficient use of factors of production increased the per capita real income and improved the diet, clothing and housing of the masses. While expenditure on food decreased as a percentage of total private expenditure, expenditure on transportation, medical care, education, amusement etc, increased more than proportionally.

In conjunction with the exploitation of new kinds of energy and new prevailing consumption patterns, one observes a general decline in the relative importance of agriculture, which also suffers from the fact that the other productive sectors tend to lose their dependence on it. The building industry substitutes steel and cement for timber. *The textile industry substitutes artificial fibers (rayon, dacron, etc.) for natural ones. The pharmaceutical industry substitutes chemical products for* spices *and* herbs. *Even the food industry follows the trend: vitamin pills replace natural* fruits, *and Coca Cola replaces* wine.

Correspondingly, both the percentage of total active population employed in agriculture and the proportion of income produced by the agricultural sector shrink markedly while a great expansion is experienced in the new key sections: the chemical, the metallurgical, and the mechanical.

In an industrial society, the contribution of science *and scientific methods to production is obviously great. Consequently, the rate of growth of an industrial society is largely influenced by the amount of resources devoted to research and education and by the efficiency at which these resources are used. In an industrial society a good deal of economic growth is due to* technological change, *better education and the training and retraining of the labor force. The growth of inputs (labor and capital) and their progressive more efficient utilization brought forward an extraordinary expansion of production. Production increased faster than population, and thus* per capita *income grew over the long run.*

The passage of a society from one type of economic organization to another also implies drastic cultural and social changes. Four generation ago more than 2/3 of the people living on earth were peasants. In ca 2050, less than 1/3 will live in the fields. The Industrial revolution is spreading all over the world. We witness changes that are not merely industrial but also social and intellectual. A new style of life is emerging, as another disappears for ever. We know what is disappearing but we do not know what to expect.

This is an age of transition as well as an age of uncertainty and anguish. Every aspect of life has to be geared to the new models of production. Family ties are not on the wane and give way to broader perspectives for larger social groups. Individual savings *gives way to collective social services, undistributed profits, and taxes. The rounded* philosophical education of the few *is set aside for the technical training of the masses.* Artistic intuition *must give way to technical precision. New juridical institutions, new types of ownership and*

management, different distribution of income, new tastes, new values, new ideals have to emerge as an essential part of the industrialization process[486].

Economic activity depends on the earth's capacity to supply raw materials, to produce food, and to absorb waste. While it took 100,000 years for the world's human population to reach 6000 million, it will now take merely 50 years to add another 6000 million. However, improvement in quality of the human species is not necessarily alternative to a growth in quantity. A larger population may mean greater possibilities in the division of labor and economies of scale. These possibilities may contribute to the growth of per capita *income, to better levels of living, and to better education. But beyond certain points,* quantity and quality may become competitive[487].

It is inevitable that, as humans beings become over-abundant in relation to other resources, their marginal value diminishes and the dignity of human life deteriorates correspondingly. 100,000 years may seem a very long span of time, but from the point of view of the whole history of the earth and mankind, this time interval is a brief fragment. It is however remarkable that during this time span homo sapiens *has turned himself from savage into the conqueror of the earth: considering that the Neolithic Revolution diffused into Europe between 5000 BCE and 2000 BCE, slightly more than 150 generation separate Europeans from their ancestors.*

Thus, within a relatively small number of generations, man has come to control his environment and to master the powerful forces of Nature. However, the selective process that favored the success and the multiplication of the

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486 When 'industrialization' occurs *gradually*, these socio-cultural changes take place in a balanced process with economic changes. But when 'industrialization' is speeded up *artificially*, the socio-cultural environment may show much greater degree of resistance to change than the economic structure and *a socio-political revolution* may emerge (e.g. as in Africa, Latin America, Iran, Turkey and the Soviet Union); All the miseries and the hardships that follow then become part of the price of industrialization.

487 In 1794, **John Barrow** (1764–1848), during a journey through China, witnessed a peculiar scene:
"Of the number of persons who had crowded down the banks of the grand canal (to Canton), several had posted themselves upon the high projecting stern of an old vessel which, unfortunately, breaking down with the weight, the whole group tumbled with the wreck into the canal. Although numbers of boats were sailing about the place, non were perceived to go to the assistance of those that were struggling in the water; one fellow was observed very busily employed in picking up, with his boat-hook, the hat of a drowning man."
This happened because men were over-abundant and hats were scarce.

aggressive type was certainly not interrupted by the Neolithic Revolution. It continued to operate well into 'civilized' times and to a large extent still operates today, when man can commend immensely powerful forces, and his efficiency – for good and evil – has increased in spectacular fashion.

Single man, like Stalin and Hitler as recent history has dramatically demonstrated, can today bring about unspeakable catastrophes that affect the entire world and the entire human species[488].

[488] The naturalist and ethologist K. Lorenz, (1903–1989) himself a Nazi supporter during 1933–1945, wrote (1966): "An unprejudiced observer from another planet, looking upon man as he is today, in his hand the atom bomb, the product of his intelligence, in his heart the aggression drive inherited from his anthropoid ancestors, which this same intelligence cannot control, would not prophesy long life for the species."

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